

## THE SYSTEM OF PROPORTIONS OF THE PARTHENON: A WORK OF MUSICALLY INSPIRED ARCHITECTURE

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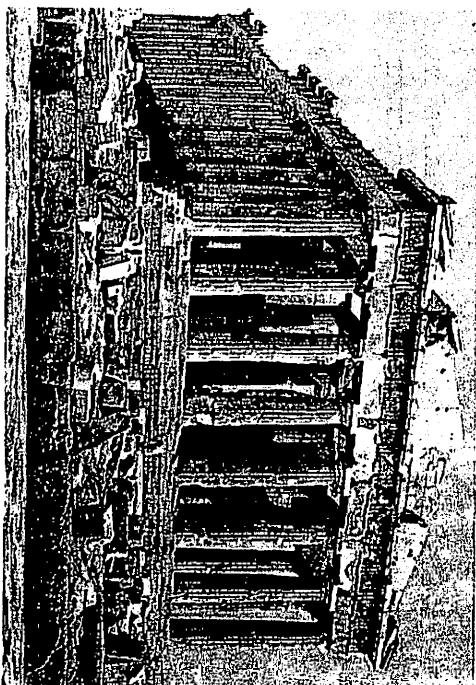
KAPPRAFF J.  
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The Parthenon was built between 447 and 438 B.C. at the height of Athenian power under Pericles, and as a lavish act of civic hubris that contributed to her downfall. It affirmed a superiority other Greek cities were not ready to concede. Major contributions of Anne Bulckens to Parthenon studies include her determination of alternate design modules and a distinctive "Parthenon foot" that enable every important dimension to be correlated in the ratios of a harmonic science the Greeks knew as "music".<sup>1</sup> Our contribution to Bulckens' work lies in trying to review it carefully within philosophical principles prevalent in the fifth century B.C. when the Parthenon was under construction, for which we rely on Philolaus, the earliest Pythagorean author, writing in Tarentum at the time of Parthenon's construction.<sup>2</sup> We believe her analysis may prove significant both to the study of other Greek temples and to a better understanding of Pythagorean influence on Greek ideals.

1. A MODERN DATA BASE. Surviving Parthenon elements were measured with great care in 1888 by Francis Cranmer Penrose — within a margin of error of 0.03% (roughly  $1/3300$ ) — relied upon by researchers ever since.<sup>3</sup> The architectural schema proposed by Bulckens, abstracted from the Penrose data, assumes a construction tolerance of no more than 0.25% (i.e.,  $1/500$ ) in deviating from her design, a figure recommended by scholars since the Parthenon congress in Basel of 1984. However, many of the measurements are within a tolerance considerably less than 0.2% (less than the difference between a perfect musical fifth of 3:2 and an equal-tempered fifth today). Thus Bulckens in her projected design — assumed to have been accomplished in two stages — permits herself no variation from Penrose's measured results as great as the inaudible subtleties in modern tuning theory.

2. THE PARTHENON IN HISTORICAL PERSPECTIVE. The Parthenon stands prominently on the Acropolis, an elevated site in the heart of Athens [fig. 1]. It is a Doric temple with some Ionic features. Three steps lead to a platform known as the stylobate. From this platform rise eight columns of the peristyle to the east and west facades, and seventeen columns to the north and south flanks. The columns support an entablature which consists of an architrave, a frieze, and a horizontal cornice. The frieze consists of a series of panels with relief work, called metopes, depicting scenes from the history and mythology of Greece. These metopes are interspersed by triple ridged triglyphs. Atop the metopes to the east and west is a pair of pediments with additional large sculptures finished in the round. Within the outer temple is an inner temple called the cella consisting of two closed spaces: the naos housing the statue of Athena by the great sculptor Phidias, and behind it a smaller room, the opisthodomos, serving as the new treasury of the Greek city-states that were members of the Delian League [fig. 2].

The Parthenon was constructed with several striking idiosyncracies. Its width of eight columns is contrasted with the normative six. Most notable is the curvature of stylobate, rising toward the middle more excessively than required for drainage, and with the axes of surrounding columns inclined slightly inwards, so that initial rectilinear alignments are intricately and purposefully deformed. Around the cella there is a

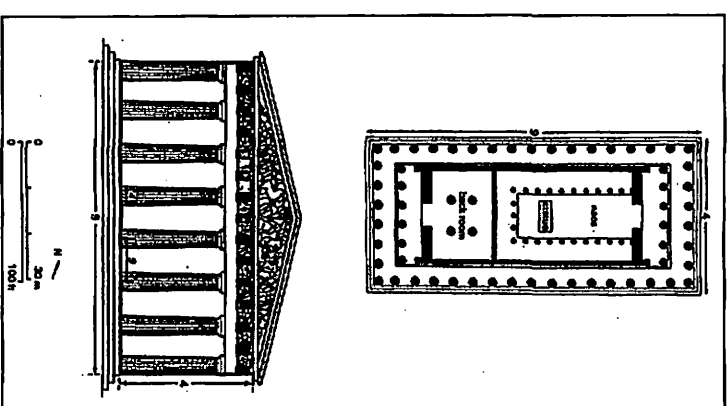


1. The Parthenon, a view from the northwest.

continuous frieze, the last feature to be completed, carved by an assembly of sculptors working within astounding limits of accuracy, but necessarily viewed through the surrounding pillars that obstruct visibility.

