

The ancient Greeks present: Rational Trigonometry

N J Wildberger
School of Mathematics and Statistics
UNSW Sydney 2052 Australia
webpages: <http://web.maths.unsw.edu/~norman/>

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Pythagoras' theorem, the area of a triangle as one half the base times the height, and Heron's formula are amongst the most important and useful results of ancient Greek geometry. Here we look at all three in a new and improved light, using *quadrance* not *distance*. This leads to a simpler and more elegant trigonometry, in which *angle* is replaced by *spread*, and which extends to arbitrary fields and more general quadratic forms.

Three ancient Greek theorems

There are three classical theorems about triangles that every mathematics student traditionally meets. To state these, consider a triangle $\overline{A_1A_2A_3}$ with side lengths $d_1 \equiv |A_2, A_3|$, $d_2 \equiv |A_1, A_3|$ and $d_3 \equiv |A_1, A_2|$.

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$$d_1^2 + d_2^2 = d_3^2.$$

Area of triangle *The area of a triangle is one half the length of the base times the height.*

Heron's formula *If $s \equiv (d_1 + d_2 + d_3)/2$ is the semi-perimeter of a triangle, then its area is*

$$\text{area} = \sqrt{s(s-d_1)(s-d_2)(s-d_3)}.$$

In this paper we will recast all three in simpler and more general forms. As a reward, we find that *rational trigonometry* falls into our laps, essentially for free. Our reformulation works over a general field (not of characteristic two), in arbitrary dimensions, and even with an arbitrary quadratic form—see [2], [3] and [4].

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Pythagoras' theorem

Euclid and other ancient Greeks regarded *area*, not *distance*, as the fundamental quantity in planar geometry. Indeed they worked with a straightedge and compass in their constructions, not a ruler and protractor. A line segment was measured by constructing a square on it, and determining the area of that square. Two line segments were considered equal if they were *congruent*, but this was independent of a direct notion of distance measurement.

Area is an *affine concept*: more precisely *proportions* between areas are maintained by linear transformations. Even with a different metrical geometry, for example a relativistic geometry in which $x^2 - y^2$ plays the role of $x^2 + y^2$, the notion of signed area defined by a determinant applies. In other words, in planar geometry area is a basic notion and may be considered prior to any theory of linear measurement.

To Euclid, Pythagoras' theorem is a relation about the areas of squares built on each of the sides of a right triangle. This insight has largely been lost in the modern formulation, but with a sheet of graph paper it is still an attractive way to introduce students to the subject, as the area of many simple figures can be computed by subdividing, translating and counting cells.

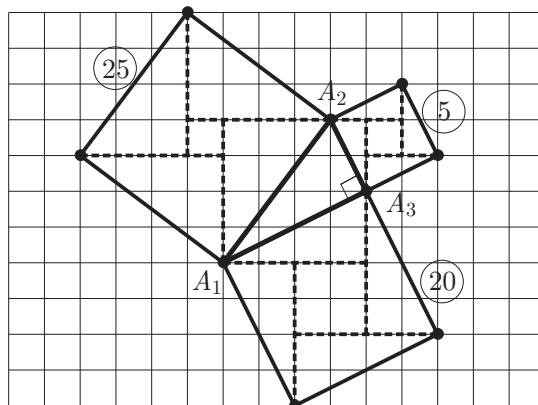


Figure 1: Pythagoras' theorem using areas

The squares on the sides of triangle $\overline{A_1A_2A_3}$ shown in Figure 1 have areas 5, 20 and 25. The largest square for example can be seen as four triangles which can be rearranged to get two 3×4 rectangles, together with a 1×1 square, for a total area of 25. So from this point of view Pythagoras' theorem is a result which can be established by *counting*, and the use of irrational numbers to describe lengths is not necessary. This applies to any right triangle with rational coordinates.

