
The science behind Europe's music scale

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Europe's first music scale was set out by Pythagoras some two and a half thousand years ago. It conformed to the algebraic theorem that two conditions can be satisfied with two variables, for its two sizes of tone were chosen to produce perfect octaves and perfect fifths. Pythagoras also discovered the simple definition of these concordant intervals as the ratios of the lengths of the vibrating strings which produce them, the ratios being $2/1$, $3/2$, and $4/3$ for octave, fifth, and fourth respectively, the last two combining, as he showed, to make an octave. These discoveries quantified the physics of music much as Newton's laws did the physics of motion. Some two thousand years on, musicians began to exploit key changes and other intervals whose concordances had become appreciated, both kinds of exploitation requiring alteration of Pythagoras's tone sizes. Some partial success was obtained by employing the skill which had developed in the logarithmic handling of intervals to produce a tuning system called 'meantone', and musicians had to be content with it for about three centuries. It used two such markedly different sizes of semitone that the wrong size in a sequence destroyed a concordance. In consequence, only half the keys played satisfactorily, eventually provoking interest in equal semitones. However, the tuning of equal semitones could only be described in a qualitative way that was not accurate enough in practice. The seventeenth century revolution in science included an enormous advance in the physics of music initiated by Mersenne. He discovered the frequencies of vibration, that Pythagoras's simple ratios apply to frequencies, the physical laws of vibrating strings, that vibrating strings and pipes produce harmonics, and that 'beats' are heard when two frequencies are slightly out of unison. 'Beats' make possible the accurate mistuning the ear requires of equal semitones. Around AD1700 Werckmeister produced a satisfactory 'equal temperament' tuning which played all keys and chords, and soon J. S. Bach had appreciated it and famously demonstrated the new power it gave composers. Since Bach's role was much like that of later inventors who made successful applications of science in technological disciplines, it seems right to regard him as the first in a long line.

In most contexts it would seem strange to compare musicmaking with steelmaking. But in the history of the application of science, J. S. Bach's demonstration in the 1720s of the power of a new tuning system looks not unlike the discovery by Bessemer a century and a quarter later of a cheap way of making steel: Bach applied recent discoveries in mathematics and physics to tuning the music scale, whilst Bessemer applied recent discoveries in chemistry to manufacture, and both had an awareness of what they were doing. A distinction of Bach's demonstration which gives it special interest is that it was arguably the very first significant application of the new sciences outside the spheres of those sciences themselves.

The evolution of Europe's music scale has often been described, but the present paper illustrates the link with science in a way that has not been attempted before. Simplifying somewhat, over the past two and a half millennia Europe's music has progressed through three tuning systems. This paper describes how each system used the mathematics and physics of the times to the limit.

Two quantitative problems of the scale

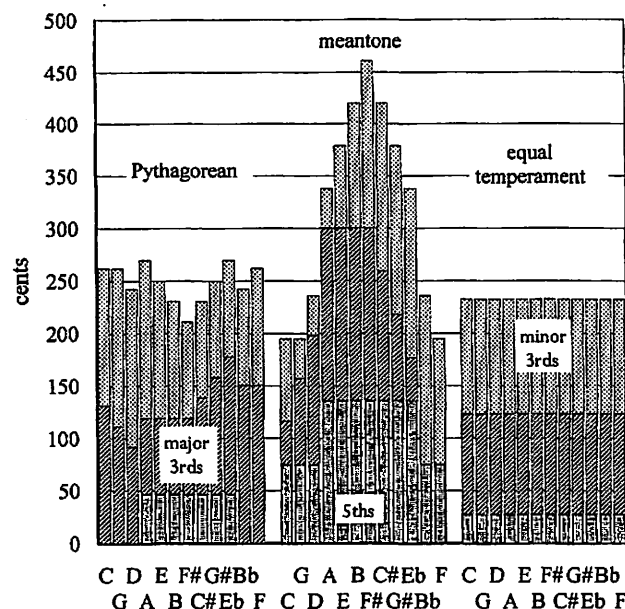
One problem of the scale concerns the principle concordant intervals known as octave, fifth, fourth, major third, and minor third. The ear finds these intervals most pleasant when the fundamental frequencies of the two notes involved have the simple ratios $2/1$, $3/2$, $4/3$, $5/4$, $6/5$ respectively, and the appeal diminishes as the actual ratios deviate from these exact values, the diminution occurring much more rapidly at the octave end of the group, i.e. $2/1$, than at the minor third end, so much so that all three tuning systems discussed have used the exact value for the octave (which is the reason why in Figures 1 and 5 the errors of fifths have all been doubled). Simple algebra explains why even genius could not provide all these ratios. In the seven note scale which Pythagoras devised and which continues today in the white notes of the piano as CDEFGABC₁, C₁ being an octave above C, the intervals EF and BC₁ are approximate semitones but all the other intervals are

full tones. The scale contains five tones and two approximate semitones. By a well known algebraic theorem, relatively recent, two equations can be satisfied with two variables, in this case the tone and the approximate semitone. In order to have a perfect octave, we must therefore satisfy the condition that five tones plus two semitones should equal one octave. Pythagoras also desired perfect fifths, which meant that in addition he had to satisfy the condition that three tones plus one semitone should equal a fifth. He could define these intervals only as ratios of string lengths on his monochord,¹ that is as the ratios 2/1 and 3/2 mentioned above. In the modern logarithmic measure of cents,² the octave is 1200 cents and the fifth 701.96. Mathematicians will know that the 'plus' sign in the two conditions just given is appropriate if cents are used, but must be replaced by a multiplication sign to follow Pythagoras and use ratios. The two equations fix the tone at the ratio 9/8 (203.91 cents) and the approximate semitone at the ratio 256/243 (90.22 cents). Having chosen these sizes for tone and semitone, all other intervals are fixed and will be satisfactory only if luck is favourable. The Pythagorean third is famously unsatisfactory, as Didymus later showed (*see below*).

There is also a sequencing problem, which can again be illustrated with the fifths of Pythagoras's scale. Musicians want them to start on every note of the scale. Unfortunately it is impossible to so arrange the two semitones among the five tones that some sequence of five notes does not include both semitones, thereby making it a badly undersized fifth. The best arrangement is fairly obviously to have the semitones as far apart as possible; then, when two octaves are put together, each semitone has three tones on one side separating it from the next semitone, and two tones on the other. Pythagoras's sequence above uses this arrangement, as did the Greek modes. It has one fifth in each octave about 114 cents too small (BF), but early musicians just had to accept this.

