

The discovery of the incommensurables and the vertigo of the infinite

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It is widely admitted that the discovery and the treatment of the incommensurability is one of the greatest accomplishments of the Greek mathematic science: the young Hellenic people, initiated for a short while to geometry and to the science of numbers, is probably the only people from the antiquity to have dared to face the theme of the incommensurability. If all that is known about the incommensurables is developed in a discursive and systematic manner in books X and XIII of Euclid's *Elements*, it is however quite possible that the discoveries in themselves may not have been made with the serenity which apparently brought life to the great productions of the Greek mind: the *Elements* technical account could hide an encounter with the infinite, first lived under the mode of vertigo. How would the Ancients conceive the mathematical infinite? Several revealing conceptions seem to have followed themselves: from the Pythagorean rational arithmetic infinite to the Platonic irrational geometric infinite, and between them, the Zenonian aporetic infinite. With this in mind, it is surprising in this regard that Plato did not publish his conception of the incommensurable infinite raised to the dimension of the ontological principle. It may be that he saw in it some sort of *vertiginous abyss* that he preferred not to reveal to the uninitiated... A few rare Platonic and pre-Socratic testimonials show that in fact this dawning of consciousness of the incommensurability, far from having been lived in the way of Archimedes' jubilation, would have rather been the object of scandal or betrayal, instantly casting the Greek consciousness into absurdity and darkness. It is this first genuinely tragic vision of the incommensurable that we will try to reconstitute.

I From the rational infinite to the logical scandal

It is in the Pythagorean setting that one can appreciate the whole extent of the scandal. Is it really about a Pythagorean discovery? It is what appears to be at least in the famous summary by Eudemus of Rhodes, the most ancient account of the history of geometry.

« Further to the Milesians, Pythagoras transformed this study [geometry], and made it liberal teaching ; as he went back to superior principles et searched the theorems abstractly and through pur intelligence ; we owe him the discovery of irrationals and the construction of the figures of

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the cosmos (τὴν τῶν ἀλόγων πραγματείαν καὶ τὴν τῶν κοσμικῶν σχημάτων σύστασιν ἀνεῖπεν) »¹.

This passage is considered by historians as relatively credible, Eudemus of Rhodes being a direct and trustworthy disciple of Aristotle. Whether it truly came from Eudemus *via* Proclus, there is little doubt, since this testimonial is relayed by the Alexandrian mathematician Pappus (IIIrd Century)² who gives a similar account stating that he owes it to Eudemus the Peripatetic, but giving out less details about Pythagoras : he only speaks about the Pythagoreans. Despite all the difficulties encountered gathering the contributions by Pythagoras and his School, one is entitled to admit, thanks to these documents, to which other converging testimonials add up (certain scholiae from Euclid's *Elements*, the legend of Hippasos, which we will examine hereafter) that the Pythagorean School played a primary role.³

At the beginning, the members of this School were intuitively convinced of a total appropriateness between figures and numbers, and so they would represent figures with numbers and numbers with figures : this is the *figured points* system. Arithmetic suites would then correspond to a progressive building of figures.

For example, the figured triangular number 6 will express itself in the following diagram :

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      a
     a a
    a a a
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This diagram corresponds to the suite 1, 3, 6, 10, 15... If we add a *gnomon* (the serie of units one must add to obtain a similar figure) equal to 4 added to this triangular number,⁴ one obtains 10, the Decade or Tetractys.

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      a
     a a
    a a a
   a a a a
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One will recognize in this diagram the presence of musical ratios 4/3, 3/2 and 2/1, starting from the base.

¹ *In Euclidem commentarius* by Proclus (Friedlein, 65, 15-21).

² Pappus, *Commentaire au X^e livre des Eléments d'Euclide*, French translation by Woepcke of an arab version, *Essai d'une restitution des travaux perdus d'Apollonius*, 1856 p. 662-663.

³ We will still be able to indicate that the Ancient pre-platonic dating, of the discovery of irrationality, is confirmed by the fact that Democritus born in 470 would have written a piece of work on irrationality (DK 68 B11).

⁴ Cf. Nicomachi Geraseni Pythagorei *Introductionis arithmeticae* II, chapter VIII (Hoche p. 89). The serie of gnomes is the suite of natural wholes (2, 3, 4, 5,...) and the following triangular number will be given by this formula : $\frac{n(n+1)}{2}$.

It is on this musical and arithmo-geometric founding that the Pythagoreans constituted their whole philosophy of the Number.⁵

Let us rapidly look at how the use of this system may have developed into the discovery of the famous theorem of Pythagoras. Did he really discover it? It is difficult to attribute it to Pythagoras in person in its Euclidian formula, but it is thought that the pre-Socratic mathematician arrived at the level of the *rational rectangle triangles* : 3 – 4 – 5 ; 5 – 12 – 13 ; 7 – 24 – 25 ; etc.⁶ His discovery is not a demonstration, but a *monstration* by counting the number of figured squared numbers that one can build on each side of a rational triangle.

The philosopher, however, must have found out very quickly that this property of square numbers could not apply itself as easily to an isosceles rectangle triangle, namely to the diagonal of a square, as one can no longer find a number or ratio measurable in unit-points for this diagonal. Such is the most elementary revelation of the incommensurable. The question is then to find out, as far as it was discovered in a very unexpected manner, how it could be integrated in a philosophical system based on the domination of the Number.

An enigmatic teaching reveals that, for the Pythagoreans, the basic numeric principles are *Monad and indefinite Dyad* (DK58 B 1a, 14, 15). Aristotle (DK 58B 13 = *Metaph. A* 987 b 22 sq.) seems to allocate this doctrine rather to *ágrapha dógmata* (the oral teachings by Plato). We have a few reasons to believe however (thanks to Theophrastus' authority in this matter) that the theory of principles did belong to the Pythagoreans, but under a much simpler form than with Plato. It would refer to, according to an important study by Paul Kucharsky,⁷ the opposition between *square numbers* and *oblong numbers* (from the table of opposites)⁸, square numbers with odd gnomes et oblongs numbers with even gnomes. The

⁵ Aristotle, *Metaph. A* 987 b 22 : οἱ δ' ἀριθμοὺς εἶναι φασιν αὐτὰ τὰ πράγματα. et 987b 10 : οἱ μὲν γὰρ Πυθαγόρειοι μιμήσει τὰ ὄντα φασὶν εἶναι τῶν ἀριθμῶν.

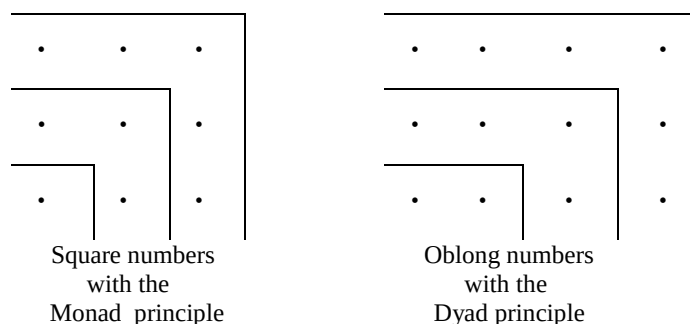
⁶ All historians remained very sceptical, and rightfully so, with regards to a formal geometric proof since the high period of Pythagoreanism. However, the Ancient Pythagoreans would have noticed the specific properties of the sacred triangle 3-4-5, by displaying pebbles in the Eurytos style, and would have drawn a *global method* of assessing the *ratios* between sides rather than an apodictic proof. Cf. W. Burkert, *Lore and Science in ancient Pythagoreanism*, Cambridge (Mass.) 1972 (English translation from a German edition, 1962), pp. 427-340.