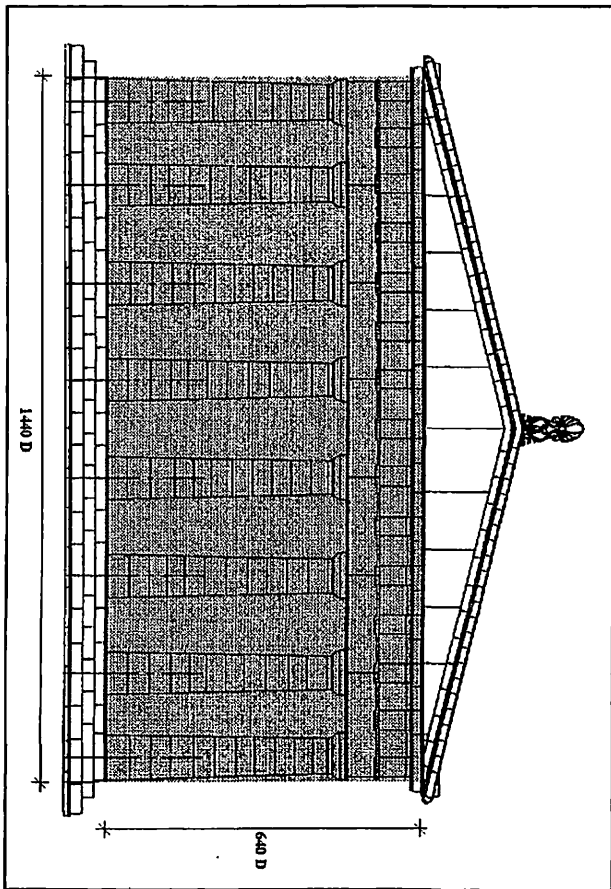
The idea of the temple was to advance Athens' claim to autochthony, historical continuity and primacy. This history began in the late bronze age with the Minoan civilization of Crete, surpassed later by the Mycenaean civilization. Parthenon sculpture reviews the stories and myths from these earlier ages as retold by Homer in the *Iliad* and the *Odyssey*. The Mycenaean civilization built palaces on elevated sites surrounded by defensive walls, and here in their Acropolis palace they built a shrine to Athena. Pericles' Parthenon, at the exact location of this old shrine, incorporated a new shrine in the area between the peristyle and the cella wall on the north side of the temple. Two earlier temples had been built on the same site. In 566 B.C. Peisistratos, the tyrant, built the Archaic Parthenon or Ur-Parthenon called the Hekatompedon or "hundred footer", dedicated to the mythical King Theseus. In 490 B.C. construction was begun on a second temple known as the "Older" Parthenon. Both were destroyed by the Persians, archenemies of the Athenians, but the idea of a "hundred footer" survived and it motivated Bulckens' effort to establish a plausible meaning for it in her own design.

Athena, sprung directly from the mind of Zeus, is depicted on the Acropolis as Nike, winged victory, but within the symbolism of Pythagorean *tern-ness* that focuses our interpretation here she is also both *motherless and virgin*, an intellectual principle associated with the number seven. Writing in the second century A.D., Plutarch makes this association with seven through analogy with the Egyptian Isis, but in an older source attributed to Philolaus and supported by Plato, symbolic seven established Athena as an intellectual principle, "ruler and teacher of all things; it is God. One ever-existing, stable, unmoving, itself like to itself, different from the rest". Within the first ten integers (that become Plato's abstract "form-numbers") 7 is a prime with no debts to any other integer but the "1" itself associated traditionally with Zeus, and 7 has no "children" (i.e. products or divisors of its own). Thus in the fifth century B.C. Athena still symbolically governed harmonic theory in *heptatonic* (seven-tone) subsets in gracious ways and in a formula that Plato preserves even while dethroning her as a goddess, for he admits reciprocal ratios of 7:5 into ratio theory as simplest approximations to the square root of 2, evident in the proportion 70:50:49:35.



2. The plan views of Parthenon and cella exhibiting the proportion 9:4.

3. DETERMINATION OF MODULAR VALUES. Vitruvius specifies the width of the triglyphs, centered above the main columns, as temple module. They alternate with square metopes "as broad as they are high", thus 1.5 times wider and matching triglyph height. This means that the 3:2 ratio of the musical fifth already embodied in triglyph height and width, also determines lateral spacing with the wider metopes. Triglyph width is divided into six parts and then given three deep vertical grooves, highly visible from a distance, and Bulckens hypothesizes that this width measures 2.5 "Parthenon feet" each of sixteen dactyls, in accord with Vitruvius' description of the Greek foot as consisting of *four palms* of *four fingers* each. These assumptions result in a modular value of  $2 \frac{1}{2} \times 16 = 40$  dactyls for the width of a triglyph and of  $2 \frac{1}{2} \times 40 = 60$  dactyls (or  $3 \frac{1}{4}$  feet) for its height (shared by sides of the square metopes). Bulckens points out that either triglyph width or height could have been used as building *modules*. Her plans subtly integrate feet and dactyls in a base 10 arithmetic with a potentially larger 60 dactyl alternative and thus with a base 60 arithmetic preferred in Greek astronomy. The rhythm of these alternating 3:2 ratios, seen from afar as we approach the building, thus anticipates how a *ratio theory* labeled "music" becomes applicable to the whole of the Pythagorean quadrivium including music, arithmetic, geometry, and astronomy, projected now as model for a more civilized



3. The east facade of the Parthenon with a 9:4 rectangle inscribed.

social order. The building's Homeric sculpture houses Phidias' elaborate statue of Athena as she projects the idea of a "civic" rule conceived by Homer but never actually developed in the *Iliad* or *Odyssey*. Thus the architecture looks both backwards and forwards; it contrasts what *was* with what *might be* in a way that miraculously still survives the monuments' physical desecration. The idea cannot be tarnished by destroying the building.