Overview

Figure 1 shows the combined effect of these two sources of imperfection for the three main tuning systems used in the past two and a half thousand years. The interval errors, i.e. deviations from the exact ratios, are depicted in units of cents along the ordinate. The calculations were made for the now familiar twelve note version of Pythagoras's scale, in other words including the black notes on the piano as well as the white, optimised for sequence (the black notes have been with us a long time – *see below*). The twelve major keys, C, G, D, ..., F, are marked along the abscissa for each of the three tuning systems. For each key in each system there is a vertical bar consisting of three portions. The length of the bottom portion equals the total error in cents of all the seven fifths in that key (doubled), the next portion the total error of all the seven major thirds,



1 Errors in the most popular chords in European music for the three main tuning systems. For the two earlier systems the error depended on the key, which is why all twelve major keys are included here. The black notes are B $\flat$, E $\flat$, and three sharps. The errors in fifths have all been doubled to reflect the ear's sensitivity to them

and the top portion the total error of all the seven minor thirds. On the left are the twelve major keys in the Pythagorean system, much the longest lived, from 400–500BC to about AD1500; in the centre are the twelve keys of the meantone system used from then until the eighteenth century; and on the right the twelve keys of the equal temperament system which replaced meantone. In judging this graphical representation of sounds it is helpful to know that our ears easily tolerate the small errors in the equal temperament fifths, but less easily the larger errors in that system's major and minor thirds. The greater errors in the thirds of most keys in the Pythagorean system, even though not very much greater, make them unsatisfactory. Of meantone's keys, the group of seven in the middle all have some large errors, and only the 'outer' keys – three on the left of the middle group and two on the right – were generally regarded as satisfactory. Equal temperament, allowing all the keys and all the chords to be used, was a formidable advance, and during the eighteenth century replaced meantone, which has practically been forgotten. In Fig. 1 the errors are shown for the major keys; similar conclusions apply for the minor keys, although the differences in detail are striking.

The first system

Pythagoras's genius could combine calculations in fractional form using the Greek number system with experiments on the monochord.³ On this instrument he could measure the ratios of string lengths, evidently to quite good accuracy, listen to the sound the two lengths made when struck separately or together, and

record the results. Starting with the rider bearing on the centre of the string with the two parts therefore equal in length (ratio 1/1), on moving the rider towards one end he discovered that the two parts sounded particularly concordant together when the ratio reached 4/3, again at 3/2, and especially at 2/1. He decided that the difference between the first two ratios, in his fractional form $(3/2)/(4/3) = 9/8$, was a suitable unit tone interval. Having fixed the tone, from his requirement for perfect fifths Pythagoras could then calculate the semitone in fractional form as 256/243. The resulting scale of five tones and two semitones has persisted, presumably because it contains the three intervals which most please the ear, and seems to have had worldwide appeal: ancient devices apparently tuned to the same scale have been found across the Mediterranean in Egypt⁴ and around the world in China.⁵ Perhaps, therefore, the scale was already used in Europe before Pythagoras. Whether he discovered or revealed its nature, in a way which enabled him to record it for posterity, seemingly the first person to do so anywhere on earth, he made a gain for the music scale somewhat akin to what Newton would later do for motion. Europe's first recorded scale was thus the outcome of interaction between 500BC music and 500BC mathematics and physics handled by a legendary genius, and perhaps came not long after the alphabet arrived in Europe.⁶

The names for the three intervals of fourth, fifth, and octave have persisted. They arose because in this scale these intervals straddle four, five, and eight notes respectively: the first note always had to be counted as 'one' because the Greek number system contained no zero, which had not then been invented. Thus in adding intervals the first note of the second interval is counted twice.⁷ The practice of regarding intervals as ratios, a natural result of using the monochord, made them independent of frequency, for instance the interval of a tone was the same at 80 Hz as at 800 Hz, although the upward tone would be 10 Hz at 80 Hz and 100 Hz at 800 Hz. Addition of intervals is equivalent to multiplying the corresponding ratios, for example a fourth and a fifth add to make an octave, and the corresponding ratios multiply to 2, i.e. $(4/3)(3/2)$. Musical intervals are thus names for ratios. Logarithms are numbers for ratios and are developed as a continuous series, whereas musical intervals exist as a limited set. Musicians had already been using this limited set of logarithms for two millennia when Napier created Europe's, and seemingly the world's, first number series and coined the term 'logarithm'.

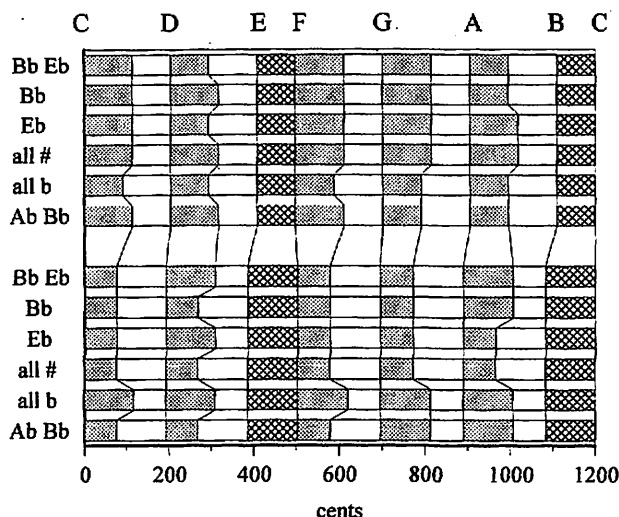
The ear complicates matters

It was during the first century BC that Didymus began to complicate Pythagoras's scheme by adding to his three concordant intervals. Didymus drew attention to the concordant monochord ratio 5/4, or major third. He realised that the nearest approach to this ratio in the Pythagorean scale was given by two tones (straddling three notes on the keyboard, whence the

name third) and he calculated their discordance in the fractional form $81/80 = (9/8)(9/8)/(5/4)$, a ratio still referred to in musicology as 'Didymus's comma' and which has been much used logarithmically. In modern terms it is 21.51 cents, and causes the discordant Pythagorean third referred to above. Ptolemy invented a new tone of ratio 10/9 which combined with a Pythagorean tone to make a perfect major third. The monochord ratio 6/5, or minor third, was also realised to be quite outside Pythagoras's scale.

