

Mathematics & Music,
after Pythagoras

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Ross, K.L.
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One of the most famous discoveries of Pythagoras, or of the Pythagorean School (it is often difficult to tell the difference), is, according to G.S. Kirk & J.E. Raven, "that the chief musical intervals are expressible in simple mathematical ratios between the first four integers" [*The Presocratic Philosophers*, Cambridge University Press, 1964, p.229]. Thus, the "Octave=2:1, fifth=3:2, fourth=4:3" [p.230]. These ratios *harmonize*, not only *mathematically* but *musically* -- they are pleasing both to the mind and to the ear. This impressed the hell out of the Pythagoreans, who also honored the "first four integers" because those add up to **ten**, the perfect number, and can be displayed in a triangle (like all "triangular" numbers), the

"Tetractys of the Decade":

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3:2

4:3

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adding a fifth to a four, which requires multiplying the ratios, results in the octave: $\frac{3}{2} \times \frac{4}{3} = \frac{12}{6} = 2$. Unfortunately, as with some other Pythagorean mathematical inquiries, the simplicity, or even the truth, of this result disappears on further investigation.

Calling intervals the "fourth," "fifth," or "octave" (i.e. "eighth"), when they are part of a system of seven tones, is a little confusing. Adding four to five doesn't even equal eight, much less seven, but nine. What is going on, however, is the device of the "inclusive" counting of ordinal numbers, where we start a new cycle of numbering (the first) with the end of the previous cycle (octava, the eighth). This form of counting is discussed elsewhere in relation to calendars. The fourth and fifth thus reduce to three and four as numbers, which then do add up to seven.

The seven note scale of the Greeks is the **diatonic** or **heptatonic** scale, to which have been assigned the letters **A** through **G**,

C	D	E	F	G	A	B	C
1/1	9/8	5/4	4/3	3/2	5/3	15/8	2/1
first	second	third	fourth	fifth	sixth	seventh	octave

although the convention as shown in the table is to run a standard octave (now meaning a sequence of notes up to a doubled ratio) from C to C. "Middle C" is the reference point for all the keys on a piano, and for the musical scale

in general. "A above Middle C" is the benchmark note for which there is a definition in terms of physics, with a sound frequently now set at exactly 440 Hz (Hertz). The table then fills out the "simple mathematical ratios" that have traditionally been assigned to all the intervals. This already creates some problems. While the fourth and the fifth add up to the octave, if we try to do the same with the third (5/4) and sixth (5/3), or the second (9/8) and seventh (15/8), the ratios do not multiply out to 2, but to 25/12 and 135/64, or to 2.08 and 2.11, respectively. This may not be what Pythagoras would have expected.

One response might be that not every interval is equal. We hear that it is only a "half step" from E to F and from B to C, while it is a "whole step" between the other notes. What this is supposed to mean we can see on a piano, where there are black keys between the white keys, but no black keys between E and F or between B and C. This may muddle the mathematics of the ratios. However, we can check. In the table all the ratios have been given the common denominator of 24, and the actual

C	D	E	F	G	A	B	C
24/24	27/24	30/24	32/24	36/24	40/24	45/24	48/24
	3/24	3/24	2/24	4/24	4/24	5/24	3/24

interval between them, in 24ths, indicated. The pattern looks irregular. Between E and F there are only two 24ths, and this is half of the following two intervals, but it is more than half of the previous ones, and the interval between B and C is

actually equal (at three) to that between C and D. There are no "simple mathematical ratios" here.

I might ask then what the scale would look like if we *wanted* the scale to *evenly divide* the octave, with equal intervals between the notes. Since the problem of the musical scale is, as John Stillwell says, "*multiplication perceived as addition*" [*Yearning for the Impossible, The Surprising Truths of Mathematics*, A.K. Peters, Ltd., 2006, p.4], what we need to do is reduce multiples to sums. This can simply be done with logarithms, which by addition give us the products of multiplication (through the "law of exponents"). The logarithm of 2 is 0.301029996. If we divide this by 7 we get 0.043004285. Adding this in successive sums through six, taking the anti-log (i.e. raised to the power of ten), and multiplying by 24, we get the values in the accompanying

C	D	E	F	G	A	B	C
4.14	4.18	4.22	4.26	4.30	4.34	4.38	4.42

table, compared to the traditional ratios and our previous scale of 24ths. As it happens, every value