Bulkens mainly follows Vitruvius to arrive at her module of 857.6 mm, the value that the contemporary archaeologist Ernst Beeger determined by computer analysis to be the most recurring dimension in the Parthenon. This could have been done at any time since the Penrose measurements were made in 1888. Partly motivating her choice of 2.5 feet as modular width, thus equating the Parthenon foot with 343.2 mm (each temple had its own measure of a foot), was the fact that the temple then can be measured in a specific way for it to become a "hundred footer" as mentioned by Hesychios. Vitruvius' foot of 16 dactyls was considered one-sixth of a man's height, so that Bulkens' foot of 343 mm ( $= \frac{1}{6} \times 294$ , the standard Attic foot at this time) is an increase by one-sixth that gives the Parthenon the distinctive numerical value (7) specifically associated with Athena. A "Parthenon foot" of 16 dactyls was anticipated in the Temple of Apollo in Delos only a few years before the Parthenon was constructed (460 B.C.). She subdivides her dactyl into halves, thirds, and quarters so that  $\frac{1}{4}$  dactyl (about 5 mm) becomes the smallest dimension considered in her descriptions, and concludes that "when a measurement (i.e., in integers) is not equally convertible into both systems" (of feet and dactyls) "it is because the measurements have changed by the application of 'retirements'" during a second stage of design. Alternate measures, in her view, must be recognized as *intentional*, and therefore invite explanation.

4. FRAMING RATIOS. The Parthenon, built by the architects Ikhnos and Kallikrates, has long been known to be framed by a 9:4 ratio both in length to width and width to height. However, little else was discovered about its proportional system. Bulkens' *modular analysis* now lets us see the ratio of stylobate length to width as 81:36 modules, and of stylobate width to the height of the entablature as 36:16 [figs. 2 & 3]. Bulkens also was alert to Renaissance architectural theory that eyes and ears are pleased by the same ratios, and that a dissonant "musical ninth" of 9:4 (an "octave double" 8:4 plus a whole tone of 9:8) is harmonized by their geometric mean at 6 into consecutive perfect fifths of 3:2 (at 9:6 and 6:4).<sup>1</sup> But all ratios possessed reciprocal meanings and thus were indifferent to physical application. The primary concept was that of a *middle (mese)*, comfortable for the voice, and here we arbitrarily assign it to *pitch class D*, center of symmetry in our modern naming system, as a fixed tonal reference to reduce confusion with the highly variable arithmetic, always sensitive to context.<sup>2</sup>

Table 1. Nicomachomachus' table for expansions of the ratio

|                                        |     |     |    |     |     |     |     |
|----------------------------------------|-----|-----|----|-----|-----|-----|-----|
| 1                                      | 2   | 4   | 8  | 16  | 32  | 64  | [B] |
| 3                                      | 6   | [D] | 12 | 24  | [A] | 48  | [E] |
| 9                                      | [C] | 18  | 36 | [D] | 72  | 144 | [A] |
|                                        |     | 27  | 54 | [C] | 108 | 216 | [D] |
|                                        |     |     | 81 | [C] | 162 | 324 | [C] |
| Subjects are viewed as expanding "from |     |     |    | 243 | 486 | [C] |     |
| the middle", named as a constant on D. |     |     |    |     | 729 | [F] |     |

Europe inherited this theory through the writings of Nicomachus in the second century A.D.<sup>3</sup> His table of proportions for expansions of the musical fifth of 3:2 analyzes the system as correlating three different views: "the duple ratio in the width (where octave doubling has no effect at all upon *pitch class*), the triple ratio in the hypotenuse", and with a succession the 3:2 ratios "in the depth". His third column (4-6-9) demonstrates Bulkens' concern with the missing *geometric means* at 6 [fig. 4a]. Bulkens realized that a ratio of 9:4 as L:W could be repeated as W:H only when these limits are "squared" as in column 5 to extend ratios of 2.3 through 16-24-36-54-81. When the tones of this sequence are arranged in scale order they form a pentatonic scale. Bulkens looked for the two missing "means" at 24 and 54 and found them as modular L x W of the *inner temple* (housing the statue of Athena) *when integral modules are measured within the thickness of the walls*. Thus her initial choices of design units seem powerfully reinforced by this integration of Pythagorean theory that they now achieve almost *naturally*.