Another complication was generated in the attempt to correct the bad fifth BF and corresponding bad fourth FB₁ in Pythagoras's scale.⁸ The solution was to create a new note between F and G called 'F sharp', written F#, such that EF# is a tone and FF# a new approximate semitone, which became named a chromatic⁹ semitone. The original semitone acquired the name 'diatonic'. In size the chromatic semitone is the difference between a tone and a diatonic semitone, which in Pythagorean tuning makes it equal to 113.69 cents, 23.46 cents larger than the diatonic semitone. The tone interval FG now consisted of two approximate semitones, a chromatic semitone FF# and a diatonic semitone F#G. Later there would be 'semitones' still further removed from the exact half tone, yet the term continued as the habitual and convenient usage. The fifth above F# is C#, which split CD into two semitones, and so on, producing finally five new sharps. There was the identical problem with descending fifths, beginning with FB, which requires 'B flat', written Bb, BBb being also a chromatic semitone, and eventually there were five new flats. Each tone could be divided into semitones in two ways, by choosing either the sharp of the note below or the flat of the note above. These five notes are the black notes on the piano, still distinguished from Pythagoras's own notes after all this time. Addition of the chromatic semitones took place slowly and mostly late during the long lifetime of Pythagorean tuning.

Figure 2 gives examples of the strong effect the choice between sharp and flat among the black notes has on the sequence of the two sizes of semitone. Each horizontal bar has twelve portions, each portion representing a semitone and being the length in cents of a diatonic or a chromatic semitone. The choice of black notes is marked at the left of the bar and is arbitrary except in the case of the topmost bar, Bb and Eb (notes not mentioned are sharp) being the eventual choice for meantone. Each choice produces a sequence different in some degree. The differences affect the concordances. Seven semitones now make a fifth, and if the seven consist of four diatonic and three chromatic semitones the fifth is perfect, but some sequences of seven contain five diatonic semitones and are about 23 cents flat, eg G#Eb in the first bar, while a few contain only three diatonic semitones and are about 23 cents sharp, such as AbD# in the bottom bar. Two bars, the third and the sixth, have three bad fifths.¹⁰ The other four bars each have only one bad fifth (in Fig. 1 this causes



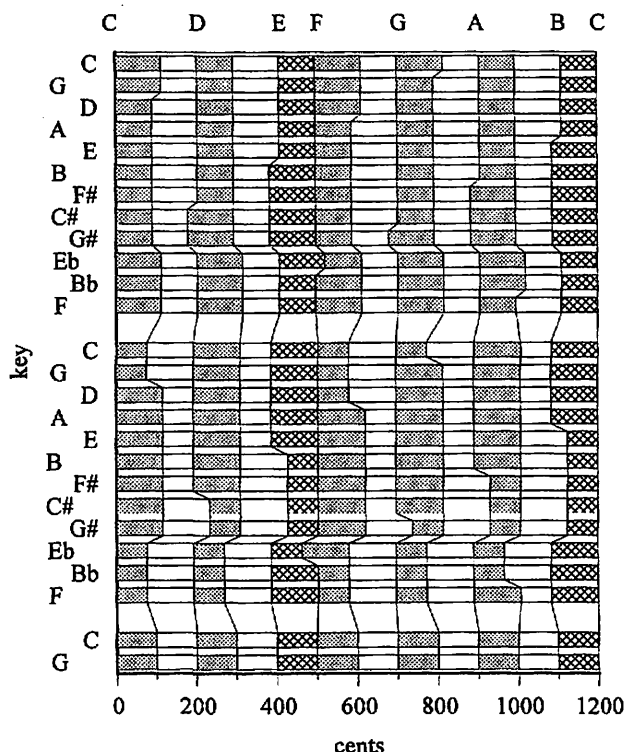
2 Plot showing differences in sequencing caused by various black notes, indicated at left of each bar, with Pythagorean tuning above and mean-tone below and the Pythagorean scale marked along the top. The twelve portions in each bar represent the twelve semitones, the length of a portion indicating the semitone's size. The hatched portions show the positions of Pythagoras's two semitones

errors among the fifths in the seven middle keys of the Pythagorean group). A choice of black notes obviously had to be made, by trial and error, and perhaps then the aim was simply to avoid the three bad fifths.

The second system

The Renaissance stimulated the wish to bring more variety to music by changing key and using thirds. As it happens, 'wrong' sequencing produces some good major and minor thirds. The unpleasant Pythagorean major third contains two diatonic and two chromatic semitones, but in all six bars of the Pythagorean group in Fig. 2 there are four sequences of four semitones where a diatonic replaces a chromatic semitone, making the sequence almost a perfect major third at about 2 cents flat. All these bars also contain three near perfect minor thirds, about 2 cents sharp, where there is a sequence of three semitones two of which are chromatic. Although the number of good thirds was less than a third of the total complement, they were readily available on twelve note church organs and may have encouraged interest in thirds.

However, key changing would be unlucky with its combinations of sequences. Since changing key means starting on a different note and then following Pythagoras's series of tones and semitones, substantial changes in sequence can be expected from key to key, and Fig. 3 shows a sample. Figure 3 is set out in the same way as Fig. 2, but each bar relates to a different key, indicated on the left, while all keys use the black notes of the top bar in Fig. 2. The order of the keys is conventional, moving up a fifth between



3 Plot showing sequencing differences caused by key changes (cf. Fig. 2), with keys marked on the left. Two equal temperament keys are included below (C and G) to illustrate the identity of all keys in this system

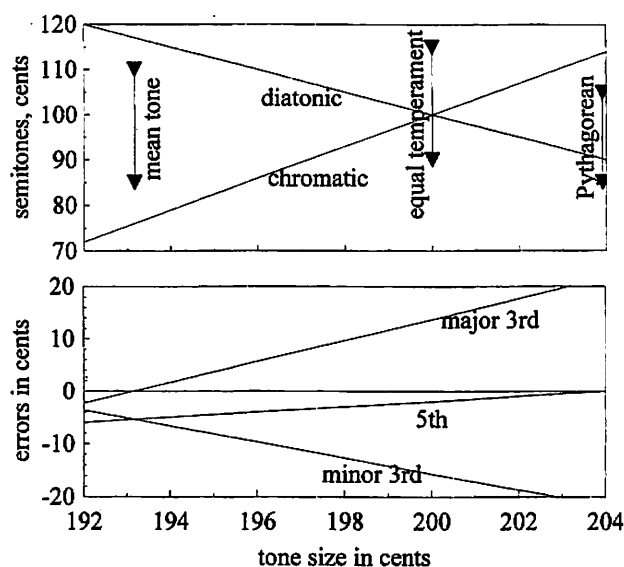
one bar and the next below.¹¹ Since each key contains seven notes we can expect each good or bad concordance to appear in seven keys and miss the other five.¹² In Fig. 3 the five keys which do not contain the bad fifth are the top three (C, G, D) and bottom two (Bb, F), but in all these five the sequencing provides mostly the correctly sequenced discordant thirds, the average of the 'wrongly' sequenced good thirds being just over one of each per key. The result is similar for the other three bars in Fig. 2 which have only one bad fifth. Pythagorean tuning could little satisfy the desired change.