⁷ Kucharsky, "Les Principes des Pythagoriciens et la Dyade chez Platon", *Les Archives de la Philosophie*, 22, Paris 1959 (première partie p. 175-191 - deuxième partie p. 385 sq.). Works issued from the extension of analysis developed by L : Robin, *La théorie platonicienne des Idées et des Nombres d'après Aristote*, Paris 1908.

⁸ Aristotle, *Metaph. A*, v, 986 a22. It is to be noted that we will not find in this table which dates, according to Aristotle, from the very early Vth Century, the opposition of the rational and the irrational. Let us cite a few opposites (among the 10° which concern us more especially:

gnome herein will be a square figure which allows to reproduce a similar figured number while framing it. One notices that by the means of these odd and even gnomes, the whole infinite series of entire numbers is run through genetically. But what matters is that the principle or basis be the Monad for the square numbers, and the Dyad for the oblong numbers. Whereas on one side, square numbers are always geometrically similar (they act in accordance with the law of the *limiting*), on the other side, oblong numbers act in accordance with the law of the *unlimited (the apeiron)*, as their shape evolves constantly, and is never strictly similar.

Figure 1



Philosophically, the first Pythagoreans would certainly see in the indefinite (simple and rational) of oblong figures the principle of all existing dissimilarity, as opposite to the similarity of the *equally equal* square numbers, the very principle of all similarity. Now, in his oral teaching, Plato, according to Aristotle, would substitute to the Pythagorean *infinite* which was conceived as a (ὥς ἐνός) a *Dyad*, i.e., an *Apeiron that comes from the Big and the Small* (τὸ ἀπειρον ἐκ τοῦ μεγάλου καὶ μικροῦ). Which would mean that Plato would enrich the theme of the Pythagorean *infinite*: he grasps it as double through the opposition of the Big and the Small. What are then the Big and the Small? They are very likely the fundamental principles of all existing variations of the more and the less, undefined substract from which all limited beings, particularly Numbers, and even Ideas, would be produced in the scope of the *genesij eij ousian* as quoted in *Philebus* en 26d (*what comes to the being through the effect of measures introduced by the Limiting*: μέτα τοῦ πέρατος). In effect, certain passages in *Philebus* (the only dialogue in which Plato resumes the notions of *Péras* and of the *Apeiron* from his oral teaching) lead us to understand that

πέρας	καὶ	ἄπειρον	limiting	and	unlimited
περριτόν	καὶ	ἄρτιον	odd	and	even
φῶς	καὶ	σκότος	light	and	dark
ἀγαθόν	καὶ	κακόν	good	and	bad
τετράγωνον	καὶ	ἑτερόμηκες	square	and	oblong

the Big and the Small correspond to the irrational basis of all reality as moving *substratum* that receives the action of the Limiting to produce the Number, as it is said that the function of the Limiting (τὸ πέρας) *is to make commensurable and consonant the opposites by introducing the Number* (τὰναντία ... σύμμετρα δὲ καὶ σύμφωνα ἐνθείσα ἀριθμὸν ἀπεργάζεται) (25e). All in all, the opposites are the Big and the Small, which under the domination of the Limiting, are transformed into Numbers. The genesis of Numbers and of finished beings, from the two fundamental principles of the Monad and the undefined Dyad, the latter one conceived as incommensurable *substratum*, such would be Plato's new conception which Aristotle indicates in a very elliptic manner in *Metaphysica*.⁹ With regards to these texts, one will understand that Plato, in his Pythagorean principles system, is going to integrate the notion of incommensurability *that the first Pythagoreans could not face* : from then on, he goes from the rational *apeiron* of figured numbers to the incommensurable *apeiron* of the Big and the Small, of the More and the Less.

Consequently, the excerpt from *Metaphysica* which we just looked at, finally suggests that with the Pythagoreans, the incommensurability remained an unresolved issue (thus impossible to integrate in their theory of principles and their *table of opposites*). In the surroundings of their first theories on figured numbers, they could not conceive the existence of such a property. Here we have all the signs of a *doctrinal contradiction*, meaning that these mathematicians knew the irrationals, but had to keep a philosophy of the principles issued from Pythagoras, who used to totally ignore this property.¹⁰ Such a hiatus can be easily understood : a deeply rooted preconceived idea cannot be destroyed from one day to the next. It is evident that in their founding conception, these first mathematicians would base themselves as any uneducated man would, upon the empirical and intuitive principle that to any length correspond inevitably a number and a measure.

⁹ At first sight, according to *Philebus*, this genesis of the beings only concerns the domain of the perceptible, but the Platonic doctrine of the two principles must be able to apply to the intelligible and perceptible reality (cf. Aristotle, *Metaphysica*, A 6, 987b 15 sq.), particularly with the genesis of ideal Numbers. Cf. P. Kucharsky, *op. cit.* p. 423 sq. The undefined Platonic Dyad is then fundamentally the *incommensurability*, on one side in the intelligible, as far as it can be totally converted into Numbers and in *lógoi*; and on the other side in the perceptible, as far as it cannot be totally converted, particularly with surfaces (diagonals of regular figures), in volumes and mainly in *the irrational and unknown founding of any reality* (*Theaetetus* 202b). Cf. *infra* n. 26 et n. 39. Consecutively to the discovery of the incommensurables, it may be that the Pythagoreans might have likened them to the *apeiron*, before Plato, as suggested in an excerpt by Iamblichus, *Vit. Pyth.* 30, §179, 8-11.

¹⁰ We will see that at the end of the 5th Century, at a time when the knowledge about the irrationals had fallen into the public domain, one Pythagorean, Eurytos (DK 45, 3), would keep the archaic unit-points system, the *psephoi*, certainly by reverence to the master's doctrine. Cf. J. Burnet, *Early Greek philosophy*, London 1919.

This is what Paul Tannery commented on :¹¹

« Comme le montrent leurs travaux sur la figuration des nombres et leur célèbre définition du point — l'unité ayant une position — les Pythagoriciens sont partis de l'idée, naturelle à tout homme non instruit, que toute longueur est nécessairement commensurable à l'unité ».

Thus, Tannery is the first erudite to have advanced, towards the discovery of the incommensurability, the idea of « scandale logique », which could also be called : « crisis of fundamentals ». He rightfully speaks about a discovery that is less problematic in its intrinsic difficulty level (since it follows directly from the generalisation of the Pythagoras theorem) than at the prejudice level, which in reality is as ineradicable as experience shows that a measurement of lengths is always possible (by practical approximation). And it does seem at first sight *inconceivable* that with a given geometric length, the diagonal of a square which the Greek would furthermore call *diametros* (which is measured from either side of the figure), could not correspond to any number, or any accurate measurement obtained from its side.

It is to be noted that the Pythagoreans would have perfectly admitted that the diagonal number might not be whole, that he might just be fractional, what they would call a *lógos*. On the other hand, they never conceived the idea of the *irrational number*, in the way that we speak today about the *golden number* or the number *pi*, but only the idea of irrational or incommensurable magnitudes. There is either commensurability and consequently a number, or incommensurability and no number. The two ideas are mutually exclusive in the Pythagoreans' mind, as in Plato's.¹² Thus, the fact that there would be no number corresponding to a given dimension, was for them a matter of the greatest scandals. It is true that these texts do not speak strictly about *scandal*. Plato and Aristotle only speak about *astonishment* (θαυμάζειν).¹³ But, as we shall see, these two thinkers had taken the position of *dedramatization* against a first much more tragic perception of the irrational. To be able to grasp this first impression of scandal, one has to go back to a legend related to the Hippasus of Metapuntum character, in which the ideas of *betrayal* and *absurdity* crop up.

¹¹ P. Tannery, *Mémoires scientifiques*, Paris-Toulouse, 1912, I, p. 268.