Following the Greek terminology of ‘quadrature’, we define the **quadrance** Q of a line segment to be the area of the square constructed on it. Pythagoras’ theorem allows us to assert that if $A_1 \equiv [x_1, y_1]$ and $A_2 \equiv [x_2, y_2]$, then the quadrance between A_1 and A_2 is

$$Q(A_1, A_2) = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

So for example the quadrance between the points $[0, 0]$ and $[1, 2]$ is $Q = 5$. The usual distance between the points is the ‘square root’ of the quadrance and requires a prior theory of irrational numbers. Clearly the irrational number $\sqrt{5} \approx 2.236067977\dots$ is a far more sophisticated and complicated object than the natural number 5.

In statistics, variance is more natural than standard deviation. In quantum mechanics, wave functions are more basic than probability amplitudes. In harmonic analysis, L^2 is more pleasant than L^1 . In geometry, *quadrance is more fundamental than distance*.

For a triangle $\overline{A_1 A_2 A_3}$ we define the quadrances $Q_1 = Q(A_2, A_3)$, $Q_2 = Q(A_1, A_3)$ and $Q_3 = Q(A_1, A_2)$. Here then is *Pythagoras’ theorem as the Greeks viewed it*—and it extends to arbitrary fields, to many dimensions, and even with general quadratic forms.

Theorem 1 (Pythagoras) *The lines $A_1 A_3$ and $A_2 A_3$ of the triangle $\overline{A_1 A_2 A_3}$ are perpendicular precisely when*

$$Q_1 + Q_2 = Q_3.$$

Area of a triangle

The area of a triangle is one-half the base times the height. Let’s see how we might remove some of the irrationalities that occur with this ancient formula by looking at an example. The area of the triangle $\overline{A_1 A_2 A_3}$ in Figure 2 is one

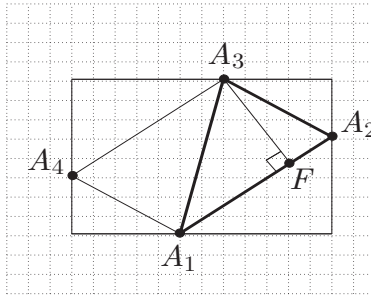


Figure 2: A triangle and an associated parallelogram

half of the area of the associated parallelogram $\overline{A_1 A_2 A_3 A_4}$.

The latter area may be calculated by removing from the circumscribed 12×8 rectangle four triangles, which can be combined to form two rectangles, one 5×3 and the other 7×5 . The area of $\overline{A_1 A_2 A_3}$ is thus 23.

To apply the one-half base times height rule, the base $\overline{A_1 A_2}$ by Pythagoras has length

$$d_3 = |A_1, A_2| = \sqrt{7^2 + 5^2} = \sqrt{74} \approx 8.602\,325\,267\,04\dots$$

To find the length h of the altitude $\overline{A_3 F}$, set the origin to be at A_1 , then the line $A_1 A_2$ has Cartesian equation $5x - 7y = 0$ while $A_3 = [2, 8]$. A well-known result from coordinate geometry then states that the distance $h = |A_3, F|$ from A_3 to the line $A_1 A_2$ is

$$h = \frac{|5 \times 2 - 7 \times 8|}{\sqrt{5^2 + 7^2}} = \frac{46}{\sqrt{74}} \approx 5.347\,391\,382\,22\dots$$

If an engineer doing this calculation works with the surd forms of both expressions, she will notice that the two occurrences of $\sqrt{74}$ conveniently cancel when she takes one half the product of d_3 and h , giving an area of 23. However if she works immediately with the decimal forms, she may be surprised that her calculator gives

$$\text{area} \approx \frac{8.602\,325\,267\,04\dots \times 5.347\,391\,382\,22\dots}{2} \approx 23.000\,000\,000\,01.$$

The usual formula forces us to descend to the level of irrational numbers and square roots, even when the eventual answer is a natural number, and this introduces unnecessary approximations and inaccuracies into the subject. It is not hard to see how the use of quadrance allows us to reformulate the result.