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One of the most famous discoveries of Pythagoras, or of the Pythagorean School (it is often difficult to tell the difference), is, according to G.S. Kirk & J.E. Raven, "**that the chief musical intervals are expressible in simple mathematical ratios between the first four integers**" [*The Presocratic Philosophers*, Cambridge University Press, 1964, p.229]. Thus, the "**Octave=2:1, fifth=3:2, fourth=4:3**" [p.230]. These ratios *harmonize*, not only *mathematically* but *musically* -- they are pleasing both to the mind and to the ear. This impressed the hell out of the Pythagoreans, who also honored the "first four integers" because those add up to **ten**, the perfect number, and can be displayed in a triangle (like all "[triangular](#)" numbers), the "Tetractys of the Decade":

$$\begin{array}{c} \cdot \\ \cdot \cdot \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \cdot \end{array} \begin{array}{l} 2:1 \\ 3:2 \\ 4:3 \end{array}$$

The Pythagoreans are supposed to have sworn their oaths by this device. In music, adding a fifth to a four, which requires multiplying the ratios, results in the octave: $3/2 \times 4/3 = 12/6 = 2$. Unfortunately, as with some other Pythagorean mathematical inquiries, the simplicity, or even the truth, of this result disappears on further investigation.

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denominator of 24, and the actual interval between them, in 24ths, indicated. The pattern looks irregular. Between **E** and **F** there are only two 24ths, and this is half of the following two intervals, but it is more than half of the previous ones, and the interval between **B** and **C** is actually equal (at three) to that between **C** and **D**. There are no "simple mathematical ratios" here.

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C	D	E	F	G	A	B	C	multiplying by 24, we get the values in the accompanying table, compared to the traditional ratios and our previous scale of 24ths. As it happens, every value is within a unit of the
1/1	9/8	5/4	4/3	3/2	5/3	15/8	2/1	
24/24	27/24	30/24	32/24	36/24	40/24	45/24	48/24	
24/24	26.5/24	29.3/24	32.3/24	35.7/24	39.4/24	43.5/24	48/24	

traditional values, except for the sixth (**B**), which is more than a unit out of step. The interval between these values does rise steadily up the scale, but we can excuse this as an artifact of the underlying multiplication. This explains the similar general trend in the traditional values. But we are embarrassed with the anomaly of the sixth and of the obvious lack of "simple mathematical ratios" when we go for precise values. Pythagoras would be mortified.

As it happens, most of the traditional ratios are not used. Why this is so we can see from the following table, where we take the interval of the fifth ($3/2$) and begin adding it successively -- where this is now done on the *twelve note scale*, the "chromatic" scale, where the black piano keys are added to the white ones (distinguished as "sharps," $\sharp$, or "flats," $\flat$, of the "natural," $\natural$, notes). We can compare the result with a baseline principle of the octave, that the ratio of *any interval* will be *doubled* in the following octave.

C	C $\sharp$ D $\flat$	D	D $\sharp$ E $\flat$	E	F	F $\sharp$ G $\flat$	G	G $\sharp$ A $\flat$	A	A $\sharp$ B $\flat$	B	C	Thus, $3/2$ times $3/2$ equals $9/4$, which is then the value for D in the second octave. Indeed, this value is the origin of the ratio $9/8$ in the first octave, which is simply divided by two from $9/4$. So far, so good. But we get in trouble by adding the next
1/1		9/8		5/4	4/3		3/2		5/3		15/8	2/1	
									10/3				
2/1		9/4		5/2	8/3		3		27/8		15/4	4/1	

fifth. This brings us up to $27/8$ for **A**. However, we can already derive a value for **A** by doubling the ratio in the first octave, which was $5/3$. So we end up with both $27/8$ (3.375) and $10/3$ (3.333) for the same interval. These are close, but not the same thing. Now, if we keep multiplying fifths, we actually get a ratio for every note in the scale, each of which then can be divided by two until we are down in the first octave. This means that all of the other traditional ratios can be discarded, and the whole system gets reconstructed on the basis of just two ratios, those of the octave, **2:1**, and of the fifth, **3:2**.