¹² This exclusive vision of the Ancients can be found in a passage by Aristotle, *Metaphysica*, D 1021a5 : ὁ γὰρ ἀριθμὸς σύμμετρος, κατὰ μὴ σύμμετρον δὲ ἀριθμὸς οὐ λέγεται. (corrupted text, correction from Apelt : σύμμετρων σύμμετρον) « en effet, tout nombre entier est commensurable, mais pour les grandeurs incommensurables, aucun nombre ne peut les exprimer » (translation from J. Tricot). On the other hand, Theaetetus will introduce in a certain way the idea of the *irrational number*. Cf. *infra* n. 32.

¹³ Cf. *infra* p. 15 sq. Plato *Laws*, VII, 819 d6, cf. *infra* n. 17 et n. 34 ; Aristotle, *Metaphysica*, A, 983 a 15. One will note that the Greek term, as well as the English term *astonishment* (to be stricken by thunder) are strong words that express an upheaval of the mind that exceeds by far a simple surprise.

II *Betrayal*

This legend is reported by Iamblichus in two excerpts from his *Pythagoras' Life*¹⁴ a Pythagorean named Hippasus of Metapuntum would have died at sea, because of his lack of respect, for having *revealed the secret* of the construction of the dodecahedron¹⁵ — a construction whose authorship came back to *That One*, which is to say Pythagoras. Another related tradition teaches us that whoever (Hippasus or someone else?) would reveal the nature of the incommensurability to people unworthy to receive such a teaching, would have to be excluded from the brotherhood, and the members of the sect would erect a tomb to signify his spiritual death...

One can understand these traditions about the secret or about the death of one or several traitors who would have disclosed it, as the first expression of traumatism, as a reflection of the shock that the discovery of irrationality would have provoked. Meaning that irrationality is something *literally unthinkable* and, in reality, one must admit that not only would the Pythagorean community have been betrayed by the divulgence of Hippasus, but the very *Greek mind in its wholeness*. Indeed, one can understand in depth this legend that it is *the very discovery of irrationality (which should go back to Hippasus¹⁶ rather than to Pythagoras)* which would have been perceived in the Greek mind like the greatest betrayal. The explanation is simple: it is the whole rational vision of the world instituted in particular by Pythagoras that could collapse in one strike. Such a discovery, such an *impiety* would only deserve death, symbolically or not, and the most severe revenge by the divinity. In other words, the legend about the betrayal by Hippasus is the first expression of scandal from the fact of the existence of the irrational: the Greeks felt deeply betrayed by the mathematical discovery and Hippasus, even though he might be a scholar, would be the only one to bear the responsibility of this humiliation of reason. In light of this, the legend of Hippasus must be taken for what it is : a legend that requires, above all, to be interpreted in a symbolic way, even if an historical background is not absent. And even though the historical content might be debatable, the

¹⁴ Iamblichus, *Vit. Pyth.*, § 88; 246; 247 et *De communi mathematica scientia*, §25 Teubner = Fragt DK 18 4. On the authenticity of a testimonial, A. Delatte sees as a source Timaeus of Tauromenium, probably because the anecdote occurs at paragraph §246 right after a remark that recalls the letter by Lysis to Hippachos, a letter which would have been used by Timaeus (cf. *Etudes sur la Littérature pythagoricienne*, Paris, 1915, p. 92, n.2) (cf. *infra* n. 22 p. 9). Timaeus lived from 356 to 260. We owe him *Story of Greeks in Sicily and in Italy* in 38 volumes of which only fragments remain.

¹⁵ It is about one of the Pythagorean *cosmical figurs*, the geometric construction of which requires the handling of an incommensurable, *the golden section*. About the knowledge of the Ancients of this irrational dimension, read Jean-Luc Périllié's : "Platon et la section d'or", in *La philosophie de Platon*, collective works under the direction of M. Fattal, L'Harmattan, Paris, 2001, p. 185 f.

legend would remain in itself deeply true by what it reveals on the troubled psychological state of the Ancient Greeks facing the incommensurable. With respect to the secret, it could only be perceived as a necessary consequence of the discovery – a consequence which is to be interpreted as well in its symbolic sense. Furthermore, this symbolic dimension of the secret would not have eluded the Pythagoreans themselves. This is what is explained in an excerpt by Pappus of Alexandria:

« Indeed the sect of Pythagoras was so affected by its reverence for this things that a saying became current in it, namely, that he who first disclosed the knowledge of surds or irrationals and spread it abroad among the common herd perished by drowning. Which is most probably a parable by which they sought to express their conviction that firstly, it is better to conceal (or veil) every surd, or irrational, or inconceivable in the universe, and secondly, that the soul which by error or heedlessness discovers or reveals anything of this nature which is in it or in the world, wanders (thereafter) hither and thither on the sea of non-identity (lacking all similarity of quality or accident), immersed in the stream of the coming-to-be and passing-away, where there is no standard of measurement. This was the consideration which Pythagoreans and the Athenian Stranger¹⁷ held to be an incentive to particular care and concern for these things. »¹⁸

It is clear that the Pythagoreans themselves had understood that this was a parabola, even if they limited their interpretative view to the secret and to the fact that the traitor would die at sea, even though the whole legend in itself was symbolic. Obviously, one can see that at the time of Hippasos (beginning of the Vth Century, B.C.), the Ancients would live essentially in the symbol: they would accomplish symbolic acts (the tomb for the one who died for the sect), they would make up symbols from start to finish, they would perceive at once the symbolic reach of a gesture or event. One will note in this respect, the importance of the *symbolon* (*pentalpha* or starred pentagon)¹⁹ and the *acousamata* (DK 58C 4-6) which are essentially symbols, the archaic formulas of which go back to the first Pythagorean times.

¹⁶ Cf. K. von Fritz in « The Discovery of incommensurability by Hippasus of Metapontum », *Annals of Mathematics*, vol. 46, No. 2, April, 1945, p. 242 sq. This magistral study on the historical and scientific levels leaves on the side the symbolic dimensions of this legend, which seems fundamental to me.

¹⁷ « The coming-to-be and passing-away » are Platonic concepts from *Theaetetus* (153a6-7): They correspond to Heraclitus', Protagoras', Homere's mobilism who state that *all things are born from flood and movement*.

¹⁸ It is about Athenian Stranger of *Laws* (819d) by Plato who comes to propose to the Cretans a better conception of the learning of mathematics and regrets not to have been initiated earlier to the incommensurability: cf. *infra* n. 34. G. Junge ("Von Hippasus bis Philolaus: Das Irrationale und die geometrischen Grundbegriffe", *Classica et Medievale*, 19, 1958 p. 53-54) points out that the legend presents a contradiction; since the Stranger from Athens militates rather in favour of a popularization of the incommensurability. This story hereby reflects the Platonic contrasted attitude in this regard – an attitude that I will try to clarify herein. Through its symbolism, the legend appears typically Pythagorean, but it became charged with more or less contradictory Platonic elements.

¹⁹ English translation by G. Junge and W. Thomson from the arab version in Pappus, *Commentary on Euclid X*, Cambridge, 1930. Also see the schooling 417 from *Elements, Euclidis, Op. t. V*, ed. Heiberg, 1888. Cf. W. Burkert, *op. cit.* p. 457-458.

²⁰ Lucien, *Pro Lapsu inter Salut.*, ed. Jacobitz i 330, II-14. The symbolic part is here overdetermined: the legend is a symbol with respect to the revelation of the mathematical structure of a symbol.