Theorem 2 (Triangle area) *The square of the area of a triangle is one-quarter the quadrance of the base times the quadrance of the corresponding altitude.*

As a formula, this would be

$$\text{area}^2 = \frac{Q \times H}{4}$$

where Q is the quadrance of the base and H is the quadrance of the altitude to that base.

Heron's or Archimedes' Theorem

The same triangle $\overline{A_1 A_2 A_3}$ of the previous section has side lengths

$$d_1 = \sqrt{34} \quad d_2 = \sqrt{68} \quad d_3 = \sqrt{74}.$$

The semi-perimeter s , defined to be one half of the sum of the side lengths, is then

$$s = \frac{\sqrt{34} + \sqrt{68} + \sqrt{74}}{2} \approx 11.339\,744\,206\,6\dots$$

Using the usual Heron's formula, a computation with the calculator shows that

$$\text{area} = \sqrt{s(s - \sqrt{34})(s - \sqrt{68})(s - \sqrt{74})} \approx 23.000\,000.$$

Again we have a formula involving square roots in which there appears to be a surprising integral outcome. Let's now give another form of Heron's formula, with a new name. Arab sources suggest that Archimedes knew Heron's formula earlier, and the greatest mathematician of all time deserves credit for more than he currently gets.

Theorem 3 (Archimedes) *The area of a triangle $\overline{A_1A_2A_3}$ with quadrances Q_1, Q_2 and Q_3 is given by*

$$16 \text{ area}^2 = (Q_1 + Q_2 + Q_3)^2 - 2(Q_1^2 + Q_2^2 + Q_3^2).$$

In our example the triangle has quadrances 34, 68 and 74, each obtained by Pythagoras' theorem. So Archimedes' theorem states that

$$16 \text{ area}^2 = (34 + 68 + 74)^2 - 2(34^2 + 68^2 + 74^2) = 8464$$

and this gives an area of 23. In rational trigonometry, the quantity

$$\mathcal{A} = (Q_1 + Q_2 + Q_3)^2 - 2(Q_1^2 + Q_2^2 + Q_3^2)$$

is the **quadrea** of the triangle, and turns out to be the single most important number associated to a triangle. Note that

$$\begin{aligned} \mathcal{A} &= 4Q_1Q_2 - (Q_1 + Q_2 - Q_3)^2 \\ &= - \begin{vmatrix} 0 & Q_1 & Q_2 & 1 \\ Q_1 & 0 & Q_3 & 1 \\ Q_2 & Q_3 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}. \end{aligned}$$

It is instructive to see how to go from Heron's formula to Archimedes' theorem. In terms of the side lengths d_1, d_2 and d_3 :

$$\begin{aligned} 16 \text{ area}^2 &= (d_1 + d_2 + d_3)(d_1 + d_2 - d_3)(-d_1 + d_2 + d_3)(d_1 - d_2 + d_3) \\ &= \left((d_1 + d_2)^2 - d_3^2\right) \left(d_3^2 - (d_1 - d_2)^2\right) \\ &= (2d_1d_2 + (d_1^2 + d_2^2 - d_3^2))(2d_1d_2 - (d_1^2 + d_2^2 - d_3^2)) \\ &= 4d_1^2d_2^2 - (d_1^2 + d_2^2 - d_3^2)^2 \\ &= 4Q_1Q_2 - (Q_1 + Q_2 - Q_3)^2 \\ &= (Q_1 + Q_2 + Q_3)^2 - 2(Q_1^2 + Q_2^2 + Q_3^2). \end{aligned}$$

Archimedes' theorem implies another formula of considerable importance.

Theorem 4 (Triple quad formula) *The three points A_1, A_2 and A_3 are collinear precisely when*

$$(Q_1 + Q_2 + Q_3)^2 = 2 (Q_1^2 + Q_2^2 + Q_3^2) .$$

The proof is of course immediate, as collinearity is equivalent to the area of the triangle being zero.

Spread between lines

An *angle* is the ratio of a *circular distance* to a *linear distance*, and this is a complicated concept. To define an angle properly *you require calculus*, an important point essentially understood by Archimedes. Vagueness about angles, and the accompanying ambiguities in the definition of the circular functions $\cos \theta$, $\sin \theta$ and $\tan \theta$ weaken most calculus texts.