Improvement was hampered by backward quantitative knowledge. For more than a thousand years Europe had suffered the Roman number system, which is only suitable for counting and has been called 'an agent of destruction' in the history of Europe's quantitative culture,¹³ implying that this part of Europe's culture would have been retarded during the Dark Ages even more than other parts. Not until the second half of the fourteenth century did the Arabic number system, more convenient than the Greek, reach Europe and begin to replace the Roman, and not until the end of the fifteenth did Euclid's geometry return to Europe, both helping quantitative thinking at last to begin catching up with that of Ancient Greece. It was thus with no better than the old Greek level of quantitative knowledge that scalemakers met the problem of changing key and using thirds. With the knowledge they had there was little they could do. Evidently they did not

realise this for they struggled hard, making (relatively) an enormous effort in scalemaking over three centuries. Barbour quotes over a hundred scales proposed during this period.¹⁴ He traces the effort from its beginning in Germany, around Erlangen University, not later than the second half of the fifteenth century according to old documents there, and attributes the first written mention of 'temperament' in the sense of modifying Pythagorean tuning to Garfurius in 1496.

The effort splits into three strands. The first quickly had partial success in producing meantone tuning and is described in the next paragraph. The second was the addition of tones like Ptolemy's 10/9 and Didymus's 81/80 and several others that have appeared during history, producing keyboards with many more than twelve notes which could play many concordances perfectly. The third also divided the keyboard more finely but according to a different principle, namely, that some equal divisions give combinations close to perfect concordance. For example, if the octave is divided into fifty-three equal units of 22.64 cents each then near perfect fifths and major and minor thirds are provided by 31, 17, and 14 units respectively, the errors in cents being only -0.07, -1.4, and 1.4. This strand grew after the invention of logarithms immensely simplified the calculations. However, none of these multinote keyboards ever became popular.

The first strand of effort simply used the logarithmic skill with intervals which the Greeks had developed. Four fifths are equal to two octaves plus a third, as can be checked on a piano keyboard. With Pythagorean tuning the four fifths would be perfect but the third would be too large by a Didymus comma. Replacing the Pythagorean third with a perfect third requires each fifth to be reduced by a quarter comma or 5.4 cents, which was found to be bearable even if undesirable. This new system with perfect thirds and flattened fifths became known as 'meantone'. The name arose because finding the meantone precisely was a prolonged problem. Scalemakers realised that the tone itself was the 'mean proportional' (i.e. square root) of 5/4, but they could not make the calculation, and it was improvised on the monochord until the return of Euclid's geometry provided the old Greek method of finding a square root. The first publication Barbour reports of the new tuning was Aron's in AD1513. The algebraic principle's second condition becomes that two tones should equal a major third, giving 193.16 cents as the new size of tone. The new diatonic semitone is 117.11 cents and the new chromatic semitone 76.05 cents, more than 41 cents *smaller* than the diatonic. This change is evident in Figs. 2 and 3, where the sequence of diatonic and chromatic semitones is identical in corresponding bars of the two groups, meantone below and Pythagorean above. In Fig. 2 it is immediately obvious that the hatched portions in each group which mark out the original Pythagorean semitones are larger in meantone, and it is fairly obvious in both figures that where the semitone is



4 Overview of changes brought about by the move from Pythagorean to meantone tuning, with semitone sizes and concordance errors plotted against tone size. The arrowed vertical lines in the upper graph mark the tone sizes in all three tuning systems discussed

small in the upper group it is large in the lower group and vice versa, and that the difference is greater in the lower group. Now the wrong size of semitone in a concordant interval introduced a bigger error than before, explaining why meantone's bad fifths and bad thirds are worse than Pythagoras's.

Although the third and the sixth bars in both tuning systems contain three bad fifths, the other four bars in both contain only one bad fifth and five keys from each have no bad fifth. For the thirds of meantone, with two diatonic plus two chromatic semitones forming a perfect third instead of a famously discordant one, the results are comparatively excellent. These five keys with no bad fifth now average nearly six good major and also six good minor thirds each. Limited to about half the keys though it was, meantone served the new thinking much better than Pythagorean tuning. The eventual choice of the black notes B $\flat$ and E $\flat$ with three sharps, used in Fig. 1, gave the most symmetrical arrangement about the key of C – in Fig. 1 it has keys G and D on one side and F and B $\flat$ on the other with matching errors.¹⁵ In a plot like Fig. 1 the other three choices in Fig. 2 giving only one bad fifth produce exactly the same set of bars as the meantone group in Fig. 1, but less symmetrically arranged.

Figure 4 gives an overview of the change from Pythagorean to meantone tuning that became available much later. The arrows in the upper graph point to the tone sizes on the bottom scale provided by the three tuning systems, that is including equal temperament here. In the upper graph it can be seen how the two semitone sizes diverge as the tone size moves away from the equal semitone value of 200 cents in either direction, steadily making wrong sequencing

more serious. The lower graph shows how the errors of the three concordances vary with tone size when calculated for the best combination of semitones. By thus showing small errors for all three concordances it makes meantone look the most attractive system – until it is realised how often sequencing does not provide the best combination. Scalemaker Salinas's proposal¹⁶ in the year 1577 of a scale using a tone size of 189.6 cents, which can be judged from the upper graph to imply that semitones differ in size by about 2:1, produces good minor thirds given the right sequence but errors in all three concordances of 55 cents or more with the wrong sequence, and a graph such as Fig. 1 shows Salinas's scale in every key to have a larger total error than meantone. The fact that a reputed scalemaker could make this proposal suggests that the sequence problem was difficult to comprehend with such limited quantitative knowledge. From Fig. 4 one can also see that in devising meantone, scalemakers moved from the accessible harbour of easily tunable fifths to the accessible harbour of easily tunable major thirds, and overshot en route the better harbour of equal semitones. At first they did not know it was there, and when they did its entrance was inaccessible.