For Philolaus "The Universe is one, and it began to come into being from the center" (fragment 17), "and from the center upwards at the same intervals as those below". He defines tone ratios in two different ways — as precise ratios, and as approximate semitones — and the latter makes it easy to map harmonies in circles (i.e., cycles), again using 7 in the *intellectual* way associated with Athena, for he accords the circle its ancient fixed value of 12 [fig. 4a]. (This "Old Babylonian" bureaucratic constant descends from a conception of day and night as 6 double hours each, linked to 6 whole tones in a double octave 4:2::2:1, and with both 6 and 12 double hours on the horizontal axis, as at the equinox). For greater visual clarity we follow Plato's practice and reduce cosmology to a single octave, using 12:00 o'clock as our "zero hour" and assigning it to pitch class D (center of symmetry in modern notation) to reduce tonal confusion while studying numerical transformations. For Philolaus the octave 2:1 is an example of *unity* as a *first composite* (i.e., twelve as cyclic constant) consisting of "five whole tones and two semitones" (the latter undifferentiated), the fifth 3:2 "of three tones and



4. (a) The tone circle of 12 tones; (b) the whole tone (6:5) defined as the difference between the fifth (3:2) and the fourth (4:3); (c) the pentatonic scale; (d) the heptatonic scale.

a senitone" and the fourth 4:3 of "two tones and a senitone". Thus his whole-tone, as the difference between a fifth and fourth, has the standard Pythagorean value of  $9/8 (= 3^2/2^3)$  slightly oversized for the octave unless coupled with these "left-over" under-sized senitones [fig. 4b]. The pentatonic scale described above is shown in fig. 4c. This pattern appears to define the Egyptian "star" glyph and served as the hieroglyph for the underworld in ancient Egyptian mythology. Plato required *continued geometric progression* to be "bent round in circles" (i.e., cycles) and leads us to the appropriate arithmetic in various formulas. Each modal pattern is "sealed" by its largest integer, and other values are doubled sufficiently to lie within its half. The diagonals in table 1 (1-3-9-27-81-243-729) prove to be modular residues in this preferred tuning, and so any of them can be doubled enough times to become largest in some modal alignment, an arithmetically trivial operation easily done mentally. Thus one set of *smallest integers* is easily adapted to all seven heptatonic modal permutations, and their patterns of whole-tones and senitones can be viewed by rotating the tone circles as "handles of radiance", an ancient Mesopotamian metaphor embodied in Athena's statue with its ivory overlaid with gold. The tones are paired symmetrically around D and defined by reciprocal ratios. Note the overlap by a whole-tone A-G in the first pair of reciprocals [fig. 4d].

Here are the first nine tones from Plato's World Soul in *Timaeus*, defined by a 9/4 ratio that integrates Greek Dorian and Phrygian modes, the only modes he permits in his seven-tone models (pentatonic modes are not named) — presumably because the others introduce no new principles whatever. The tone circles shown here can be rotated to "enthone" each radial value in turn, every mode "sealed" by its own largest integer, but with string names, mode names, and appropriate integers swirling around it in various modal and arithmetical transformations. *The mode must be imagined in motion.* Plato's two Greek modes are commonly noted on the white digitals of modern keyboards, with the Greek Dorian octave on E and Phrygian on D respectively, framed 9/4 (=864/348) as an "octave" 2:1 (=768/384) plus a whole-tone of 9/8.

| E   | D   | C   | B   | A   | G   | F   | E   | D   |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 384 | 432 | 486 | 512 | 576 | 648 | 729 | 768 | 864 |

In other words the value of 96 for E from table 1 is multiplied by 8 and 4 to obtain the octave limits of  $768/384$  for the Dorian mode, while 216 for D is doubled and quadrupled resulting in the octave limits  $864/432$  of the Phrygian mode. All other tones are raised to the intervening octave by appropriate powers of 2. Fig. 4c illustrates the Phrygian mode around a tone circle. The circle can be rotated to "enthone" any tone as tonic of a unique pattern. Plato explained this tuning (we suppress most of his process) as resulting initially from a *continued geometric proportion*, his "world's best bonds", that is "bent round into a circle" (i.e., a cosmic cycle) so that "beginning and end" of the octave coincide, as we see here in both the E octave and D octaves, where imagination must alternate between perspectives.

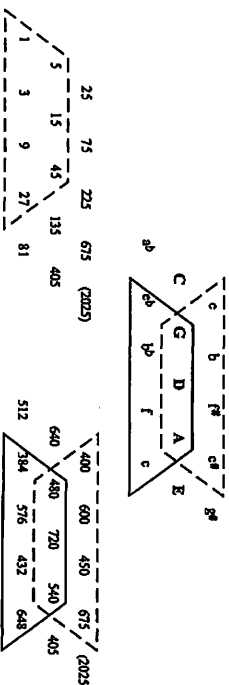
Harpis can be given enough strings to extend any tuning across several octaves, but lyres are normally returned to present some portion of a mode within a singer's best compass, and thus in practice the seven-tone system quantified by Nicomachus is actually extended across *reciprocal* seven tone systems encompassing thirteen pitch classes.

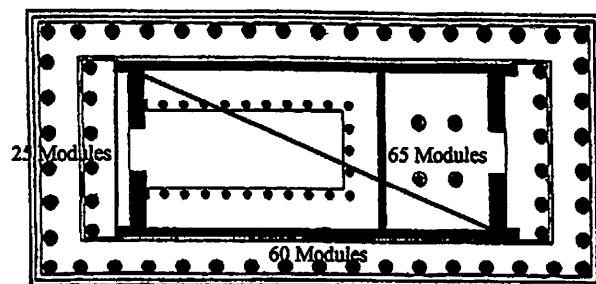
A<sub>1</sub> E<sub>1</sub> B<sub>1</sub> F C G D A E B F# C# G#

Athena's number "7" is the distance in senitones between successive fifths in this tuning sequence. Any five consecutive members constitute a pentatonic subset, and any seven consecutive members constitute a heptatonic subset. (Many other subsets are available but are less commonly codified.) Athena as civic goddess and deified "7" must be imagined as disciplining this extended community where, in Plato's language, "nothing is private but the body", all components symbolize equally "needy citizens", the primary "necessity" ("with whom not even the gods can argue") is the perfection of the octave 2:1 (never questioned ancient practice, but often times in ours), so that "political" decisions must be made in the interest of "what is best for the community". *Athenian justice therefore cannot be blind*, but must remain alert to time and circumstance, and Phidias' statue radiated that complex of values.

