

CONTINUITY AND IRRATIONAL NUMBER.

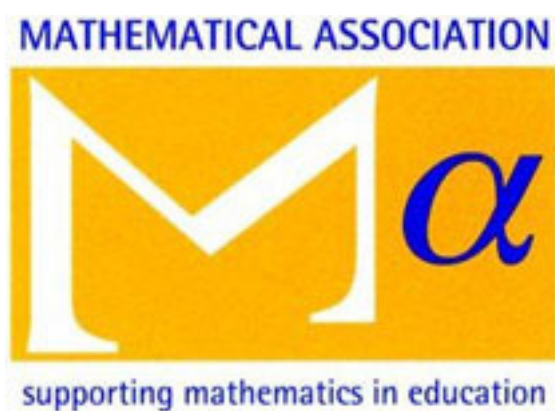
By G. J. WHITROW.

IN a previous article * I have outlined my conception of the rôle of history in the exposition of mathematical technique. In this article I attempt to provide my bare thesis with a respectable clothing of practicability. The treatment of Number and Continuity which follows is a short and very sketchy historical supplement to the technical treatment of the same subjects in, say, the earlier chapters of Hardy's *Pure Mathematics*.

Primitive man's conception of the world was based on his perception of universal difference and dissimilarity. In the prelogical stage "space" did not appear as homogeneous—in fact to this day the less developed races still retain the faculty of "orientation". Without the idea of the homogeneity of space, however, a science of geometry was impossible. Thus the preliminary step to the construction of such a science was the creation of a fictitious world. Moreover, between any two distinct shapes in visual space, despite every quantitative similarity, there is at least a qualitative difference in so far as they make separate contributions to the pattern presented to us. It was necessary, therefore, that the geometer should construct a space which was not the image of the space of sight, but "the projection and continuation of that impulse to spatiality already visible in visual perception to its ideal limit". Similarly an act of fiction was necessary to the development of arithmetic. Cardinal number and tallying arose from "equating the unequal" and assuming that there is a certain homogeneous element common to all discrete objects—a numerical element. In order to estimate and to give a name to any cardinal number it was ultimately necessary, historically if not logically, to develop the idea of ordinal number, and this involved an analysis of the term "many". Thus, at the very threshold of mathematics, man was faced with the fundamental polarity of synthesised space and analysed number, and so, in a sense, of continuity and discontinuity.

We must look to the Greeks for the earliest rational speculations on number and continuity. It is important to realise how keen was the Greek insight into the order and harmony of nature. As a French writer has remarked: "Le Grec aperçoit l'ordre dans les choses, le moderne l'y introduit". For the early Greek philosophers space and number were essentially *material* concepts. Thales' "water", Anaximenes' "air", Heraclitus' "fire" were anticipations of the "infinite"—by which was implied the notion of cosmic matter diffused throughout and identical with space. Pythagoras conceived the idea of the condensation of the primitive infinite around nomads as centres so that points became delimited. This idea may have been suggested to him as a result of his astronomical studies, for the framework of the heavens appeared to be an

* "The Importance of the History of Mathematics in relation to the Study of Mathematical Technique", *Math. Gazette*, XVI (October 1932), p. 225.



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Author(s): G. J. Whitrow

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IN a previous article * I have outlined my conception of the rôle of history in the exposition of mathematical technique. In this article I attempt to provide my bare thesis with a respectable clothing of practicability. The treatment of Number and Continuity which follows is a short and very sketchy historical supplement to the technical treatment of the same subjects in, say, the earlier chapters of Hardy's *Pure Mathematics*.

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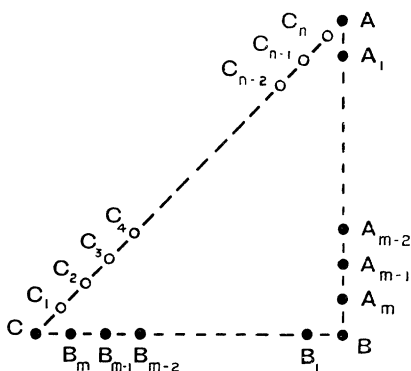
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aggregate of points. Thus we see that for Pythagoras the point was "concrete", it was "unity having position" and was *not* dimensionless.

The central doctrine of Pythagoras' philosophy was his assertion that "things are (like) numbers". Much useless verbiage has resulted from attempts to analyse this semi-mystical declaration, and it is well to bear the following points in mind :

- (1) that a corpuscular theory of matter was implied, and this involves the existence of indivisibles or "spatial units";
- (2) that by a "number" Pythagoras implied a "concrete number", i.e. an assemblage of points in definite order and formation, and thus
- (3) that in less epigrammatic language the dictum becomes, "All things are composed of spatial units which taken together constitute a number".