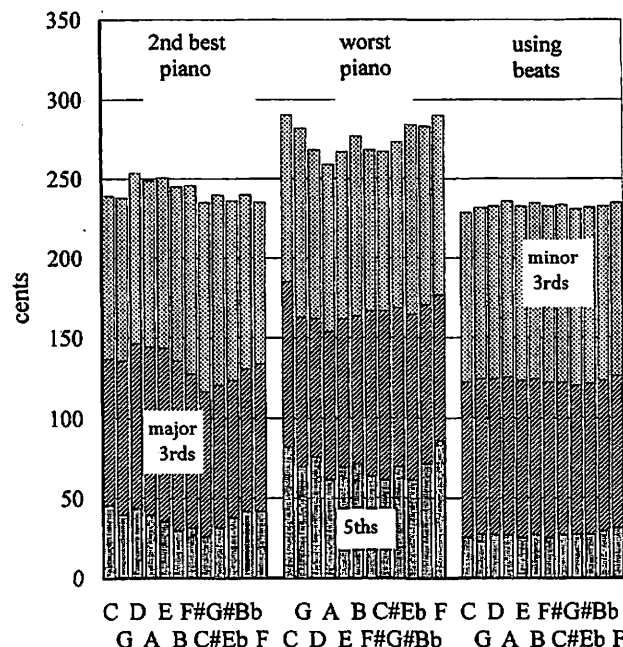
The meantone system was the outcome of interaction between sixteenth century music and the mathematics and physics of Ancient Greece, combined with trial and error among the black notes. The search for a better system continued unabated but fruitlessly until the knowledge to improve it became available.

The third system: equal temperament

Tuning of the first two systems could be checked by their precise fifths or thirds, but equal temperament requires precision in *mistuning* all intervals that could scarcely have been reliably achieved before the huge progress in physics and mathematics during the seventeenth century. Some items in this surge very directly helped the precise tuning of musical instruments. This section describes these items and how they helped. But first let us consider the tuning difficulty.

The tuning problem

We now know that with twelve perfectly equal semitones all major thirds (four semitones) are 13–14 cents sharp. Since the 21–22 cents sharp thirds of Pythagorean tuning were discordant, there is little latitude for mistuning in the direction of larger semitones. And because the minor thirds (three semitones) of equal temperament are 15–16 cents flat, the latitude for error in the direction of smaller semitones is also small. Consequently the semitones have to be equal to within a very few cents and errors must not increment over successive semitones. In the early 1800s 'No Friend to Tuning Quacks' emphasised the value of 'beats' when he wrote in a journal which has



5 Chord errors in instruments tuned to equal temperament by experts (errors in fifths are doubled): the two pianos have been tuned ignoring beats; the remaining instrument, a harmonium, has been tuned making use of beats

since become prestigious: 'most practical tuners, if they do not actually count the beats while tuning, make use of their recollection from habit'.¹⁷ Ellis¹⁸ demonstrated the practical difficulty in achieving the necessary equality by ear without help from beats, even for experienced professional tuners, as late as the early 1900s, nearly a century after most English pianos were nominally tuned in equal temperament. He possessed a set of a hundred and five tuning forks capable, he believed, of measuring frequency to an accuracy of much less than one cent, and with it tested five pianos and two harmoniums which had been tuned by professional tuners. Figure 5 shows three results in a chart like Fig. 1. On the left is the second best piano, next the worst piano, and on the right the one instrument tuned by beats, a harmonium. Comparison with the ideal equal temperament group on the right of Fig. 1 shows that only the beat tuning has produced little extra chord error.

The new science

It is well known that during the seventeenth century people such as Galileo, Descartes, and Newton revolutionised quantitative culture, taking it far beyond the position Ancient Greece had reached. Two parts of this revolution, the foundation of a new science of acoustics and a great advance in the power of calculation, transformed the scene for musicology. They are now described in some detail to make clear how great the transformation was.

Marin Mersenne founded the new science with his discovery of the laws governing the vibration of strings. In his main work he refers several times to these laws and writes of frequency with familiarity,¹⁹

while Mills describes some of the experiments systematically although briefly.²⁰ Mersenne's experiments began with strings, cords, or ropes long enough (between 16 and 1000 feet) to see the vibrations and time them – with his pulse – and thus he measured frequency of vibration for the first time. He discovered that frequency increases in the same ratio as length decreases, or for a given length increases as the square root of the load on the string, or inversely as the square root of the mass of the string. He also provided directly observed proof of what had always hitherto been assumed, that the frequency of vibration is independent of the amplitude of the vibration. He realised that the ratio of lengths measured on the monochord which had been used since Pythagoras's time to describe an interval was also the ratio of frequencies above and below the interval, but inverted, that is to say if the numerator represented the length of the lower pitched string, it represented the frequency of the higher pitched string. As the one way open to him of transferring these experiments from the visible vibration of long cords to the much faster vibration of musical strings he next compared his monochord with different lute strings, adjusting the tension or length of the monochord for unison sound, and found the same relations. Consequently his laws were valid over a wide range of conditions and he could calculate musical frequencies for the first time. The physics of stringed instruments were beginning to be understood.

One set of experiments gives an idea of the experimental accuracy Mersenne could achieve. He measured the vibration frequencies of wires of iron, copper, silver, and gold equally loaded. The densities and therefore masses increase substantially in the order just given, gold being about two and a half times as dense as iron. Mersenne appears to have assumed that all four wires had the same thickness, in which case the frequencies would be proportional to the inverse of the square root of density and, to the nearest percentage point, would be 6, 13, and 36% less than for iron, somewhat greater reductions than the experimental results of 5, 11, and 33%. Given the sensitivity to thickness this agreement is impressive. About the same time Galileo had come to similar conclusions,²¹ although his experiments were generally less comprehensive than Mersenne's and those connecting length and frequency were less direct. The later part of the century saw the first direct measurement of frequency when Hooke devised a rotating cogwheel the teeth of which struck a flexible strip. The first mathematical description of a vibrating string was given in 1713 by Brook Taylor.²² During the eighteenth century the problem of fully treating the vibrating string with its harmonics was popular among European mathematicians mainly because it was central to European music and the new differential calculus invented in Europe could handle it.