There is a reason that classical trigonometry is painful to students—it is *based on the wrong notions*. As a result, mathematics teachers are forced to continually rely on $90 - 45 - 45$ and $90 - 60 - 30$ triangles for examples and test questions, which makes the subject very narrow and repetitive.

Rational trigonometry, developed in [2], see also [4], shows how to simplify and enrich the subject, leading to greater accuracy and quicker computations. We want to show that the basic results of this new theory *follow naturally from the above presentation of Pythagoras' theorem, the Triangle area theorem, and Archimedes' theorem*.

The key innovation is to replace angle with a completely algebraic concept. The separation between lines l_1 and l_2 is captured rather by the notion of *spread*, which may be defined as the ratio of two quadrances as follows.

Suppose l_1 and l_2 intersect at the point A . Choose a point $B \neq A$ on one of the lines, say l_1 , and let C be the foot of the perpendicular from B to l_2 . Then

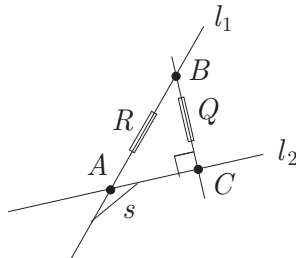


Figure 3: Spread s between two lines l_1 and l_2

the **spread** s between l_1 and l_2 is

$$s = s(l_1, l_2) = \frac{Q(B, C)}{Q(A, C)} = \frac{Q}{R}.$$

This ratio is independent of the choice of B , according to Thales, and is defined *between lines, not rays*. Parallel lines are defined to have spread $s = 0$, while perpendicular lines have spread $s = 1$.

The spread between the lines l_1 and l_2 with equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ turns out to be

$$s(l_1, l_2) = \frac{(a_1b_2 - a_2b_1)^2}{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}. \quad (1)$$

Since this is a rational expression, spread becomes a useful concept also over general fields, although in this paper we stick to the usual situation over the decimal numbers.

You may check that the spread corresponding to 30° or 150° or 210° or 330° is $s = 1/4$, the spread corresponding to 45° or 135° etc. is $s = 1/2$, and the spread corresponding to 60° or 120° etc. is $3/4$.

The following *spread protractor* was created by M. Ossmann [1].

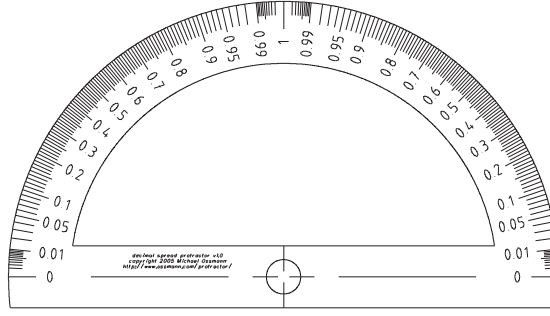


Figure 4: A spread protractor

We use the notation that a triangle $\overline{A_1A_2A_3}$ has quadrances Q_1, Q_2 and Q_3 , as well as spreads s_1, s_2 and s_3 , labelled as in Figure 5. Note the diagrammatic conventions that help us distinguish these quantities from distance and angle.

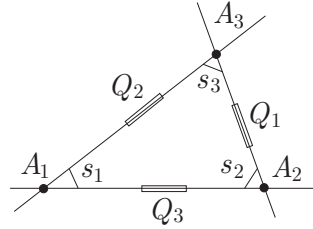


Figure 5: Quadrances and spreads of a triangle

Rational trigonometry

Let's see how to combine the three ancient Greek theorems as restated above to derive the main laws of rational trigonometry, independent of classical trigonometry, and without any need for transcendental functions. If H_3 is the quadrance of the altitude from A_3 to the line A_1A_2 , then the Triangle area theorem and the definition of spread give

$$\text{area}^2 = \frac{Q_3 \times H_3}{4} = \frac{Q_3 Q_2 s_1}{4} = \frac{Q_3 Q_1 s_2}{4}.$$

By symmetry, we get the following analog of the Sine law.