In their interpretation of the drowning of the traitor, that which the followers of the sect would feel beyond betrayal and scandal, was the necessity of *hiding the knowledge* because of a great anxiety about irrationals perceived in themselves as *absurd and inconceivable*. The Pythagoreans would have feared, as Plato would later, that this discovery would cast doubt in their minds. The question was to know whether the Pythagoreans would veil the knowledge because it would contradict their theory of the Number or whether this was only so as not to lose the minds who risked, further to this revelation, to sink into the abyss of *the sea of non-identity*, as it is said in this passage. We are inclined to believe in the second solution, since, paradoxically, the members of the School really would not have adopted among themselves an attitude of censorship or repression when facing the irrational (since their symbols, which were signs of acknowledgement, were figures the diagonals of which were obviously irrational such as the pentagon, the square or the dodecahedron). On that point, P.-H. Michel, continuator of P. Tannery, distinguishes two conceptions of scandal : one is logical and legitimate, the other is rebellious and unhealthy (it is the obstruction of a fact that challenges a pre-established doctrine).²⁰ This author shows that we cannot accuse the Pythagoreans to have hidden the irrationals for doctrinal contradiction, through some sort of bad conscience. Furthermore, it would be carrying against them a very serious and insufficiently founded accusation: an unhappy conscience does not necessarily mean a bad conscience.²¹ As indicated in the passage by Pappus, the Pythagoreans and Plato would have rather had a *reverence*, a *veneration* for the figures of which they knew the intimate irrationality. What might have confirmed them was that all figures were first regular, perfect and would express primarily the whole power of the number. Hence, as *cosmic figurs*, they were perceived as gifted with a high symbolic power through their first expression of harmonic or transcendent numbers : 12 (dodecahedron), 5 (pentagon), 4 (square, tetrahedron), 6 (cube). Being conscious of the internal irrationality of these figures, these should reveal to their eyes all the mystery of the real, the subsumption of the infinite through the finite, and the incommensurable through the commensurable. Should this knowledge remain hidden from the uninitiated, it is first because there used to be in these times with the Pythagoreans, as well as with Plato later, a selective conception of oral transmission of knowledge which is not ours.²² Moreover, these mystic philosophers could rightly so suspect, that the

²⁰ P.-H. Michel, *op. cit.* p. 486.

²² See on the subject the *Lettre de Lysis* to Hipparchos (Jamblique, *V.P.* §75-78) written by a IVth Century B.C. traditionalist, according to A. Delatte (*op. cit.*, p. 103), invoking Lysis' authority (Vth Century). This letter would have been transmitted to us by Timaeus of Tauromenium. One will note, however, the very Platonic

uninitiated would not understand it and, neglecting the domination of the number on the incommensurable, he would fall onto the breach of irrationality. Consequently, it is the whole cosmo-theological conception of the *rational communion* of the beings, of the Pythagorean *κοινωνία* of which Plato speaks about in the *Gorgias* (508a), which could be put in difficulty.

III Absurdity

Thus the legend of Hippasus gives an instructive general idea of the importance of the shock and uneasiness facing the emergence of the irrational. However, to go further ahead in the analysis of the emotions that the Greeks might have felt, and moreover on the *absurd* point of view, of which Pappus speaks (*surd*)²³, we must question the proofs of incommensurability as they were conceived during this high period of Greek science.

A genuine mathematic proof of the incommensurability of the diagonal of the square was formulated by the Greeks (probably in the Vth Century). It is a proof that inaugurates the purely deductive mathematic, called the apagogic proof (*ἀπαγωγή* : absurd deviance). Aristotle makes an allusive statement in the *Premier analytics* 41a 21-37 — but it is later exposed in due form in an appendix in book X of Euclid's *Elements*. It succeeds by putting down $\sqrt{2} = m/n$ (with m and n premier between them), taking into consideration that the numbers m et n are both even and odd.

$\sqrt{2}$ situated between 1 and 2 = m/n with m and n commensurable and premier between them.

Let us suppose that m is even and n is odd.

$(m/n)^2 = 2 = m^2/n^2 \Leftrightarrow m^2 = 2n^2$. If m^2 is a square even, it is divisible by 4, since all square evens are divisible by 4.

Same thing for $2n^2$ which is equal to m^2 , then n^2 is divisible by 2, then n is even.

We have here a very Ancient example of a *demonstration by the absurd* and this is to explain what an *hypothetic* proof is for Aristotle to give this example (the absurdity of the conclusion : n both even and odd, implies an inaccurate premiss that postulates that m and n are commensurable : then m and n are incommensurable). One may have noticed the Pythagorean character of the terminology of the proof (table of opposites with the

content of the sophistic criticism : « Les sophistes ... attrapent les jeunes gens dans leur nasse sans convenance ni justesse. C'est pourquoi ils rendent malveillants et téméraires leurs auditeurs. En effet, ils déversent en des personnalités instables et troubles des savoirs et des doctrines divins, tout comme si quelqu'un versait dans un puits profond rempli de boue une eau pure et limpide » (translation by L. Brisson). Cf. M.-D. Richard, *L'enseignement oral de Platon*, Paris, 1986, p. 49 sq.

²³ The term *surd* is in fact a Latin originated English term which translates the Arabic *asamm* which means *deaf*, *ineffable* : arabic translation from Greek *ἄρρητον* (inexpressible). But, etymologically, absurd is what is deaf, or outside of tune.

opposition of the even and the odd ; and fragment 5 by Philolaus m^2 : being evenly even — divisible by 4 —, while $2n^2$ is supposed to be an odd-even)²⁴; one may have also noticed the *arithmetism* of the proof: another proof of the Pythagoreanism. However, this apagogic proof cannot be easily brought back to the first Pythagoreanism (before the dissolution of the sect, in the middle of the Vth Century): for most historians, it would be later since it integrates the reasoning by the absurd which was certainly developed by the Eleatic School.

In any case, this is what the proof reveals, there are no identifiable numbers : the incommensurable is a ἄρρητον, an *inexpressible*. However, the Pythagoreans did not just stay there. They found at least one algorithm of a progressive approximation, it is the *diagonal and lateral numbers* method.

Let a triangular figure be: an isosceles rectangle pseudo-triangle. By using the properties of the figure hereunder, it is a matter of introducing a sort of artificial commensurability by postulating the side and the diagonal as $a = d = 1$ and by building the figures $d_1, d_2, d_3...$ which draw nearer and nearer to an isosceles rectangle triangle.

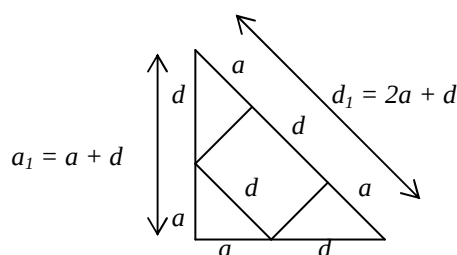


Figure 2

$$\begin{aligned} d_1 &= 2a + d = 2 \times 1 + 1 = 3 \\ d_2 &= 2a_1 + d_1 = 2 \times 2 + 3 = 7 \\ d_3 &= 2 \times 5 + 7 = 17 \\ d_4 &= 2 \times 12 + 17 = 41 \end{aligned}$$

$$\begin{aligned} a_1 &= 1 + 1 = 2 \\ a_2 &= 2 + 3 = 5 \text{ (approximations by Plato)} \\ a_3 &= 5 + 7 = 12 \\ a_4 &= 12 + 17 = 29 \end{aligned}$$

This is to indicate the differences between the squares of the *rational diagonals* (διαμέτρων ῥητῶν) and those of the *irrational diagonals* (ἄρρητῶν), by taking Pythagoras' theorem :

$d_1^2 = (2a + d)^2 = 9$	$a_1^2 + a_1^2 = 2a_1^2 = 8$	difference : + 1
$d_2^2 = (2a_1 + d_1)^2 = 49$	$2a_2^2 = 50$	difference : - 1 (irrational diagonal by Plato)
$d_3^2 = (2a_2 + d_2)^2 = 289$	$2a_3^2 = 288$	difference : + 1
$d_4^2 = (2a_3 + d_3)^2 = 1681$	$2a_4^2 = 1682$	difference : - 1

²⁴ Also see DK44A13, DK58 B2, 58B5, 58B22.