The following table works out this process. The blue ratios are our reference values for the octaves (the firsts and the eighths, "inclusively" counted). In red we follow the additions (by multiplication) of the fifths. Once we get a red value for any key, then we divide it down octave by octave to the first one. This gives us values for all the intervals and all the notes, although most are now very far from being "simple mathematical ratios." But, unfortunately, the problem of inconsistency found in the previous table occurs again.

C	C# D $\flat$	D	D# E $\flat$	E	F	F# G $\flat$	G	G# A $\flat$	A	A# B $\flat$	B	C
1/1	$\frac{2187}{2048}$	9/8	$\frac{19683}{16384}$	81/64	$\frac{177147}{131072}$	$\frac{729}{512}$	3/2	$\frac{6561}{4096}$	27/16	$\frac{59049}{32768}$	243/128	2/1
2/1	$\frac{2187}{1024}$	9/4	$\frac{19683}{8192}$	81/32	$\frac{177147}{65536}$	$\frac{729}{256}$	3	$\frac{6561}{2048}$	27/8	$\frac{59049}{16384}$	243/64	4/1
4/1	$\frac{2187}{512}$		$\frac{19683}{4098}$	81/16	$\frac{177147}{32768}$	$\frac{729}{128}$		$\frac{6561}{1024}$		$\frac{59049}{8192}$	243/32	8/1
8/1	$\frac{2187}{256}$		$\frac{19683}{2048}$		$\frac{177147}{16384}$	$\frac{729}{64}$		$\frac{6561}{512}$		$\frac{59049}{4096}$		16/1
16/1	$\frac{2187}{128}$		$\frac{19683}{1024}$		$\frac{177147}{8192}$			$\frac{6561}{256}$		$\frac{59049}{2048}$		32/1
32/1			$\frac{19683}{512}$		$\frac{177147}{4096}$					$\frac{59049}{1024}$		64/1
64/1					$\frac{177147}{2048}$							128/1
												$\frac{531441}{4096}$

After seven octaves, adding twelve fifths brings us to a value for C, $\frac{531441}{4096}$, which is different from the value we get, $\frac{128}{1}$, derived directly from our original value of the eighth, at $\frac{2}{1}$. The ratio between these two values is 1.013643265 ($\frac{531441}{524288}$, or $3^{12}/2^{19}$), which gets called the "*Pythagorean comma*" [cf. Stillwell, p.4] -- it is the dislocation between the system of octaves and the system of fifths.

This means that the whole Pythagorean project is now in shambles -- although, as Stillwell says, "the Pythagoreans may never have noticed" [p.20]. These scales *cannot* be constructed with "simple mathematical ratios." And it creates a dilemma for real musicians, which is how the scales are to be constructed at all. To be sure, music can be played using the intervals derived from adding fifths, or even using the original ratios, and the ear may not object -- despite using notes created by systems that are ultimately inconsistent. The differences are, after all, rather small, even for the original and traditional ratios. But it is annoying. There is a sort of Pythagorean itch that keeps us thinking that there should be a proper mathematical solution to the matter. This is not going to be as simple as what Pythagoras expected, but the belief continues that the fundamental ratio, the octave, $2:1$, can be reconciled with the division of the scale into other intervals.

The solution to the problem, or at least one solution, is to adjust the ratio of the fifth so that *it is commensurable* with seven octaves. Seven octaves is $128:1$, or 2^7 . We can derive a value for the fifth from

this simply by taking the 12th root, $2^{7/12}$, or **1.498307077**. This is certainly close to $3/2$ (**1.5**), but it is not exactly the same number. Stillwell says that this approach, "equal semitones" or "*equal temperament*" [p.21], was developed almost simultaneously in Europe and in China, by the Dutchman **Simon Steven** in 1585 and by **Zhu Zaiyu** (Chu Tsai-yü) in 1584 (during the **Ming Dynasty**). The Chinese cannot be said to exactly be following the Pythagorean project, but obviously they encountered similar paradoxes and were looking for an equally satisfying solution. That a solution was found simultaneously in both East and West is remarkable and rather wonderful.

There is some solace for Pythagoras here. The intervals are mostly no longer simple ratios of integers. The fifth is not $3:2$. But now it is $2^{7/12}$. That is a notation unfamiliar to the Greeks, who didn't get all that far thinking about powers and roots, but we can be sure that Pythagoras would have eaten it up. The red ratios in the table above thus can be reconstructed with successive powers of $2^{7/12}$. Not as simple as the original idea, but certainly just as mathematical. The following table works this out, and it does give us a simpler looking system than with the integer ratios.