It has taken long ages for "number" to be deprived of its many non-mathematical associations. Leaving the theological and kabbalistic aspects out of account, one is still left to distinguish between logical and organic unity. Pythagoras' number includes both. Moreover, as will be seen, it includes measurable as well as numerical unity.



Apart from theoretical arguments, Pythagoras had at least one very important piece of experimental evidence for his doctrine. By employing a monochord, consisting of a string stretched over a resounding board with a movable bridge, he was able to measure the lengths corresponding to high and low notes on the same string. The significance of this experiment lies in the fact that it demonstrated a correlation between numbers and sound (one of the most volatile of phenomena). Nevertheless this deduction was based on the confusion of ratios of lengths with ratios of numbers. The importance of this confusion cannot be overemphasised. Unit length is a magnitude qualitatively indistinguishable from other magnitudes of the same kind, and if we identify it with numerical unity we are led either to treat the number one as divisible or to admit an

indivisible length. Either of these eventually would involve a series of antinomies.

At this stage we ought to remind ourselves that fractions were never regarded by the Greeks as numbers in themselves, but as ratios of integers. Pythagoras' doctrine of proportion was based on his corpuscular theory of matter—for him there existed an indivisible length, "the point", and any two lines stood to one another in a definite numerical ratio. For a time geometry became a branch of arithmetic, until the implications of Pythagoras' theorem came to light, and it was discovered that the diagonal of a square is "incommensurable" with respect to the sides. There was only one logical escape from this dilemma, unless one rejected the entire fabric of Pythagoras' system, and that was to assert that the diagonal of a square cannot be constructed.*

This difficulty was the result of the conflict of continuity and discontinuity—of homogeneous space and discrete number. It is important to notice that the Pythagoreans explicitly recognised the gaps in the number system. The logical consequence, therefore, should have been to recognise gaps in lengths "between the points". But they never seem to have faced this "fact", and it was reserved for Parmenides and his pupil Zeno to examine it with scientific ruthlessness. Zeno's so-called sophisms or paradoxes were simply *reductiones ad absurdum* of the Pythagorean thesis. They reveal that the existence of gaps in space necessarily implies the existence of gaps in time. Strictly speaking, Zeno was not talking the language of geometry but of kinematics, so to the Greek geometers his paradoxes may have appeared irrelevant. To modern mathematicians, however, the particularly relevant aspect is to be found in the ϵ argument. Most of modern analysis is ultimately based on inequalities of the form $x < \epsilon$, where ϵ is arbitrarily small. In Zeno's paradoxes the fundamental inequality is $x > \epsilon$, where ϵ is a (theoretically) fixed lower limit. Now for the Greek, I think it was an integral part of his outlook that there should be a fixed lower limit, ϵ . I think that to him the source of the dilemma in these paradoxes appeared to be the law of ordinal arithmetic, viz. that every integral number has a successor, for was it not this which made possible the existence of an infinite sequence? In him it was not "horror of the infinitesimal and indivisible" but "horror of the infinite process" which the Eleatic arguments inspired. Arithmetic, *quâ* arithmetic, was definitely brought into disrepute.

In this rapid survey of the Greek attitude towards our subject there are two other names which must not be omitted. Archimedes is famous for many things, but in this context he is chiefly famous for his brilliant use of the "method of exhaustions", originally devised by Eudoxus, one of the greatest pure mathematicians the

* My diagram illustrates what this sentence means. A comes between C_n and C_{n+1} on $CC_1C_2\dots$ and thus the constructed diagonal "falls short of" or "goes beyond" A .

N.B.—To the Pythagoreans the "strips" C_1C_2 , etc., and *not* the partitions C_1 , $C_2\dots$, etc., were the "points" of the diagonal.

world has ever seen.* This method was the Greek equivalent to our Newtonian Integral Calculus, and depended on the notion of upper and lower approximations to the required answer. It was based on the important axiom (due to Eudoxus)—now known as Archimedes' axiom: "however small x may be and however big y there exists an integer n such that $nx > y$ ". One notices that x may never be zero nor y infinite, or, more precisely, any number system to which this axiom applies must include or exclude infinity and zero together. As we know, the Greeks excluded both, and it was reserved for the Hindus and Arabians to pave the way for the modern "solution" of the problem of correlating the spatial and the numerical.