One can then distinguish the *rational and irrational diagonals* in book VIII of the *Republica* (546c) – an obscure passage strongly impregnated with the Pythagorean mysticism.²⁵ From that time on, an absolute periodic proof appears (+1, -1, +1...) from the incommensurability. Furthermore, one will notice the following property :

The irrational $\sqrt{2}$ is always framed in excess or by default by the rational ratios (*lógoi*) : 1 by default, 3/2 (1,5) in excess, 7/5 (1,4) by default, 17/12 (1,417) in excess, 41/29 (1,413) by default, 99/70 (1,41428) in excess, 239/169 (1,41420...) by default,....²⁶

Here we have, it seems to me, an unsettling link between the irrational approximation procedure (which progresses indefinitely towards a never reached limit) and the Achilles' apory and Zeno of Elea's Tortoise, in which *one can see clearly two positions* (the most and the least or the fastest and the slowest) that keep indefinitely approaching each other without ever meeting...

Indeed, what may have given Zeno such an idea as to compare the paradoxical run of Achilles and the Tortoise, but the example of a never reached limit, pursued by two evolutive movements which should normally meet but do not succeed? To have been able to imagine that the fastest runner will never pass the slowest; this idea had to be suggested by an algorithm with a double progression approximation. In and of itself, this idea is not natural and cannot come spontaneously to the mind without a mathematical cause.²⁷ It is true that the approximation of the most and the least does not follow the same direction in the algorithm, while Achilles and the Tortoise go the same way. But it can be easily understood that Zeno, having got to know the approximation algorithm, found it so inconceivable that he sought any means to caricature it by revealing what, according to him, was complete absurdity. It is not surprising in itself that this algorithm might have provoked such a reaction, as the Greeks never really accepted the idea of a progression

²⁵ The algorithm of the lateral and diagonal numbers is supported by the Pythagorean principle of the Monad, as the principle of all figures, according to the supreme and generating reason (Théon, *Expos.* : πάντων τῶν σχημάτων κατὰ τὸν ἀνωτάτω καὶ σπερματικὸν λόγον ἡ μονὰς ἀρχεῖ). I, xxxi p. 70 = Hiller p. 43 (5-7).

²⁶ This algorithm could provide us with further explanations on the mathematic sense of the Platonic Pythagorean notion of the *indefinite Dyad of the Big and the Small*. It is that there is for Plato, a More and a Less where the Big and the Small are transformed into Numbers by an approximation procedure. According to him, the pure irrational is not a number but a *substratum* moving from the More and the Less towards excess and default that discrete numbers surround more and more with *lógoi*, without ever really grasping it. The incommensurability is in that sense a veritable indefinite Dyad and, and the commensurable number that approaches the incommensurable is a number constantly evolving between the more and the less: it is the *production of a metrion*, as quoted in *Politicus* (284c1).

²⁷ Abel Rey (*La jeunesse de la science grecque*, Paris, 1933, p. 197) particularly analyses the infinite divisibility argument, as expressed in fragment 3, which supposes a preliminary awakening to the incommensurable: « Zeno's argument, while aiming to the indefinite divisibility, reveals that, contrarily to an archaic representation, the mind would have raised itself to the conception of something that the figured and stated number cannot reach nor measure, to the conceptual construction of an incommensurable relationship ».

towards a limit : this would totally hurt their sense of mathematic correctness, according to which perfection can only reside in the finite, the balanced, the measured.

In any case, it was noticed that the aporetic arguments by Zeno were to fight two contemporary Pythagorean conceptions : on the one hand, that of the unit-points, the indivisible figured numbers, on the other hand, the conceptions of the *ad infinitum* division of the line, which may only result from the discovery of irrationals.

It is a famous thesis that was formerly defended (in 1933) by Abel Rey:

« L'argumentation de Zénon vise deux interprétations pythagoriciennes : celle qui s'appuie sur le nombre nommé, fini, sur les indivisibles, l'interprétation archaïque, et celle qui, plus savante, fait état des innovations de la mathématique, mais des innovations tombées déjà dans le domaine courant, car Zénon, pas plus que Parménide, n'a fait figure de mathématicien ».²⁸

With the irrational, the indefinite division of the straight line ceases to be arbitrary, useless, in the same way as when someone tries to divide indefinitely a measured unit that should not be divided (cf. Plato, *Republica*, VII, 525e) : it imposes itself in its whole ontological density. It becomes necessary, as an unavoidable mathematic constraint. To establish an approximate measure of the diagonal, one has to determine intervals, or more and more precise *lógoi*, indefinitely. However, such a property, even though necessary, could not have been accepted without violent reactions... One will notice furthermore that Zeno seems to have sought to lay down the basis of the incommensurability, as it is said in certain testimonials that he would have laid the principle of indivisible lines (DK29A22). In this way, he would represent the passage of the Pythagoreans punctual measure to the linear measure²⁹ (while defending the principle of the absolute *plenum* of the Parmenidian Being) : as it is mostly the theory on the multiplicity of figured points of the Pythagoreans that he would have had in his line of sight. May be he also had in mind with the indivisible

²⁸ A. Rey, *op. cit.*, Paris, 1933, p. 199. The idea, consensual from the start, of a relationship between Zenon's apories and the conception of figured numbers, was contested by several Pythagorean hypercritical commentators (cf. A. Szabó, *Les Débuts des Mathématiques grecques*, 1969, trad. franç. 1977, p. 288-289 ; Kirk and Raven, in G.S. Kirk, J.E. Raven and M. Schofield, *Les Philosophes présocratiques*, (English edition, 1983), Fribourg-Paris, 1993, p. 298 n. 9 ; W. Burkert, *op. cit.* p. 285-288 ; L. Zhmud, "All is number, basic doctrine of Pythagoreanism reconsidered", *Phronesis*, 34, 3 1989 p. 277). There would not be, according to them, neither conceptual proof, nor tradition concerning an opposition from Zeno to the Pythagoreans. It is to forget that Zeno wrote a piece of work entitled *against the Philosophers* (which may only designate the Pythagoreans) (DK29 A2) ; on the other hand, the term ὅγκοι in the apory of mobile and immobile ranges of masses reported by Aristotle (*Physica*, IV, IX 239b33) reminding us the ἀριθμοὶ ὅγκοι, the entire numbers of *Timaeus* (31c4), according to the Ancient Pythagorean conception (cf. M. Caveing, *Revue d'Etude Philosophique*, 15, 6, 1965 and L. Brisson, *Le Même et l'Autre*, Academia Verlag Sankt Augustin 1994, p. 373). In fact, the Tannery-Rey thesis was resumed with new arguments by M. Caveing in *Zénon d'Elée, Prolégomènes aux doctrines du continu*, Paris, 1982, chap. IV and *La figure et le Nombre* Lille, 1997, p. 315 : « Le pythagorisme ancien a certainement donné une signification physique immédiate à l'arithmo-géométrie, aussi bien qu'aux proportions numériques et à la géométrie des grandeurs rationnelles, sans soupçonner les apories qu'une telle conception devait nécessairement susciter ».

lines theory to solve the issue of incommensurability of the diagonal expressed in unit points...