Mersenne also reported the discovery which would change *mistuning* by ear from guesswork into a

precise operation. He detected beats when there was slight mistuning from concordance. He heard, and felt, beats from slightly mistuned wind instruments.²³ He evidently, and crucially, noticed that the frequency of the beats increased with the degree of the mistuning, for Barbour quotes him as recommending their use for accurate mistuning.²⁴ Mersenne could hear four harmonics above the fundamental and identified them correctly as the octave, the fifth above that, the next octave, and the third above that, i.e. two, three, four, and five times the fundamental frequency. Soon Wallis studied 'The trembling of consonant strings' and removed the mystery around harmonics.²⁵ He tuned one string on a viol an octave above another. On striking the first string the lower pitched string 'trembled' in two halves with the centre point at rest, as shown by 'a little bit of paper, lightly wrapped about it' which was moved successively from one end to the other. With the first string tuned an octave plus a fifth above the lower pitched, the latter 'trembled' in three parts, in four parts when the first string was tuned two octaves above, and so on. Evidently a plucked or bowed string normally vibrates in all these modes simultaneously and emits the fundamental frequency plus frequencies two, three, four, etc. times higher. A wind instrument could replace the first string if tuned to its note, proving that the second string was set in motion by vibrations travelling through the air. Wallis had proved that a vibrating string normally emits harmonic notes together with the fundamental. At the end of the century Sauveur was well aware that vibrating strings and pipes emit several harmonics. His proposal for a standard of frequency implies a clear idea about calculating beat rate.²⁶

The seventeenth century saw the first measurements of the speed of sound. One method was to observe the explosion of a gun from a considerable distance and measure the time between the arrival of the flash (taken as instantaneous) and the sound. Another method was to make a noise at some distance from a reflecting wall and find the frequency at which repetition coincided with arrival of the reflection. Gassendi made the first measurement (1570 feet per second) using the first method probably about 1630. Derham²⁷ quotes the following results in feet per second²⁸ spread over the next seventy years: Boyle 1200, Flamsteed and Halley 1142, Florentine Academy 1338, French Observatory 1172, Mersenne 1474, Roberts 1300, Walker 1338, all noticeably above the now accepted value of 1087. Nevertheless, it typifies the spirit of that century that several measurements were made of a feature whose existence had not previously been thought about quantitatively, and maybe not at all. In 1709 Hauksbee with particularly thorough experiments proved that sound does not travel through a vacuum.²⁹ No doubt stimulated by these measurements, Newton made the first theoretical calculation of the speed of sound and arrived at 979 feet per second.³⁰ His calculation includes the explanation, for the first time, of the conversion of a

local to and fro motion of the 'particles of air' (today, molecules) into a travelling pulse,³¹ as he called it, one part of which is compressed air and one part rarefied air, the speed of travel being the speed of sound. The length of the pulse is what today is called the wavelength. Galileo had also used the word 'pulse'³² (and also 'beat') in connection with the travel of sound through air, rather as though he thought of sound as travelling like tiny projectiles (whose motion he had spent much time studying), which indicates that there was conceptual progress between these two thinkers. To Newton it was now obvious that the wavelength was equal to the speed of sound divided by the frequency, and he correctly deduced 'that the lengths of the pulses in the sounds of all open pipes are equal to twice the lengths of the pipes'.³³ The physics of wind instruments were now too beginning to be understood. At the end of the century Sauveur proposed the name 'acoustics' for the growing body of coherent knowledge about sound.

Other inventions would enormously enhance the power of calculation and assist a strategic view of tuning. By the late 1500s European mathematicians had our present number system, and the zero had been included. A large advance began in 1594 when the Scot Napier and shortly after the Swiss Bürgi invented logarithms. During the 1620s convenient tables of logarithms to the base 10 were published. These solved the musicologist's basic problem of finding the twelfth root of 2, the semitone ratio in equal temperament. For the first time the equal temperament ratios of string length on the monochord could be written down.³⁴ Then in 1608 Pitiscus invented decimal notation. And soon it was realised that decimal notation allowed realistic approximation, at first by simply dropping unnecessary decimal places and, eventually, by rounding the final figure upwards if that was closer. Irrational numbers, so called because they cannot be written as whole number fractions, could now be handled to any desired precision. This seemingly simple advance jolted scalemakers into a new world. Their practice of writing enormous string lengths in order to be able to state the string length ratios of each interval as whole number fractions – Mersenne's 'enharmonic' scale (a result along the second strand of development in scalemaking mentioned above) required an octave on his monochord to be recorded as 28 800 units to 57 600 units to accommodate all his interval ratios, although he knew that an accuracy of one in a thousand only was necessary – could be replaced by strings of realistic length. Descartes and Fermat linked algebra and geometry – Descartes published 'La Géométrie' in 1637 – eventually making possible an overall view of scale tuning, of which Fig. 4 is an example.

The new science provided musicologists with a coherent body of knowledge in which the familiar monochord ratios were connected with new concepts such as frequencies, beats, speed of sound, wave-

length, equal temperament ratios, and some strategic oversight of a scale's tuning. The radical change in thinking which accompanied it is epitomised by three books dealing respectively with the period shortly before the seventeenth century, the early part of that century, and its end. Palisca's account depicts how important it still seemed in the 1570s to leading people to discover what the Greeks really thought.³⁵ Mersenne's 1630s treatise is alive with description of current practice and experiment to discover novel truths. Rasch depicts Werckmeister in the 1690s as looking firmly to the future, wishing to help the modern desire to change keys freely.³⁶ On the scale of a human lifetime, adoption of the new possibilities was gradual, nor did every musicologist need to know all the new details. For example, when Mersenne wrote his treatise in 1636 it is clear that he personally was still not acquainted with logarithms, but he could call on the expert Grandhomme. Werckmeister still used string length ratios rather than frequency ratios and did not calculate equal temperament ratios, but he could look them up in a book. He knew that beats were connected with mistuning and whether or not he could calculate their rate his practical experience of monochord ratios and associated beat rates would help him to tune with them. The knowledge was now available that could provide the optimum tuning system.

Emergence of equal temperament

The sequencing problem with the two very different semitone sizes of meantone drove some early scalemakers to wish for equal semitones, but they could not give monochord ratios. However, in 1581 Vincenzo Galelei, father of Galileo Galilei, pointed out that the ratio 17/18 is close to being correct. However the ratio produced by twelve such intervals, i.e. 17/18 to the twelfth power (0.50363), is 12–13 cents from the perfect value (0.5), an unacceptable octave error, while a fifth is 9 cents flat, also too large an error. Barbour describes a scale using this ratio created in 1619 by the astronomer Kepler, who applied the ratio to the first eleven intervals and accumulated all the large corrections in the twelfth, just what should be avoided. Not surprisingly, even the most expert brains could then be unclear about the overall strategy of a tuning system. Around 1550 a new type of lute appeared on which the separate wood/ivory frets for each string were replaced by lengths of catgut tied around the neck and which were adjustable in position. Since each length served as a fret for all the strings and the positions of the two sizes of semitone do not coincide for all strings, the new construction suited an equal semitone scale. That thought may have been encouraged when Vincenzo Galilei discovered his ratio. However, the gut fretted lute failed to popularise equal semitones. As late as 1636 Mersenne could report only one hearsay case of a keyboard instrument being tuned to equal semitones, and in 1643 J. Denis described the gut fretted lute as an imperfect instrument.³⁷