5. HOW ALTERNATE PARTHENON MEASURES ENCODE AN ALTERNATE "JUST" TUNING SYSTEM. Parthenon width of 36 modules conceived as  $40 \times 36 = 1,440$  dactyls defines the double octave 4:2::2:1 of a related "just tuning" that Plato analyzes carefully while restricting the double octave of 1440:720::720:360 to its "small" limit, vigorously rejecting the notion that Delphi makes more than *one* of anything. We show Plato's "small" octave but with a reminder that all members enjoy reciprocals in a neighboring "great" octave (1440:720) twice as large and therefore relevant to the Philolaus theory presumably assumed by Parthenon builders. The design also *limits* the principles it embodies.

The "just" system is conveniently imagined as resulting from splitting the "spiral of fifths" shown above into three equal parts, severing C and E into alternate tunings represented here in lower case by c and e, and their overlap creates a 15-element set that is centered on Parthenon framing ratios, with the five central tones as common elements of both tuning systems.<sup>8</sup> It is easy to imagine that a sequence of "triples" 1-3-9-27-81 is repeated in the second row 5 times greater and in the third row 25 times greater, and that *any* integer can then be doubled enough times to embrace all the others. We follow Philolaus and Plato by doubling 45 in the initial layout four times into 720 (leaving further doubling into 1440 to imagination). The "small" limit of 720 is  $12 \times 60$  (the alternative Bulckens' module), relevant to time and the stars, while it defines the senitones of a related twelve-tone octave (or 13 under reciprocity). It contains alternate tunings for C and E as c and e. With ratios of E and e taken as 9/8 and 10/9, we see that these tones differ by the ratio 81:80, known as the syntonic comma, the ratio by which spiral fifths and just intervals disagree — safely ignored in some circumstances but not in others. There is no g# in the third row except as a reciprocal of a b, so that the twelfth tone in either system is necessarily asymmetric with the others, and the thirteenth is normally suppressed.



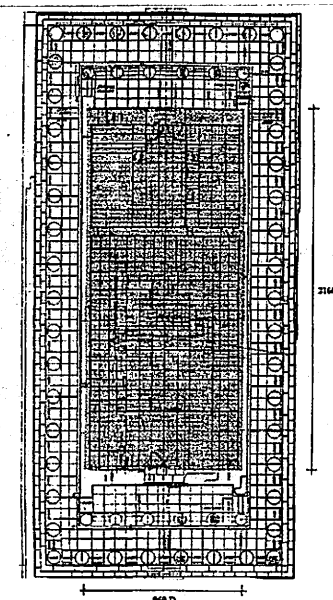


5. The cella with a 5:12:13 triangle inscribed.

approximation was still considered more tolerable, that the 9:8 whole tone ratios in the Nicomachus table were split arithmetically in the "great" second octave to produce the missing five chromatic pitches as "tritones" of 17:12 and 17:24 lying closer to the modern equal temperament value of  $\sqrt{2}$  than in either just tuning or an extended spiral fifths tuning. We have no evidence for this except that the perfection of the octave as 2:1, important to Plato, may be alluded to by the eight pillars on each end of the Parthenon representing the eight tones of the heptatonic scale within the octave limit, and if so then seventeen pillars on each side may allude to the sum of its harmonic and arithmetic means ( $8 + 9 = 17$ ) which functions as the tritone in the "great" second octave that we are suppressing here. Half of the expense of the Parthenon went into leveling the rock on which it stands, and the most expensive marble went into shoring up the base along one edge by several feet to withstand the enormous weight of the pillars, so that our speculation ties together several obvious concerns. Both *proportional size* and *absolute measures* were powerfully motivic concerns involving tremendous cost over-runs during the construction; and 8 is tonally significant. Could 17 also be?