With regards to Achilles' apory and the Tortoise, one will note that the by-the-absurd argument herein is taken in the opposite way from that of the apagogic proof. Indeed, according to the thesis being considered, Zeno would have used the by-the-absurd process to show the impossibility of the irrational, while the by-the-absurd argument will be used by the Pythagoreans or by one of their heirs to demonstrate the existence of the incommensurable. One can then infer that upstream from eleatism, the proof of the incommensurability was not yet totally established or accepted, while presenting itself as highly probable, implied by the approximation algorithms; downstream, it will be demonstrated without ambiguity by the *apagogic proof*. Consequently, the elaboration of the apagogic proof must have been done from a polemic dialogue between the Pythagorean community and the Eleatic School towards 450. This teaching was then transmitted to Hippocrates of Chios 450-430.³⁰ From Hippocrates of Chios it came to Plato, the go-between certainly was Theodorus de Cyrene. Plato reveals, indeed, a mathematic teaching of the incommensurability on behalf of Theodorus in the *Theaetetus*.

IV Vertigo

Until now, we have been speaking about *logical scandal*, about *doctrinal contradiction*, about *betrayal*, about *inquiétude* and about *absurdity*, but not exactly yet about *vertigo*: with respect to this notion, the Platonic dialogue in *Theaetetus* could bring us a few elements. Let's first take note of the important passage 147d which is the most ancient text to make a direct reference to a mathematic proof of incommensurability, towards the end of the Vth Century :

« Here we have Theodorus who had drawn (ἔγραψε), before us, *something about powers* (περὶ δυνάμεων τι), and *had shown* (ὑποφαίνων) that the powers three and five foot are, according to their length, not commensurable (μήκει οὐ σύμμετροι) to that of one foot, and having taken them one by one, until the seventeen foot ones, *for a particular reason* (πῶς), he had stopped here. It came to our mind, the number of powers appearing infinite (ἄπειροι τὸ πλῆθος αἱ δυνάμεις ἐφαίνοντο), to try to gather them under a unique term that could help to designate all that powers can be ».

The *dunámeis* are the considered lengths from the square they generate. Some are commensurable in length, others are only through their square. For example $\sqrt{3}$ is not commensurable in length but remains commensurable through a square equal to 3 that this

²⁹ A. Rey, *op. cit.*, p. 204. « (...) Zénon clôt la conception archaïque du nombre, l'indivisible conçu comme ayant grosseur et épaisseur, et étant point, unité ponctuelle tout de même. Elle est définitivement abattue pour le plus grand bien de la logique géométrique, de la saine mathématique ».

length generates. Theodorus' aim was then to demonstrate the incommensurability of the *dunámeis* of 3, and 5, and of certain numbers up to 17. Without pretending, in this study, to settle the difficult question concerning which demonstration process Theodore would use, what is of interest to us, is that the text lets a certain familiarity with the incommensurable appear: the knowledge seems acquired and so does the infinite number of *dunámeis* (being possibly incommensurable) which does not seem, in this passage at least, to make the young Theaetetus shudder. Several reasons could explain this.

This knowledge has been assimilated for some time and does no longer directly give rise to astonishment. We are at the second stage as cited by Aristotle in *Metaphysica* (983 a 15-20), according to which the mathematicians would be most surprised if the diagonal became commensurable. One will notice that it is not even a matter of the *dúnamis* of 2. The only detectable reason of this eviction is that this property has been known for quite some time and that its incommensurability needs no longer to be proven. We have here a sufficient element of proof against the partisans of late dating. The knowledge is now digested and even vulgarised. In this regard, Á. Szabó noticed Theaetetus' neglected vocabulary, with the abbreviated formulas of μήκος (148a8) and δυνάμεις (148b1), instead of μήκει σύμμετρος and δυνάμει σύμμετροι.³¹ Theaetetus is then introduced by Plato like a young neophyte who does not yet master the vocabulary,³² even though Theodore is veritably the master who demonstrates by construction (ἔγραφε ... ἀποφαίνων) and who uses the correct expression μήκει οὐ σύμμετροι (147d5-6).

It is revealing that in the rare passages where Plato tackles the incommensurability or the mathematic irrationality, he often seeks to dedramatize it. And we can understand this effort to dedramatize only in relation to an initial *pathos*, or to a first traumatic *thauma*. In this passage of *Theaetetus*, Plato *mathematically* dedramatizes the very concept of the incommensurable by showing that what is incommensurable in length is finally commensurable by its *dúnamis*. Besides, the other times when he exposes knowledge on the irrationals, he does it with humoristic or mathematically playful examples, as in *Politicus* (266ab) and in book VIII of *Republica*, or in a very allusive manner as in *Hippias major* (303b). If the vertigo does not appear directly in these texts it is because Plato seeks to

³⁰ Iamblichus, *De com. mathema. scientia*, §25. The source could be the *Historic Résumé* by Eudemus.

³¹ Á. Szabó, *Les débuts des mathématiques grecques* (trad. Federspiel) Paris, 1977, p. 70-71.

³² This lack of mastery does not keep Theaetetus' mind from coming across a new intuition that consists of *create the notion of irrational number*: dealing with rational and irrational dimensions as numbers, which implies a generic *non distinction* between entire numbers and irrationals conceived as numbers in the same way as the others. Cf. M. Narcy, Introduction to his translation from *Théétète*, Paris, 1995, p. 62. Plato, in his oral teachings will reject this geometrical conception of the number which rather anticipates the Aristotle vision (cf. *infra* n. 39 et 40). Cf. M. Narcy, (*ibid.*) p. 65.

think and to *compensate it* : the incommensurable having to be perceived before any thing else, under its positive, logic, playful angle, finally like an object of intellectual research.

Actually, as Henri Joly stated, *Plato assists or participates in an attempt to "rationalisation des irrationnels"*,³³ which explains his contempt for the ignorance of the incommensurables [in book VII from *Laws*] and his commitment to the city to learn science. It is then a matter of going from the old *pathological thauma of the scandal and the betrayal* of the Pythagoreans, to a pure intellectual amazement that would only belong to the philosopher, as said in *Theaetetus*: μάλα γὰρ φιλοσόφου τοῦτο τὸ πάθος, τὸ θαυμάζειν : *it is rather belongs to the philosopher this pathos to be astonished* (*Theaetetus* 155d).³⁴ In which precise context was this important sentence pronounced? Socrates (en 155bc) just indicated a paradoxical dialectic about the opposition between being and becoming : Socrates, who does not get greater or smaller in the space of one year, can, from greater than he is now, become smaller than Theaetetus, since the latter is becoming greater. He is then *a posteriori* what he was not earlier, without having become, without having changed so much. In fact, Socrates states, this is only one paradox among a myriad of others, of which, Theaetetus, you are not unaware. Curiously, Socrates said : you are not *unlimited* (οὐκ ἄπειρος) : since in Greek the same term means being infinite, limitless and being ignorant ; which only shows the pejorative vision which the Greeks had of the infinite, but also corresponds to a total inversion of the meaning in relation to our vision : to be *bounded, limiting* for them means being acquainted with something ; on the other hand, being unlimited, limitless (or *non-bounding*) is to be drowned without distinction, being totally overtaken. Facing the unlimitation of paradoxes, Socrates said to Theaetetus, you are not supposed to be *non bounded*, meaning : *ignorant*.

The young man is going to answer that he is completely overtaken : « By the Gods, Socrates, I am astonished (θαυμάζω) at what these things can be, to the extent that it is out of bounds (ὑπερφυῶς), and I have to say, that each time I look at them, I am *swirling in the darkness* (βλέπων εἰς αὐτὰ σκοτοδινίω) ». ³⁵ Here we have a verb σκοτοδινιῶ that A. Diès, and E. Chambry translate every rightly so by the notion of *vertige*. In fact, it is no coincidence if Aristotle in *Metaphysica* (983 a 10-20) makes of the diagonal incommensurable a characteristic object from the philosophic *thaumazein* : one can then

³³ Henri Joly (*Le Renversement Platonicien*, Paris, 1974 p. 204-205) borrows this expression to A. Virieux-Reymond, *Platon ou la géométrisation de l'univers*, Paris, 1970 p. 38.