C	C# D $\flat$	D	D# E $\flat$	E	F	F# G $\flat$	G	G# A $\flat$	A	A# B $\flat$	B	C
2^0	$2^{1/12}$	$2^{1/6}$	$2^{1/4}$	$2^{1/3}$	$2^{5/12}$	$2^{1/2}$	$2^{7/12}$	$2^{2/3}$	$2^{3/4}$	$2^{5/6}$	$2^{11/12}$	2^1
2^1	$2^{13/12}$	$2^{7/6}$	$2^{5/4}$	$2^{4/3}$	$2^{17/12}$	$2^{3/2}$		$2^{5/3}$	$2^{7/4}$	$2^{11/6}$	$2^{23/12}$	2^2
2^2	$2^{25/12}$		$2^{9/4}$	$2^{7/3}$	$2^{29/12}$	$2^{5/2}$		$2^{8/3}$		$2^{17/6}$	$2^{35/12}$	2^3
2^3	$2^{37/12}$		$2^{13/4}$		$2^{41/12}$	$2^{7/2}$		$2^{11/3}$		$2^{23/6}$		2^4
2^4	$2^{49/12}$		$2^{17/4}$		$2^{53/12}$			$2^{14/3}$		$2^{29/6}$		2^5
2^5			$2^{21/4}$		$2^{65/12}$					$2^{35/6}$		2^6
2^6					$2^{77/12}$							2^7

It is now evident from this table that the interval between successive values is always $2^{1/12}$. This is the perfect marriage of addition and multiplication. By adding the exponent $1/12$, we increase the ratio by the multiple $2^{1/12}$. In the end, we do actually get "simple mathematical ratios" of integers, but as *exponents* rather than as the final values themselves.

Finally, we can calculate the actual frequencies of sound for the octave above Middle C. This is based on the standard of 440.0 Hz for A above Middle C. We can multiply that by the reciprocal of $2^{3/4}$ to get the frequency of Middle C. That can then be multiplied in turn with each given ratio to derive all the values. The frequencies are given rounded off to the first decimal place.

C	C# D $\flat$	D	D# E $\flat$	E	F	F# G $\flat$	G	G# A $\flat$	A	A# B $\flat$	B	C
2^0	$2^{1/12}$	$2^{1/6}$	$2^{1/4}$	$2^{1/3}$	$2^{5/12}$	$2^{1/2}$	$2^{7/12}$	$2^{2/3}$	$2^{3/4}$	$2^{5/6}$	$2^{11/12}$	2^1
261.6	277.2	293.7	311.1	329.6	349.2	370.0	392.0	415.3	440.0	466.2	493.9	523.3

These values sometimes differ by one or two tenths of a Hertz in comparison to the values I originally copied from Isaac Asimov and that I cite in the section on the [Electromagnetic Spectrum](#). These small differences are insignificant but I am not aware why they would occur --Asimov's values may have been rounded off in a different way.

[Triangular Numbers](#)

[Pythagorean Triplets](#)

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Triangular Numbers

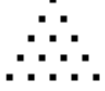
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One of the most intriguing areas of mathematics that drew the interest of the Pythagoreans was in **number series** (where *series* in Latin is both the singular and the plural). Our practice today of speaking of "squares" and "cubes" for numbers that are multiplied by themselves two or three times is an artifact of the way that the Pythagoreans liked to *visualize* their numbers as patterns of little dots. Beginning with **4**, **square** numbers can always be represented by little squares: $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$. The Pythagoreans then discovered that the addition of successive odd numbers produces successive squares.

integers	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
odds	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29
squares	1 $\begin{smallmatrix} \square \\ \square \end{smallmatrix}$	4 $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$	9 $\begin{smallmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{smallmatrix}$	16 $\begin{smallmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{smallmatrix}$	25 $\begin{smallmatrix} \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square \end{smallmatrix}$	36	49	64	81	100	121	144	169	196	225