When the lute was superseded by the violin during the seventeenth century it had failed to leave any impression on the tuning system. This phase of interest in equal semitones came too early to benefit from the new science. Figure 5 indicates that without any of the new knowledge and totally without experience of equal temperament, only by accident would tuners get close enough to a serviceable equal temperament tuning. There was a similar sequence of events in a quite different field which progressed simultaneously with equal temperament, the calculation of longitude at sea, for its evident usefulness produced suggestions during the first half of the sixteenth century of methods long before they were practicable.³⁸

Near the century's end Werckmeister (1645–1706) began to apply the new knowledge. Some fifty years after Mersenne published his discoveries he evolved a tuning system that was better organised overall than meantone. It used four sizes of semitone, of 90, 96, 102, and 108 cents, more nearly equal than the two meantone semitones of 76 and 117 cents. Werckmeister's semitones were alternated along the scale and minimised the sequence problem. Small errors do not continually accumulate as in Kepler's tuning, nor are there errors in a concordant interval as large as in meantone. This tuning made playable the seven central keys, that is A to E $\flat$ in Fig. 1, without much worsening of the outer keys. Moreover, for the first time twelve fifths were equal to seven octaves, a theoretical but impossible ideal for Pythagoras but practically necessary for Werckmeister in order to pass through all twelve keys and return smoothly to the first.⁴⁰ He had a realistic attitude to the accuracy required, used beats as a guide, and could think of the tuning of a scale as one whole. The system fulfils his claims, proving that his tuning was precise.

However, the new scale still has four fifths that are 6 cents flat, three major thirds 22 cents sharp, and four minor thirds 22 cents flat, errors which Werckmeister had intentionally located mainly in the middle range of keys in Fig. 1. According to Rasch he gradually realised that equal temperament tuning, of which the equality just mentioned is an inherent property (as is obvious today since twelve fifths of 700 cents are evidently equal to seven octaves of 1200 cents), and which he could evidently produce accurately enough, is better suited to music that treats all keys equally. It is simply a piece of good luck for European music that when only the first algebraic condition is satisfied by making twelve equal semitones add up to a perfect octave, seven semitones make fifths that are nearly perfect, being less than 2 cents flat, while four make major thirds about 14 cents sharp and three make minor thirds about 16 cents flat, which are accepted by most ears.⁴¹ The equal semitones completely eliminate the problem of sequence, as the two bottom bars in Fig. 3 illustrate, and allow very free movement between keys.

Christoph Wolff, J. S. Bach's most recent biographer, writes that 'Bach's music – his search for truth

– was affected more, both subconsciously and consciously, than that of any other contemporary musician by the spreading culture of Newtonianism and by the spirit of discovery that followed the Scientific Revolution',⁴² which suggests that Bach (1685–1750) appreciated the new knowledge not less well than Werckmeister. Bach in fact used Werckmeister's organ tuning manual (his being forty years younger would have helped in this respect) and they lived no more than a hundred kilometres apart during most of the twenty-one years their lives overlapped. And later (pp. 229–230), Wolff writes:

Bach's primary purpose in writing *The Well-Tempered Clavier*, then, was to demonstrate in practice the musical manageability of all twenty-four chromatic keys ... Before and around 1700, the general spirit of discovery spurred by the Scientific Revolution had prompted a new spurt of mathematical and physical research, predominantly by German scholars like Werckmeister, to expand and systematize the conventional tonal system. Johann David Heinichen ... had by 1710 devised the circle of fifths, clarifying the harmonic inter-relationships within a system of twenty-four modes or keys, and several composers wrote small experimental pieces in remote keys. But as late as 1717, Johann Mattheson still deplored that 'although all keys can now, per temperament [tuning], be arranged in such a way that they can be used very well, diatonically, chromatically, and enharmonically,' a true *demonstratio* was lacking. It fell to Bach, who accepted this challenge, to demonstrate the compositional practicability of the new system of twenty-four keys, and he did so on an unparalleled level of compositional refinement and technical perfection ... More than any other of Bach's works composed before 1722, the preludes and fugues of *The Well-Tempered Clavier* manifest his resolve to leave nothing untried, even if it meant exploring avenues where no one had gone before. In demonstrating that the tonal system could be expanded to twenty-four keys not just theoretically but practically, Bach set a milestone in the history of music whose overall implications for chromatic harmony would take another century to be fully realised.

With Book 2 (1744), Bach provided the evidence which has convinced most musicians since his time that the complete freedom to change key provided by equal temperament 'far outweighed what was lost in sonic elegance'.⁴

Bach's success came some ninety years after Mersenne had started the new science. In the 1760s came the successful application of the methods of finding longitude about eighty years after Newton's breakthrough. Bessemer's was the first truly science based success in manufacturing, fifty years after Davy discovered the affinity of carbon for oxygen. Thereafter followed a string of such successes thirty to forty years after the science on which they were based.⁴² Bessemer was not as farsighted as Bach but had a slice of luck. His aim had been to find a quicker, cheaper way of making the metal now called wrought iron, but he made mild steel instead, a new alloy which became, and still remains, the backbone of mechanical engineering. Since Bach's achievement was as productive in his own sphere and needed no

luck it seems fair to place him as chronologically the first of these very successful appliers of modern science.

None of this will help musicians with music, but it may progress a separate matter. Like many scientists I have benefited much from music and therefore from Bach's application of science. If his position as chronologically the first major applier of science is generally recognised, it may also be widely realised that benefit between musicians and scientists has been mutual.

Notes and literature cited

1. The monochord was the principal tool of musicologists for well over two thousand years. It consists of a board carrying a scale above which a string is held taut by clamps at either end of the board. A rider moving along the scale bears on the string, dividing it into two parts which vibrate separately when struck, the ratio of the lengths of the two parts being measured from the scale. The same tension is in both parts, for which reason the monochord gave more reliable results than two separate strings.
2. Cents were invented by A. J. Ellis, who gives a method of calculating them in *Proceedings of the Royal Society*, 1881, 31, 382; for a table of values see his 1912 translation of Helmholtz's 'On the sensations of tone as a physiological basis for the theory of music' (Ref. 18, p. 450). The cent system was invented for equal temperament tuning and also provides convenient measurement for other tuning systems. Equal temperament has twelve equal semitones, each being 100 cents with the octave therefore 1200 cents. Modern conversion to cents uses the equation