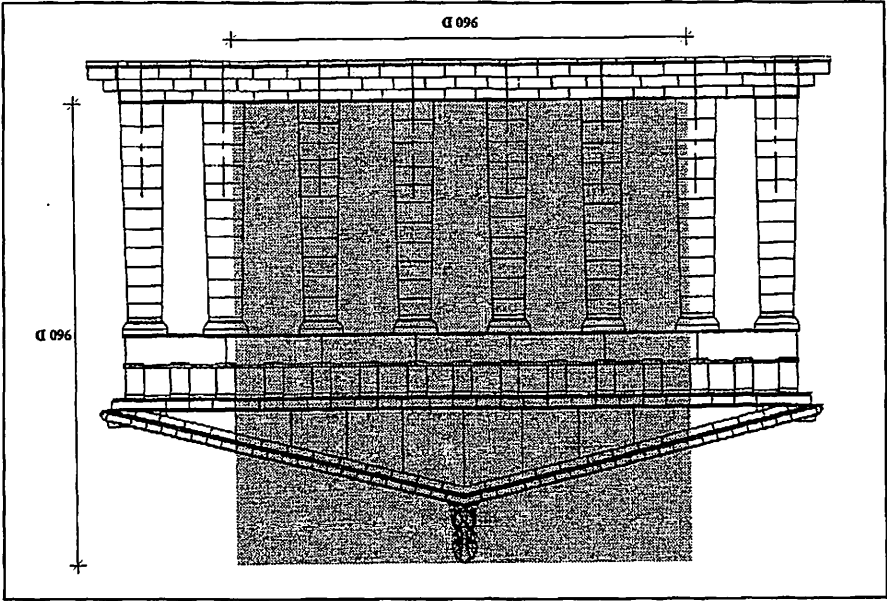
**7. THE PARTHENON AS A HUNDRED FOOTER.** Anne Bulckens hypothesizes that the Parthenon was constructed in two phases. In the first phase of design the interaxial lengths were all equal and taken to be five modules. Bulckens noticed that the cella width of five column interaxials and length of twelve interaxials allows a diagonal of thirteen modules to define a Pythagorean triple, 5, 12, 13 [fig. 5], a design feature that appeared in the Temple of Athena at Paestrum constructed only a few years before the Parthenon. And in this particular triangle the perimeter is 6000 dactyls suggestive of base 60 arithmetic, while the largest inscribed circle has a diameter of 10 modules, so that *ten-ness* is also invading the implied geometrical considerations. The length of the cella is further subdivided into eight interaxials for the naos housing the statue of Athena, and four for the opisthodomos (the treasury), yielding an octave ratio of 2:1 for these two inner rooms. Therefore, since each interaxial is five modules in length, the length of the naos was 40 modules in the first stage of the design, and because each module is  $2\frac{1}{2}$  Parthenon feet, the naos length is 100 Parthenon feet, thereby relating it to the Hekatompedon ("hundred footer") of a Parthenon forerunner mentioned in an ancient quote by Hesychios. Figured as dactyls, the naos measures 1600 dactyls in the first phase of the design, but 1620 dactyls in the second phase, a ratio of 81:80 equal to the syntonic comma. An error of this size could be comfortably buried in the wall between the naos and the treasury.

Bulckens locates the statue of Athena at almost precisely two-thirds ( $66\frac{2}{3}$  feet) from the entrance. The measurement from the outside of the naos wall to the east, not counting the antae wall, to the inside of the wall of the opisthodomos to the west is 2160 D or 54 modules, 54 being the hidden harmonic mean as two-thirds of 81, the stylobate length, as illustrated earlier.



6. The cella exhibiting a width of 960 D (dactyls) and a length of 2160 D.

But why is a factor of 100 important to harmonic theory, as it seems to have been in establishing Parthenon length? The answer lies in "just" ratios of 5:4 (defining Greek "ditone" major thirds) that functioned as an early Mesopotamian cube root of 2, and require a base of  $4^3 = 64$  from the Nicomachus table to be integrated. Nothing smaller than 64 permits 3 consecutive ratios of 5:4 to be defined in integers, and from Plato's example in the *Republic* we see for ourselves that 64:80::80:100::100:125, so that three consecutive steps fall short of the octave double at  $2 \times 64 = 128$  by three units. It so happens that a simple correction factor of  $\frac{1}{125}$  produces an excellent cube root for musical temperament, but it had to be calculated as  $100 + \frac{1}{4}$  of itself +  $\frac{1}{100}$  of itself (i.e., in Egyptian *unit fractions*). It is the arithmetic that must be "hundred footed". Twelve equal divisions of the octave require both square and cube roots of 2, and Plato made both a central concern of fourth-century B.C. mathematics. Here again we are trying reinforce Bulckens' suspicion that *temperament* was a conscious concern in Parthenon design before Plato took up this issue. The difference between spiral fifths tuning and just tuning lies in the comma of 81:80 between them, and the correction factor of  $\frac{1}{100}$  produces a cube root value of 1.26 lying between them and that is correct through five decimal places. Plato is not interested in better approximations, but rather in gaining acceptance for the idea that irrationals like the square and cube roots of 2 are legitimately "numbers" in their own right. Bulckens' Parthenon may merit recognition as direct inspiration for many of Plato's ideas. Our *idealist* philosopher could have been arguing partly from the example of a brilliant *empirical* design familiar to his own students, for in his own time Plato was accused of "plagiarizing" his musical ideas from Philolaus — who seems the more appropriate philosopher for fifth-century B.C. Parthenon architecture.