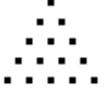
The favorite series of the Pythagoreans, however, were the **triangular** numbers. As we might imagine, these are the numbers with which we can construct little triangles of dots, beginning with **3**: $\begin{smallmatrix} \square \\ \square & \square \\ \square & \square & \square \end{smallmatrix}$. The rule for the construction of this series is very simple: add all the successive integers up to the desired point. **Three** is **1+2**, **15** = **1+2+3+4+5**, etc. Triangulars draw some small attention in modern mathematics, for instance in the proposition of **Pierre Fermat** (1601-1665) that *every number* "is either triangular or the sum of two or three triangular numbers" [cf. Constance Reid, *From Zero to Infinity*, 1955, A.K. Peters, Ltd., 2006, p.76]. However, a series with a *very similar rule* is encountered constantly in calculus and elsewhere. "Factorials" are the series produced by the successive *multiplication* of integers, rather than addition as with the triangulars. Factorials quickly become very large. In the table below, I have resorted to the notation for factorials, with an exclamation mark (!), because the actual numbers would make the

table too big. The factorial of **13** ($13!$) is already **6,227,020,800**. One of my calculators cannot go over **69!** without displaying an error message for a number that is too large to be calculated -- **69!** itself is already **1.711×10^{98}** . The great virtue of factorials is that their reciprocals rapidly become exceedingly small. One use of factorials may be seen in the continued series for the constant **e** and the trigonometric functions explored [elsewhere](#) at this site.

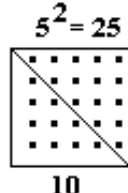
integers	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
triangulars	1 	3 	6 	10 	15 	21	28	36	45	55	66	78	91	105	120
factorials	1	2	6	24	120	720	7!	8!	9!	10!	11!	12!	13!	14!	15!

The Pythagoreans particularly honored the triangulars because one of them is the number **10**, and the Pythagoreans honored **10** as the perfect number. They are supposed to have sworn their oaths by the "Tetractys of the Decade," i.e. the little triangular image, . Their reasons for taking **10** as the perfect

number now seem a little silly -- that **10** is perfect because we have ten fingers and toes -- or redundant -- that most peoples count in tens (which is probably because we have ten fingers and toes -- although the [Babylonians](#) counted in *sixties* and the [Maya](#) in *twenties*). Nevertheless, their reverence for **10** had various interesting consequences, such as in their cosmology, where they thought there needed to be ten planets. It also gave to the triangulars a prominence that they have not enjoyed since. One Pythagorean discovery about triangulars is that every **square** number is the sum to two successive triangulars, e.g. **$10+15 = 25 = 5^2$** .

integers	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
triangulars	1 	3 	6 	10 	15 	21	28	36	45	55	66	78	91	105	120
squares	1 	4 	9 	16 	25 	36	49	64	81	100	121	144	169	196	225

The Pythagorean proof for this discovery gives us some important insight into how the very idea of proof developed in Greek mathematics. The Pythagoreans liked to *visualize* their proofs, like their numbers. Thus, they reasoned that, as any square can be divided by a diagonal into two triangles, any *square*

number should be divisible into two *triangular numbers*: . The truth of the proposition can

be read off the diagram. The earliest proofs for the Pythagorean Theorem were evidently of this kind, with the proofs using more abstract argument coming later. The triangulars thus occupy a critical position in the earliest days of the history of mathematics.

A proposition much like the Pythagorean discovery relative to squares and triangulars is presently one of the most famous unsolved problems in mathematics. This is "Goldbach's Conjecture," named after **Christian Goldbach** (1690–1764), which is that *every even number greater than two can be expressed as the sum to two prime numbers*. Actually, in 1742 Goldbach himself wrote a letter to the great mathematician **Leonhard Euler** (1707–1783) in which he proposed that every integer greater than two can be expressed as the sum of three primes. Since **1** is no longer regarded as a prime number, this is currently restated as *every integer greater than five can be expressed as the sum of three primes*. Euler replied to Goldbach that this proposition would actually be a *corollary* of one about even numbers as the sum of two primes. Thus "Goldbach's Conjecture" about even numbers (the "strong" conjecture) is directly due to Euler, but Golbach gets the credit for suggesting this *kind* of proposition. Thus, like the Pythagorean work, modern mathematics is still looking at relationships between number series, in this case between evens and primes. But, unlike the Pythagoreans with their diagram, Goldbach's Conjecture remains unproven -- although by 1995 the French mathematician Olivier Ramaré proved that every even number equal to or greater than **4** is the sum of no more than six primes. So we get closer.

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