³⁴ In *Laws* 819d5-6, Plato makes another link between the incommensurability, *pathos* and amazement : *I was greatly astonished by our passiveness in this regard* (παντάπασι ... τὸ περὶ ταῦτα ἡμῶν πάθος ἐθαύμασα).

³⁵ One will note that Plato uses the vocabulary of the Pythagorean table of the opposites with the darkness (*skótos*) placed on the side of the *apeiron*. Cf. *supra* n. 8. Is it only a coincidence?

understand the Aristotelician passage like a commentary from this excerpt in *Theaetetus*. And this is really what this is all about just beneath the surface, in Theaetetus' mind in our excerpt. Moreover, at the time of the scene of the dialogue, Theaetetus is still young (smaller in size and in mind than Socrates) : he has not yet had time to master the dialectic philosophic subtleties. On the other hand, He has already been initiated by Theodore to the mathematic paradoxes and, as *small* as he is for now, he will become the *big* mathematician who will mainly work on the irrationals, who will outsmart in this domain all his contemporaries, including Socrates.³⁶ Consequently, even though the Socratic argument context may seem a bit external, one can retain that Theaetetus' great breathtaking amazement (dark, non playful) is really about the incommensurability of reality; in the same way, by his youth, this character is rightly placed to represent the naïve Greek consciousness particularly that of the Pythagoreans who were the first to be facing this problem, being completely overtaken.

V Solutions and occultations

Theaetetus' retort can then be taken for an evident echo of the *vertigo* of the first mathematicians placed before the discovery of the incommensurability.³⁷ One could not then say that there was no crisis of fundamentals, or deep vertigo of the Ancients facing the incommensurable. And it is an illusion to believe that progress of mathematic sciences generally only proceeds in a linear and cumulative manner, mostly for a discovery in itself as problematic, as paradoxical as the one of the incommensurability. The famous *Greek miracle*, this dazzling progress of the Greek mathematic in the Vth and the IVth Centuries, on the contrary was accomplished, not through light and serenity, but darkness, in the confrontation of apparently insoluble problems, by coming across a crisis without precedent, a most vertiginous situation for the mind. The nascent Greek mathematic mind is then as much a mind of the abyss than a contemplative quest of essences. But *in fine*, this crisis of fundamentals, far from having paralyzed research, developed it tremendously: the Greek mind summoned up all its resources to find intellectually acceptable solutions, rationally, philosophically. What was called the *Greek miracle*, from a mathematic point of view, is nothing else than this extraordinary adventure of the human spirit which was the intellectual discovery and understanding of the irrationals, the exemplary result of which is

³⁶ What hit Theodorus, is that Theaetetus, through his ugliness and his quick wits, seemed to be a perfect replica of Socrates (143e-144a) and the latter also perceived an *happy nature* and predicted celebrity (142cd).

³⁷ One should not expect to find direct testimonials: outside Plato and Theaetetus, the crisis of the founding only touched particularly the Pythagorean and Eleatic circles, of which very few writings remain; we saw that it is rather in the more or less intentional symbolic dimension of certain texts that traces can be perceived.

the publication, at the end of the IVth Century, of this monument of the rational thought that were the Euclid's *Elements*. In reality, this *Greek miracle* presents all the characteristics of an *epistemological revolution* in the full Bachelardian³⁸ sense : It is not Reason which comes down from the sky and manifests itself clearly, in all serenity, in the contemplative mind of the Ancients, it is on the contrary the contradicted run of the mind which finally succeeds to reach unequalled heights, but which all the more mobilised unhoped for resources that it was submerged with the most inextricable apories : one may believe that it is by exposing oneself to vertigo, like the acrobat in *Zarathoustra*, that the mind can be drawn to overpass itself.

However, in terms of epistemological revolution, the result of this painful process settles through the passage of a mind system that would identify in all naivety numbers and things, to a geometric *continuum* thought in which rationality is being discovered on new bases, with infinite possibilities of extension. This is how a deep "*change of episteme*" gets accomplished : we go from the Ancient *arithmetism* to the system of unit points, to the new Theatetian and Euclidian spirit, founded on geometry. What Plato seems to have intuitively understood, is that through Theaetetus a veritable *change of paradigm* gets accomplished: in this, *Theaetetus will become grand*, he represents the new tendency. Arithmetic which, in the Ancient Pythagorean vision summarised in itself the whole *mathesis universalis*, loses its privileged place to become only a particular case of mathematic. In this regard, we could even consider the Platonic system of exposed principles in the *ágrapha dógmata* as retransmitted by the Aristotelician tradition, as an ultimate tentative to ontologically keep the Pythagorean pre-eminence of the Number, in a world representation which becomes *more geometrico*, and which integrates the linear incommensurability in a vast procession of the four intelligible perceptible dimensions of the real, from the two primordial principles of the Monad and the indefinite Dyad. If Euclid's geometry (rallying the works by Theaetetus and Eudoxus) is the great mathematic response to the crisis of the foundations in the Vth et IVth Centuries, the doctrine of the principles of oral teachings by Plato constitutes the first great philosophical response.³⁹ The second will be Aristotle's.⁴⁰ If the Platonic

³⁸ H. Joly, *op. cit.* p. 206-207.

³⁹ Cf. K. Gaiser (*Platons Ungeschriebene Lehre*, Stuttgart 1963 p. 136 sq.) : the whole Platonic system (*open*, in dialectic research, not closed) would organise itself according to the four dimensions Point-Line-Surface-

reply was not disclosed in writing, it is that instead of raising the minds towards transcendental Reason in a misunderstood and misinterpreted manner, it risked, on the contrary, to throw down the lost in the darkness and the drowning in the *sea of non-identity*... We have here a non-neglectable supplementary explanation of the non-publication of the Platonic conception of the *indefinite Dyad*: rather than revealing the abyss of the *apeiron* of the incommensurability in its whole ontological dimension, Plato, in his writings, preferred to emphasise the conversion of the incommensurable into commensurable (*Philebus*, 25th, *Polit.* 284 b-c). Such an *apeiron* was perceived by Plato as a fascinating abyss : there was to avoid at any cost that which the common man would throw himself down for by showing what still could be saved as rational in the irrational, and by hiding as much as possible the existence of the abyss.⁴¹ Such a precaution may seem strange and pathetic nowadays, but does not modern man himself have the tendency to overshadow the problem by covering it up behind a word or under a mathematical formula thinking to have solved the matter once and for all? Plato had reasons to believe that it was better to hide the abyss of the irrational to the uninitiated, while making it for himself and his disciples, the object of a veritable metaphysical meditation. Does the actual man really interrogate himself as to why, in the most simple figures, the most regular, and apparently the most rational (square, pentagon, circle, etc.), it is not possible to find the least common

Volume which cross hierarchically the structure of the intelligible and perceptible reality. (cf. M.-D. Richard, *op. cit.* p. 192-3, 231 sq.). The stumbling block between lines and surfaces, between the intelligible and the perceptible, is nothing other than the incommensurable itself present in the *atomon eidos*, the lowest degree of the intelligible (cf. our study: *Platon et la section d'or*). Finally, in reply to Zeno's apories, Plato maintains the two aporetic terms that Zeno would oppose without bringing any solution : the indivisible and rational units will be stationed in the intelligible and the infinite divisibility and incommensurable of geometric lengths in the perceptible. It is to be noted to what extent the Platonic reply to the apories is closely linked to the treatment of the irrationals: another supplementary reason to show that we cannot conceive the ones without the others.