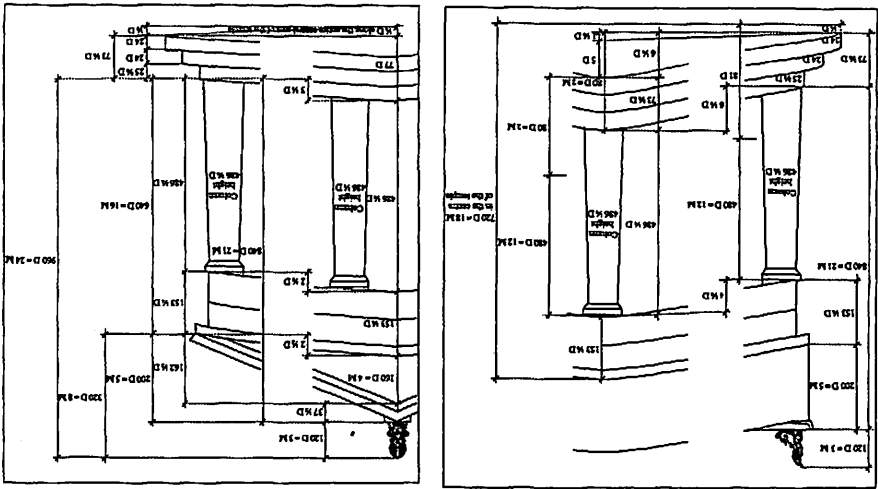


7. The east facade of the Parthenon with a 6:6 square inscribed.

8. WIDTH OF THE CELLA. In the first phase of design the cella was 5 interaxals in width, amounting to 1000 dactyls as measured between the outer walls. In the second phase of the design this was altered to 1001.5 dactyls due to slight variations in the interaxals. Bulkins noticed that when the radius of the corner column whose diameter was 83.5 dactyls was subtracted from the width, a measurement of exactly 960 dactyls or 24 modules was obtained, and this width ran through the cella walls [fig. 6]. This width played a role in the pentatonic scale of measurements central to the system of Parthenon proportions discussed above. Bulkins has found temperament to be connected once again to Parthenon dimensions. If the interaxal of 200 dactyls is multiplied by the  $\sqrt{2}$ , the result is 283.33 dactyls the difference being the column width of 83.5 dactyls, off by 0.199%, from the preset margin of error of 0.2%.

At an earlier time, there was a standard temple decoration at the crown of the pediment known as the acroterium which no longer exists. Bulkins sides with a school of thought that placed the measure of the acroterium at 120 dactyls making the height from the corner of the stylobate to the top of the acroterium to also be 960 dactyls. But 24 forms the mean proportional between 16 and 36 or  $16:24 = 24:36$  just as  $4:6 = 6:9$  where 16 modules (640 dactyls) is the height from the stylobate to the top of the entablature and 36 modules (1440 dactyls) is the width of the stylobate. As a result the area of the east facade [fig. 3] equals the area of the square built on the width of the cella to the top of the acroterium or  $16 \times 36 = 24 \times 24$  as  $4 \times 9 = 6 \times 6$  [fig. 7].

9. STYLOBATE GEOMETRY. The stylobate can be divided also into six 3, 4, 5 right triangles measuring 27, 36, 45 in modules, 27 being one-third of its length, integers also important to spiral fifths tuning. The area of each triangle is then  $777,600$  square dactyls. Imagining these numbers as relevant to tuning theory, "head



8. Elevation of the northeast corner and center of east facade of the Parthenon exhibiting exaggerated curvature of the facade exhibiting a variety of elevations. Drawing not to scale. seen from the maximum point of the stylobate is 480 D. Drawing not to scale.

digits" of 7,776 "seal" a set of 9 pitch classes in spiral fifths tuning (the largest value is  $3^6 = 5561$ ), but the seal is on the sixth value ( $3^5 = 243$ ) so that reciprocal meanings define eleven pitch classes (D C C F B and E in one direction and D A B F F and C# in the other). The full area of 777,600 easily circumscribes the entire 13-tone set defined by  $3^{12} = 531,441$ , so that we encounter significant tonal limits in various ways. A circle inscribed within each 3-4-5 triangle has a radius of 360 D and a diameter of 720 D, thus Parthenon width of 36  $\times$  40 = 1440 dactyls has numerical resonances with various tonal and calendrical sets. We have to wonder where numerical speculation may have ended, for the building can be studied, it seems, as a treatise on number theory integrating all of the Pythagorean sciences.

Bulkins notices that when the distance from the stylobate edge to the partition wall between the naos and the opisthodomos is imagined to be a string length of 2000 D then the statue of Athena lies at two-thirds the interval of a musical fifth 3:2, and the old shrine to Athena from Mycenaean times lies at a musical fourth of 4:3.

10. COLUMN HEIGHT. The important front corner columns with a height of 486 ( $= 2 \times 243$ ) dactyls (with margin of error 0.1% from the value of  $486\sqrt{2}$  dactyls) define the sixth pitch class in the Nicomachus set of 7, center of symmetry in the maximum set of 11. Bulkins subtracts the  $6\sqrt{4}$  dactyl rise to the middle of the stylobate along the north facade to propose an alternate theoretical height of 480 [fig. 8], "conforming to Pliny's statement that an old rule for the height of the column was that it should measure one third of the temple's width". In other words, the height of 480 dactyls is the projected height of the corner column as seen from the middle of the stylobate along the north facade. This ratio of 81:80 between the measured 486 and her theoretical 480 is another instance of the syntonic comma between just and spiral fifths tuning, and the variation signals to her a concern with musical temperament that helped to shape the extreme rise in the stylobate. We suggest that the measured value of 486 for the corner be considered a major design focus.

emphasizing the perfect tonal symmetry of "Pythagorean" tuning through eleven pitch classes, the normative limits of perfect inverse tonal symmetry.

E♭ B♭ F C G D A E B F♯ C♯

**11. ELEVATION MEASURES.** In the various elevations measured from the steps or stylobate at the front façade to the east and the side façade to the north, Bulckens finds all the ratios of the first ten integers (2:1, 3:2, 4:3, etc.) that constitute Philolaus's *ten-ness* [figs. 8 & 9]. We call attention to the 640 dactyls from the corner of the stylobate including column plus entablature and the 960 dactyls that reaches from the stylobate to the top of the acroterium. Perhaps more significant is  $7 \times 120 = 840$  dactyls measuring elevation from the stylobate to include column + entablature + pediment, integrating Athena's 7 into the whole edifice. Since 640 and 840 were measured from the same datum, namely the stylobate, their difference of 200 dactyls is the height of the pediment while the difference between 960 and 840 is the 120 dactyl height of the acroterium.