$$\text{cents} = (1200/\log 2) \log(\text{ratio})$$
 Since an octave covers a frequency range of 2:1, the logarithmic base c of cents is given by $c^{1200} = 2$, whence $c = 2^{1/1200} = 1.00057779$. A single cent is generally regarded as the smallest interval of interest to the human ear.
3. C. A. Taylor describes his likely procedure in 'Physics of musical sounds'; 1965, London, English Universities Press.
4. R. JOURDAIN: 'Music, the brain, and ecstasy', 69; 1997, New York, NY, W. Morrow.
5. 'The new encyclopaedia Britannica', 15th edn, vol. 12, 672; 1993, Chicago, IL/London, Encyclopaedia Britannica.
6. J. MAN: 'Alpha beta'; 2000, London, Headline.
7. This habit has generally been avoided, for example when counting banknotes, since Europe's number system acquired a zero.
8. B_1 is the octave above B, just as C_1 is the octave above C.
9. Aristoxenos used this term writing in the fourth century BC and also used the adjective 'enharmonic' for smaller intervals such as $B\flat A\sharp$. See 'The harmonics of Aristoxenos', (ed. and trans. H. S. Macron); 1902, Oxford, Oxford University Press.
10. For example in the third bar those starting on $E\flat$, $G\sharp$, and $A\sharp$, that on $E\flat$ being about 23 cents sharp because it contains only three diatonic semitones.

11. Moving up two fifths from C reaches D in the next octave, moving up a fifth and down a fourth reaches D in the same octave as the starting C; both are equivalent key changes.
12. This is easier to see if one imagines keys changed in arithmetical order. Number the twelve notes C to B consecutively 1 to 12. Let key 1 be the sequence with the first note of Pythagoras's scale on note 1, key 2 with the first note on note 2, and so on. Since Pythagoras's scale contains seven notes it follows that each of the twelve notes, and therefore the first note of every concordance, appears seven times among the twelve keys. From this point of view musicians choose keys in the order 1, 8, 3, 10, 5, 12, 7, 2 (14), 9, 4 (16), 11, 6 (18) – the set obtained by moving up a fifth are the odd numbered keys here and the set obtained by moving down a fourth (or up a fifth and then down an octave) the even numbered.
13. M. KLINE: 'Mathematical thought from ancient to modern times', 178; 1972, Oxford, Oxford University Press.
14. J. M. BARBOUR: 'Tuning and temperament: a historical survey'; 1951, East Lansing, MI, Michigan State College Press.
15. Each row of twelve keys along the abscissa in Fig. 1 is 'the circle of fifths' set out in a straight line. In the 'circle', key G is next to key C in the clockwise direction and key F is next in the anticlockwise direction.
16. M. LINDLEY: in 'New Grove dictionary of music and musicians', vol. 18, 662; 1980, London, Macmillan.
17. *Philosophical Magazine*, 1806/7, 26, 187.
18. H. L. F. HELMHOLTZ: 'On the sensations of tone as a physiological basis for the theory of music', (trans. A. J. Ellis), 485; 1912, London/New York, NY, Longmans, Green (first German edition 1877).
19. MARIN MERSENNE: 'Harmonie universelle'; 1963, Paris, CNRS. This is a facsimile of the original Paris 1636 edition.
20. JOHN MILLS: 'A fugue in cycles and bells'; 1936, London, Champion and Hall.
21. GALILEO GALILEI: 'Two new sciences', (trans. Henry Crew and Alfonso de Salvio), 94–107; 1914, New York, NY, Macmillan (first Italian edition 1638).
22. BROOK TAYLOR: *Philosophical Transactions of the Royal Society of London*, 1713, 376, 291.
23. MARIN MERSENNE: 'Harmonie universelle', book 3, proposition 28 (see Ref. 19).
24. J. M. BARBOUR: 'Tuning and temperament: a historical survey', p. 47 (see Ref. 14).
25. J. WALLIS: *Philosophical Transactions of the Royal Society*, 1677, 134, 839.
26. J. SAUVEUR: 'Collected writings on musical acoustics (Paris 1700–1713)', (ed. R. Rasch); 1984, Utrecht, Diapason Press.
27. DERHAM: *Philosophical Transactions of the Royal Society*, 1708, 5, 380.
28. All Europe at that time used the Roman foot, although countries' standards varied.
29. F. HAUKSBEY: *Philosophical Transactions of the Royal Society*, 1709, 5, 500.
30. ISAAC NEWTON: 'Principia mathematica', (trans. I. B. Cohen and Anne Whitman), book 2, 770–778; 1999, Berkeley, CA, University of California Press. Newton used the 'isothermal' elasticity of air. Later the larger 'adiabatic' elasticity was discovered and Lagrange and Laplace realised that it was the correct quantity to use.

- Its use raises the calculated value to 1160 feet per second, in somewhat better agreement with experiment.
31. Laplace labelled the concept 'a monument to his genius'.
 32. More precisely, their translators used the word 'pulse'.
 33. ISAAC NEWTON: 'Principia mathematica', p. 777 (see Note 30).
 34. The mathematician Simon Stevin had quite independently calculated these ratios in around 1600, but his manuscript remained unpublished until 1884.
 35. C. V. PALISCA: 'Letters on ancient and modern music to Vincenzo Galilei and Giovanni Bardi by Girolami Mei'; 1960, Leawood, KS, American Institute of Musicology.
 36. A. WERCKMEISTER: 'Musicalische Temperatur', (ed. R. Rasch); 1983, Utrecht, Diapason Press.
 37. J. DENIS: 'Treatise on harpsichord tuning', (ed. and trans. V. J. Panetta); 1987, Cambridge, Cambridge University Press.
 38. See WILLIAM J. H. ANDREWES (ed.): 'The quest for longitude'; 1996, Cambridge, MA, Collection of Historical Scientific Instruments, Harvard University.
 39. The difference between twelve fifths, i.e. $(3/2)^{12}$, and seven octaves, i.e. 2^7 , was known in Ancient Greece and is called a Pythagorean or 'ditonic' comma. Its value is 1.01364, or 23.46 cents. In equal temperament the fifth is flat by a twelfth of a ditonic comma. Werckmeister called this unit a 'grad', which he used as a convenient size of logarithmic unit anticipating the later cent. One grad is equal to 1.955 cents.
 40. This does not prevent players not tied to fixed notes (for example violinists) from improving the concordance when circumstances permit.

41. CHRISTOPH WOLFF: 'Johann Sebastian Bach. The learned musician', 7; 2000, Oxford, Oxford University Press.
42. D. MCLEAN: 'Structural materials', (ed. E. D. Hondros and M. McLean), 296; 1986, London, Institute of Materials.



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