⁴⁰ Aristotle proposes the distinction of the infinite as power (domaine of mathematic idealities) and the infinite as an act, the existence of which he rejects. J. Toussaint-Desanti, ("Une crise de développement exemplaire : la découverte des nombres irrationnels", in *Logique et connaissance scientifique*, La Pléiade, Paris, 1967, p. 451 sq.) shows that Aristotle resolves the irrationals crisis in that the fundamental *distance* between, on one hand, the substantial representation of the number (the real as a unit multiplicity) and, on the other hand, the *continuum* in right of indefinitely divisible operations on lengths, first thought as incompatible by Zeno of Elea (who would suppress the two aporetic terms). Meanwhile Aristotle, through his conceptual distinction of the *act* and the *power*, would find an elegant solution to guarantee the possibility of mathematical operations by maintaining them in the non real ideality of the infinite in power.

⁴¹ Thus, the incommensurability has, according to Plato, a double positive and negative aspect : positive as a concept that can admit a geometric rationality, negative as an ontological principle of the rebellious wrong to any rationality. This ambiguity may have been the source of confusion with the disciples of the Ancient Academy. One can understand then that contradictory Platonic elements could have cropped up in connection with the Pythagorean legend, as reported in the *Commentary* by Pappus and Euclid's *scolia*. One will note that, following the example of the Pythagoreans, the *apeiron* by Plato bears indeed the stamp of evil: *the shape of virtue is one, those of evil are infinite* : ἐν μὲν εἶναι εἶδος τῆς ἀρετῆς, ἅπειρα δὲ τῆς κακίας (*Rép.* IV, 445c5-6). Even though this sentence is a resonance of the theme of the *Monad* and the *indefinite Dyad*, Socrates does not seek to elucidate the principle cause of evil in the *Republic*. Cf.

measure between the different lengths that constitute them? Isn't there here an incomprehensible enigma that persists beyond the reassuring formula?

Through the figures of *scandal*, *betrayal*, *absurdity* and *vertigo* that we were able to perceive more or less in certain testimonials, it is the whole psychological, moral and philosophical drama, of one of the greatest conquests of the human spirit, which can be reconstituted. In view of the shortage of purely historical data, we had to resort to the deciphering of the imaginary, and of the implicit, to get a clear idea (if not distinctive) of the whole tragic extent of the adventure that played itself during the Vth Century B.C. It turns out that the *vertigo* of the irrational may have only struck an elite of thinkers previously convinced of an existence of the rationality of the world, of an almighty measured order of the number. Even if it corresponds first to a troubling implication of a popular and empirical bias, one further understands the importance of the confusion of those who had pushed to the highest degree the trust in Reason. There is here finally an encounter with the infinite (as an impossibility to assign a numeric limit) that seems to have been particularly problematic and could well have been as painful, as vertiginous as the dawning of the consciousness of the universe infinite in the XVIth et XVIIth Centuries.⁴² The Greeks did go through a first *Pascalian crisis* as they faced the infinite — such an unbearable anxiety that they could only react with disastrous desires: the death of Hippasus by water and through divine anger singularly anticipates the death of Giordano Bruno by fire and through human revenge. However, the Greeks' metaphysical ordeal could not be perceived in all its significance without a quick examination of the global consequences upon later thinking.

Trouble thrown into the minds by reason of the existence of the incommensurable, drew certain thinkers, the sophists, to doubt the value of the *Lógos* and its inherent truth.⁴³ We should not in this regard underestimate the link there was between Theodorus, the

Th. A. Szlezák : "L'idée du Bien en tant qu'*arché* dans la *République* de Platon", in *La philosophie de Platon*, *op. cit.*, § 3.

⁴³ Cf. Henri Joly, *op. cit.* p. 380 : « En présence de ce *thauma*, où se mêlent l'étonnement intellectuel et une sorte de terreur tragique suscitée par le spectacle d'une réalité sans rationalité, deux attitudes étaient possibles. L'une consistait à douter de l'idée de science, d'où le conventionnalisme, voire le scepticisme de la sophistique. L'autre attitude, platonicienne celle-là, consistait à maintenir l'idée de la science *en opérant un déplacement de l'épistémè et en inaugurant, autour de la géométrie, un nouvel esprit scientifique* » (underlined by the author). Concerning this interpretation of Platonism, it seems to me that, taking into account the oral doctrine of principles, Plato maintains the arithmetic pre-eminence over geometry, by making possible, in the perceptible domain, the change of *episteme* as described by H. Joly: the incommensurability does not exist in itself, but supposes a fundamental commensurability (for example, the diagonal is incommensurable only in relation to the commensurable side): then, arithmetic precedes geometry.

mathematician, and the well known sophist, Protagoras (see *Theaetetus*, 161b, 162a).⁴⁴ Curiously, between these two characters, one decided to remain silent (146b, 165a) and to only do geometry, while the other abandons the theory of the *lógos*, but decided to speak, as indicated by his name : Protagoras. In both cases, it is the truth that is threatened, it is the *Lógos* that is affected. With one there is a *lógos* without the discourse (the purely mathematical *lógos*), with the other there is a discourse but no *lógos*, without the science. Through the example of these emblematic characters, one notices a fracture, which is the *ratio*-unit discourse which is broken in one way or another. Furthermore, the Zenonian apories appeared to us as directly linked to the irrational crises. Thus, irrationality was demonstrated with the absurd. In other words, through the irrationals, the Greeks were faced to the absurd and they even mastered through their own feat of strength to transform the absurd into a real principle of demonstration that would merge into the irrational truth of the incommensurable and the inexpressible. Will we have sufficiently weighed all the notions which knock each other out and which deploy themselves in negativity and the unthinkable? Throughout the convergence of these aporetic notions, how would one not perceive the tracks of a veritable drama of the mind flickering on his logical foundations? Therefore, we will not be surprised to see one of the masters of the by-the-absurd argument, Zeno of Elea, be associated with the sophists by Plato⁴⁵. He is the one who got flogged by the crisis of the irrational, who fought it with derisory arguments and who probably had to submit himself to it by reason of the indisputable apagogic proof, as far as he knew it. But the consequence was disastrous on the *lógos*-discourse level: no more truth could then hold for the first sophists. No matter which thesis, even the most absurd, could be defended... It was the beginning of the ruin of the *Lógos*. Now we find ourselves in a situation which may not be so different : on one hand the disciples of Theodore, those who calculate but do not practice philosophy win; on the other hand, the continuators of the sophists reduce philosophy to a rhetoric discourse. Further to the crisis of the irrationals, we finally lost the sense of the *Lógos* which would gather in one notion the Rapport, the Discourse and the Cosmos, a unitary vision developed by Pythagoras and which we can find with Heraclitus. Despite Plato's efforts, as well as Aristotle's and the Stoicists' to keep under various forms this vision of the fundamental adequation between the Reason and the Real, the doubt was definitely introduced in the minds.

⁴⁴ The links between the master of geometry and the sophists are evident: Theodorus, as well as Protagoras, land in Athens and attract a crowd of young people (*Theaetetus* 143de). M. Narcy (*op. cit.* p. 52-53) sees in Theodorus' geometry a link with Protagoras' sensualism.

⁴⁵ Plato, *Alcibiades major*, 119a ; *Phaedrus*, 261d.

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Article summary :

This study aims to show how the first vertiginous vision of the incommensurability deeply moved the Greeks' mind in the Vth et IVth Centuries B.C., before becoming a new geometric conception of the infinite that will integrate the irrationals. This crisis and its epistemologic passing seem to have occurred in three main stages: the rational Pythagorean infinite, the aporetic Zenonian infinite and the Platonic irrational infinite. Now, curiously, Plato did not publish his conception of the infinite raised to the dimension of an ontological principle. Isn't it because he would always see in it some kind of *vertiginous abyss* ? Through the tragic figures of *scandale*, *betrayal*, *absurdity* and of *vertigo*, it is the psychological, moral and philosophical drama of one of the greatest conquests of the human spirit that one can try to find in the texts.

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