This emphasizes that most major elements of the construction can be reckoned as integers when the rises of the stylobate and entablature are taken into account. In fact Bulckens suggests that the principal function of these rises were to enable the lengths to be represented as integers. The height from the ground level to the top of the corner column is 560 dactyls also divisible by 7. When the column height of 480 dactyls is subtracted from 560 the height of the steps including the  $6\frac{1}{4}$  dactyl rise to the center of the stylobate at the north façade is found to be exactly 80 dactyls. The stylobate along the east façade rises  $3\frac{1}{4}$  dactyls and this rise makes the height of the steps exactly 77 dactyls so that the visitor is greeted by Athena's number as she enters at the east. It should be noted that the ratio of  $840:560 = 3:2$ , the defining ratio of the Parthenon. Thus Bulckens offers an inspiring vision of the whole with which future Parthenon studies must compete.

**12. CONCLUSIONS.** The Parthenon succeeded in offering to the Athenians a sense that they had always occupied the land of Attica. The iconography and proportions of the temple supported this claim through myth and symbols. The musical scale inspires a *geometrical algebra* to which architects seem to have been loyal, and with equal attention both to motivic inspiration and its disciplined limitation. We are tempted to suppose that Phidias's imposition of extremely high standards of metric correlation on the assembled sculptors — especially evident in the frieze surrounding the cella, last to be completed, may have been inspired by the "civic tolerance" over which Athena presides — emphasizing that tolerances are by aesthetic choice, motivated by theory and not by the limitations of craft. While pentatonicism provides principal dimensions, Athena's heptatonicism pervades the whole with her divine perspective from "7". The Parthenon thus can be seen as an architectural and sculptural testament to the goddess, embodying the skills of contemporary Greek mathematics, music, and philosophy as an inspiration for the future.

## NOTES

<sup>1</sup> Arne Bulckens, "The Parthenon's Main Design Proportion and Its Meaning" (Ph.D. diss., Deakin School of Architecture, Geelong, 2001); idem, "The Parthenon's Symmetry", *Symmetry: Art and Science* (Fifth Interdisciplinary Symmetry Congress and Exhibition of the ISIS-Symmetry, Sydney, 9–14 July 2001), 1–2 (2001), 38–41; idem, "The Parthenon Height Measurements: The Parthenon Scale With Roots of 2". A presentation at the Matomi-um Congress, Brussels, 8–12 April 2002 (in preparation).

<sup>2</sup> Ernest G. McClain, *The Pythagorean Plato* (York Beach, ME: Nicolas-Hays, 1978, 1984); idem, *Myth of Invariance* (York Beach, ME: Nicolas-Hays, 1976, 1984).

<sup>3</sup> Francis Cranmer Penrose, *An Investigation of the Principles of Athenian Architecture* (2nd ed., London: MacMillan, 1888); A. Bulckens, "The Parthenon's Main Design Proportion"; idem, "The Parthenon's Symmetry".

<sup>4</sup> Rudolf Wittkower, *Architectural Principles in the Age of Humanism*. Studies of the Warburg Institute 9 (London: Warburg

Institut, 1949; repr. ed., New York: John Wiley, 1998); Jay Kappraff, *Beyond Measure: Essays in Nature, Myth, and Number* (Singapore: World Scientific, 2002).

<sup>5</sup> A complete survey of harmonic law can be found in J. Kappraff, *op. cit.*

<sup>6</sup> Nicomachus of Gerasa, *Introduction to Arithmetic*. Trans. by Martin Luther D'Ooge (Ann Arbor: University of Michigan Press, 1938); J. Kappraff, *op. cit.*; idem, "The Arithmetic of Nicomachus of Gerasa and Its Applications to Systems of Proportion", *Nexus Network Journal* IV/3 (2000).

<sup>7</sup> A translation of Hermann Diels, "Fragmente der Vorsokratiker", in: Kathleen Freeman, *Ancilla to the Pre-Socratic Philosophers* (Cambridge, Mass.: Harvard University Press, 1966).

<sup>8</sup> E.G. McClain, *Myth of Invariance*; F.C. Penrose, *op. cit.*; J. Kappraff, *Beyond Measure*.

# Music in Art

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**Jay Kappraff** (New Jersey Institute of Technology, Newark) & **Ernest G. McClain** (Brooklyn College, City University of New York), *The system of proportions of the Parthenon: A work of musically inspired architecture.*

The architecture historian Anne Bulckens has studied the proportions of the Parthenon based on the measurements of Francis Cranmer Penrose (the measurements referred to by most scholars). In her work she discovered the length of a module and measure of a Parthenon foot used throughout the structure conforming well to the architectural principals described by Vitruvius. The principal result of her studies is that all of the significant measurements can be reckoned as integers within the preset tolerance of 0.2% although the load bearing elements have deviations of far less. We have discovered that the integers can be correlated with the musical scale of Pythagoras. Most notably the length, width, and heights of the outer temple and the length and width of the cella form a pentatonic scale. Our contribution to Bulckens' work lies in trying to review it carefully within philosophical principles prevalent in the fifth century B.C. when it was under construction-for which we rely on Philolaus, the earliest Pythagorean author, writing in Tarentum while the Parthenon was under construction. We believe her analysis may prove significant both to the study of other Greek temples and to a better understanding of Pythagorean influence on Greek ideals.