During the sixteen or more centuries which elapsed between the death of Archimedes and the rise of the Renaissance mathematicians, the concept "number" was gradually extended. Symbolism was the primary cause. The idea of indeterminate number, revealed par excellence in Algebra, compelled mathematicians to resort to symbolic solutions, irrespective of whether they had a meaning or not. But to write down the meaningless is to endow it with some meaning, and so gradually zero, fractions, and later negative numbers were admitted on equal terms with integers. It was not unnatural that such an extension of the idea of number should have been slow and painful, and it is not surprising to discover that unity was not regarded as a number until a very late date. The Greeks regarded it as the origin of numbers. For them a number was essentially a plurality; unity was therefore never conceived as a number in the full meaning of the word. This idea survived for a long time, and we find in the Arithmetic (1585) of no less celebrated a mathematician than Stevinus of Bruges a long logical discussion establishing one as a number. Despite his logical pedantry on this point he was not squeamish about treating irrationals as if they obeyed the same laws as ratios. The time was approaching, in fact, for a new coalescence of geometry and arithmetic.

In the early seventeenth century Descartes invented his Analytical Geometry on the assumption that anything which is to unity as one linear segment to another may be regarded as a number. It is important that we should be clear on the following:

- (1) that in the winter of 1619-20 Descartes studied the mathematics of Pythagoras and thus was fully aware of that system;
- (2) that the Pythagorean and Cartesian number concepts were entirely different. Pythagoras correlated *lengths* and *numbers*. Descartes (though he used the language of mensuration) implicitly, at least, correlated *points* and *number-sets*, or coordinates. Just as unity may be regarded numerically and metrically, so zero may be regarded on the one hand as the symbol for the null-class, and on the other as the origin of a frame of reference. It was the latter idea which Descartes

* Eudoxus and Archimedes are, in a sense, parallel to Newton and Gauss, respectively.

introduced and as a result of which he was able to attach a specific numerical label to every point in virtue of its position with respect to zero and the axes. Despite superficial similarity, the analytical geometries of Descartes and of Apollonius were fundamentally different, and that of Descartes was by far the more fruitful. As a result of this, guided by intuition, and unperturbed by logic, geometers saw their ancient science develop an amazing second youth.

Meanwhile, with just as little respect for logic, the problem of infinitesimals was treated by Cavalieri, Kepler, Fermat, Barrow and eventually by Newton and Leibnitz. In the Differential Calculus, as in Analytical Geometry, the practical results were so astounding that these alone seemed more than sufficient justification for the validity of the theory. Nevertheless, the logical difficulties were still present. Newton spoke of a flow and of fluxions but he determined these fluxions by treating the flow as if it were a succession of minute jumps. A Zeno, however, was destined to arise to point out this and many other analogous antinomies. Bishop Berkeley smote the mathematicians hip and thigh, pointing out with much logic and no less wit that infinitesimals had to fulfil simultaneously the contradictory properties of being both equal to zero and different from zero. As a result of his polemics the subsequent treatment of the calculus in England, where it was regarded as a branch of geometry, was, from the standpoint of rigour, far in advance of the continental, though achieving far fewer spectacular results.

On the continent, however, logic was almost abandoned. As Spengler ironically remarks, "The age of refined scepticism saw the emergence of one seemingly impossible truth after another". Thus, for example, infinite series were used irrespective of whether they were convergent or divergent, and infinite processes such as integration and summation were inverted purely at the whim of the writer and independent of any logical criteria. With the turn of the century logic began slowly to emerge out of the welter of new tricks and devices with which the mathematical market had been flooded. Cauchy paved the way, but the fundamental problem, with which we are concerned here, was not really tackled until the middle of the nineteenth century by Dedekind, Weierstrass and Cantor. Meanwhile the number domain had undergone more extensions. Abel's work on the quintic in 1826 revealed the "existence" of algebraic numbers not expressible either as rationals or as radicals, and in 1844 Liouville discovered the class of transcendentals, to which e and π were later shown to belong.

Cauchy defined an irrational number as a limit. By a limit he understood a certain fixed value towards which a variable approaches indefinitely near. If this limit is not a rational number, then if we start from the field of rational numbers the limit is fictitious, and thus if we define irrationals as limits they become logically non-existent in our arithmetic. The first person to evade this difficulty successfully and to divorce the apparently incompatible notions of irrationals and limits was Dedekind—in his famous pamphlet,

Stetigkeit und irrationale Zahlen (published 1872). The essential idea occurred to him in 1858, but it was dependent, implicitly at least, on the “principles of permanence” first stated by Hankel in 1867. These may be stated, as Danzig (*Number: the language of Science*) states them, in the form of the following definition :

“A collection of symbols, infinite in number, is called a number field, and each individual element in it has a number

- (1) if among the elements of the collection we can identify the sequence of *natural numbers* ;
- (2) if we can establish criteria of rank which will permit us to tell of any two elements whether they are equal or, if not equal, which is the greater, these criteria reducing to the natural criteria, when the two elements are natural numbers ;
- (3) if for any two elements of the collection we can devise a scheme of addition and multiplication which will have the commutative, associative and distributive properties of the characteristic operations bearing these names and which will reduce to the natural operations when the two elements are natural numbers ”.