Theorem 5 (Spread law) *For a triangle with quadrances Q_1, Q_2 and Q_3 , and spreads s_1, s_2 and s_3 ,*

$$\frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3} = \frac{4 \text{ area}^2}{Q_1 Q_2 Q_3}.$$

By equating the formulas for 16 area^2 obtained from the Triangle area theorem and Archimedes' theorem, we get

$$\begin{aligned} 4Q_2Q_3s_1 &= (Q_1 + Q_2 + Q_3)^2 - 2(Q_1^2 + Q_2^2 + Q_3^2) \\ &= 2Q_1Q_2 + 2Q_1Q_3 + 2Q_2Q_3 - Q_1^2 - Q_2^2 - Q_3^2. \end{aligned}$$

Rearranging gives the following analog of the Cosine law.

Theorem 6 (Cross law) *For a triangle with quadrances Q_1, Q_2 and Q_3 , and spreads s_1, s_2 and s_3 ,*

$$(Q_1 - Q_2 - Q_3)^2 = 4Q_2Q_3(1 - s_1).$$

Now substitute $Q_1 = s_1D$, $Q_2 = s_2D$ and $Q_3 = s_3D$, where $D = Q_1Q_2Q_3/4 \text{ area}^2$ from the Spread law into the Cross law, and cancel the common factor of D^2 . The result is the relation

$$(s_1 - s_2 - s_3)^2 = 4s_2s_3(1 - s_1)$$

between the three spreads of a triangle, which can be rewritten more symmetrically as follows.

Theorem 7 (Triple spread formula)

$$(s_1 + s_2 + s_3)^2 = 2(s_1^2 + s_2^2 + s_3^2) + 4s_1s_2s_3.$$

This formula is a deformation of the Triple quad formula by a single cubic term, and is the analog in rational trigonometry to the classical fact that the three angles of a triangle sum to $3.14159265359\dots$

The *Triple quad formula*, *Pythagoras' theorem*, the *Spread law*, the *Cross law* and the *Triple spread formula* are the five main laws of rational trigonometry.

We now see these are closely linked to the geometrical work of the ancient Greeks.

As demonstrated at some length in [2], these formulas and a few additional ones suffice to solve the majority of trigonometric problems, usually more simply, more accurately and more elegantly than the classical theory involving transcendental circular functions and their inverses. As shown in [3] and [5], the same formulas extend to geometry over arbitrary fields (not of characteristic two) and with general quadratic forms.

In retrospect, the blind spot first occurred with the Pythagoreans, who initially believed that all of nature should be expressible in terms of natural numbers and their proportions. When they discovered that the ratio of the length of a diagonal to the length of a side of a square was the incommensurable proportion $\sqrt{2} : 1$, legend has it that they tossed the expositor of the secret overboard while at sea.

Had they maintained their beliefs in the workings of the Divine Mind, and stuck with the *squares of the lengths as the crucial quantities in geometry*, then mathematics would have had a significantly different history, Einstein's special theory of relativity would possibly have been discovered earlier, algebraic geometry would have quite another aspect, and students would today be studying a simpler and more elegant trigonometry—much more happily!

References

- [1] M. Ossmann, 'Print a Protractor', download online at <http://www.ossmann.com/protractor/>.
- [2] N. J. Wildberger, *Divine Proportions: Rational Trigonometry to Universal Geometry*, Wild Egg Books, Sydney, 2005, <http://wildegg.com>.
- [3] N. J. Wildberger, Affine and Projective Rational Trigonometry, 2006 arXiv:math/0612499.
- [4] N. J. Wildberger, A Rational Approach to Trigonometry, *Math Horizons*, Nov. 2007.
- [5] N. J. Wildberger, One dimensional metrical geometry, *Geometriae Dedicata*, **128**, no. 1, 145-166, 2007.