Summarising all this, we may say that if any system submits to operations *as if* it were composed of bona-fide numbers then from the mathematical point of view we may actually regard it *as* a system of numbers. With this idea latent in his mind Dedekind framed the axiom now known after him and Cantor conjointly : “It is possible to assign to any point on a line a unique *real* number, and conversely any real number can be represented in a unique manner by a point on a line ”. This axiom, like so many axioms, was simply a definition—the arithmetical definition of a straight line. In essence it is fundamental in Cartesian geometry.

To my mind the striking thing about Dedekind’s theory is his conception of a point. I think that for him, as for Clifford, a point “is not an idea got at by assuming a small particle to become smaller and smaller without any limit, but it is the boundary between two adjacent portions of a line, which is the boundary between two adjacent portions of a surface, which is the boundary between two adjacent portions of space ”. The classical notions of a point either as the generator of a line or as the limit of a decreasing magnitude had involved a host of antinomies. By introducing the notion of a point as a partition of a line, and correlatively of an irrational as the partition of rational numbers, Dedekind appeared to have abolished completely the infinitesimal and the infinite process. Actually this was not the case, for, as we can easily see, the partition process becomes trivial if applied to only a finite set of rationals. It is essential that we should partition *all* the rational numbers. We seem inevitably driven to the conclusion that in order to connect the continuous and the discontinuous, geometry and arithmetic, we must invoke the infinite.

Cantor’s theory, of which Weierstrass’s is a particular case, proceeds entirely differently. Just as Frege and Russell define the cardinal number 2, say, as the class of all couples, so Cantor regarded

the irrational as defined by, and therefore equivalent to, an infinite convergent sequence of certain specified rationals. Thus an irrational was really not one number but a specified infinite set of rational numbers. Moreover, convergence was defined in a purely arithmetic manner, making no reference to infinitesimals.

In conclusion, let us examine the Cantor-Dedekind notion of the linear continuum, on which so much modern analysis is based. Any real number is represented by a point on a line. The totality of real numbers obeys the following rules :

- (1) it is well ordered ;
- (2) it is infinite in extent ;
- (3) it is everywhere dense (between any two distinct numbers a third may be constructed) ;
- (4) it contains all its limit points ;
- (5) it contains as a sub-domain the totality of rational numbers.

This sub-domain obeys (1), (2) and (3), but not (4). It is therefore said to be "compact", whereas the real domain is said to be "perfect", and is thus said to be a "continuum". This notion of the continuum is a highly ingenious theoretical substitution for the vague empirical notion of the continuous. Indeed, from the point of view of the physicist and applied mathematician, no guess could have been more wild and further from the truth than Pythagoras' cry, "All things are numbers". In any but a highly academic sense, irrationals are *not* numbers. As Sir Oliver Lodge in his British Association address on "Continuity" (1913) pointed out, "Whenever a commensurable number is really associated with a natural phenomenon there is necessarily a noteworthy circumstance involved in the fact". Bearing this in mind it is not so paradoxical as Danzig (*vide Number: the language of Science*) seems to imagine that since the present acts as a partition and is neither part of the past nor part of the future it is truly irrational in the Dedekind sense.

G. J. WHITROW.

920. Warren, picking up a stone flung it out far. It described its semi-circle, and sank into oblivion again, in the wilderness of thorns and scrub and other stones.—Edward Thompson, *Lament for Adonis*, p. 173. [Per Mr. R. C. Fawdry.]

921. Some faint smattering of Arithmetic, or the everlasting laws of Numbers ; faint smattering of Geometry, everlasting laws of Shapes ; these things, we guess, not altogether in the dark, Rodriguez Francia did learn, and found extremely remarkable. Curious enough : that round Globe put into that round Drum, to touch it at the ends and all round, it is precisely as if you clapt 2 into the inside of 3, not a jot more, not a jot less : wonder at it, O Francia ; for in fact it is a thing to make one pause ! Old Greek Archimedeses, Pythagorases, dusky Indians, old nearly as the hills, detected such things ; and they have got across into Paraguay into this brain of thine, thou happy Francia. How is it too, that the Almighty Maker's Planets run, in those heavenly spaces, in paths which are conceivable in thy poor human head as Sections of a Cone ? The thing thou conceivest as an Ellipsis, the Almighty Maker has set his Planets to roll in that.—T. Carlyle, *Dr. Francia : Miscellanies* (1888), iv, p. 266.