

GREEK GEOMETRY FROM THALES TO EUCLID.¹

IN studying the development of Greek Science, two periods must be carefully distinguished.

The founders of Greek philosophy—Thales and Pythagoras—were also the founders of Greek Science, and from the time of Thales to that of Euclid and the foundation of the Museum of Alexandria, the development of science was, for the most part, the work of the Greek *philosophers*. With the foundation of the School of Alexandria, a second period commences; and henceforth, until the end of the scientific evolution of Greece, the cultivation of science was separated from that of philosophy, and pursued for its own sake.

In this Paper I propose to give some account of the progress of geometry during the first of these periods, and

¹ It has been frequently observed, and is indeed generally admitted, that the present century is characterized by the importance which is attached to historical researches, and by a widely-diffused taste for the philosophy of history.

In Mathematics, we have evidence of these prevailing views and tastes in two distinct ways:—

1° The publication of many recent works on the history of Mathematics, *c. g.*—

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also to notice briefly the chief organs of its development.

For authorities on the early history of geometry we are dependent on scattered notices in ancient writers, many of which have been taken from a work which has unfortunately been lost—the *History of Geometry* by Eudemus of Rhodes, one of the principal pupils of Aristotle. A summary of the history of geometry during the whole period of which I am about to treat has been preserved by Proclus, who most probably derived it from the work of Eudemus. I give it here at length, because I shall frequently have occasion to refer to it in the following pages.

After attributing the origin of geometry to the Egyptians, who, according to the old story, were obliged to in-

and is not free from inadvertencies and even errors, yet I have derived advantage from the part which concerns Pythagoras and his ideas. Hankel's book contains some fragments of a great work on the History of Mathematics, which was interrupted by the death of the author. The part treating of the mathematics of the Greeks during the first period—from Thales to the foundation of the School of Alexandria—is fortunately complete. This is an excellent work, and is in many parts distinguished by its depth and originality.

The monograph of M. Bretschneider is most valuable, and is greatly in advance of all that preceded it on the origin of geometry amongst the Greeks. He has collected with great care, and has set out in the original, the fragments relating to it, which are scattered in ancient writers; I have derived much aid from these citations.

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vent it in order to restore the landmarks which had been destroyed by the inundation of the Nile, and observing that it is by no means strange that the invention of the sciences should have originated in practical needs, and that, further, the transition from sensual perception to reflection, and from that to knowledge, is to be expected, Proclus goes on to say that Thales, having visited Egypt, first brought this knowledge into Greece; that he discovered many things himself, and communicated the beginnings of many to his successors, some of which he attempted in a more abstract manner (*καθολικώτερον*), and some in a more intuitional or sensible manner (*αἰσθητικώτερον*). After him, Ameristus [or Mamercus], brother of the poet Stesichorus, is mentioned as celebrated for his zeal in the study of geometry. Then Pythagoras changed it into the form of a liberal science, regarding its principles in a purely abstract manner, and investigated its theorems from the immaterial and intellectual point of view (*ἀύλως καὶ νοερῶς*); he also discovered the theory of incommensurable quantities (*τῶν ἀλόγων πραγμάτων*), and the construction of the mundane figures [the regular solids]. After him, Anaxagoras of Clazomenae contributed much to geometry, as also did Oenopides of Chios, who was somewhat junior to Anaxagoras. After these, Hippocrates of Chios, who found the quadrature of the lunule, and Theodorus of Cyrene became famous in geometry. Of those mentioned above, Hippocrates is the first writer of elements. Plato, who was posterior to these, contributed to the progress of geometry, and of the other mathematical sciences, through his study of these subjects, and through the mathematical matter introduced in his writings. Amongst his contemporaries were Leodamas of Thasos, Archytas of Tarentum, and Theaetetus of Athens, by all of whom theorems were added or placed on a more scientific basis. To Leodamas succeeded Neocleides, and his pupil was Leon, who added much to what had been

done before. Leon also composed elements, which, both in regard to the number and the value of the propositions proved, are put together more carefully; he also invented that part of the solution of a problem called its determination (*διορισμός*)—a test for determining when the problem is possible and when impossible. Eudoxus of Cnidus, a little younger than Leon and a companion of Plato's pupils, in the first place increased the number of general theorems, added three proportions to the three already existing, and also developed further the things begun by Plato concerning the section,¹ making use, for the purpose, of the analytical method (*ταῖς ἀναλύσειν*). Amyclas of Heraclea, one of Plato's companions, and Menaechmus, a pupil of Eudoxus and also an associate of Plato, and his brother, Deinostratus, made the whole of geometry more perfect. Theudius of Magnesia appears to have been distinguished in mathematics, as well as in other branches of philosophy, for he made an excellent arrangement of the elements, and generalized many particular propositions. Athenaeus of Cyzicus [or Cyzicinus of Athens] about the same time became famous in other mathematical studies, but especially in geometry. All these frequented the Academy, and made their researches in common. Hermotimus of Colophon developed further what had been done by Eudoxus and Theaetetus, discovered many elementary theorems, and wrote something on loci. Philipus Mendaus [Medmaeus], a pupil of Plato, and drawn by him to mathematical studies, made researches under Plato's direction, and occupied himself with whatever he thought

¹ Does this mean the cutting of a straight line in extreme and mean ratio, "*sectio aurea*"? or is the reference to the invention of the conic sections? Most probably the former. In *Euclid's Elements*, Lib., xiii., the terms *analysis*

and *synthesis* are first used and defined by him in connection with theorems relating to the cutting of a line in extreme and mean ratio. See Bretschneider, *Die Geometrie vor Euklides*, p. 168.

would advance the Platonic philosophy. Thus far those who have written on the history of geometry bring the development of the science.³

Proclus goes on to say, Euclid was not much younger than these; he collected the elements, arranged much of what Eudoxus had discovered, and completed much that had been commenced by Theaetetus; further, he substituted incontrovertible proofs for the lax demonstrations of his predecessors. He lived in the times of the first Ptolemy, by whom, it is said, he was asked whether there was a shorter way to the knowledge of geometry than by his Elements, to which he replied that there was no *royal road* to geometry. Euclid then was younger than the disciples of Plato, but elder than Eratosthenes and Archimedes—who were contemporaries—the latter of whom mentions him. He was of the Platonic sect, and familiar with its philosophy, whence also he proposed to himself the construction of the so-called Platonic bodies [the regular solids] as the final aim of his systematization of the Elements.⁴

I.

The first name, then, which meets us in the history of Greek mathematics is that of Thales of Miletus (640–546 B. C.). He lived at the time when his native city, and Ionia in general, were in a flourishing condition, and when an active trade was carried on with Egypt. Thales himself was engaged in trade, and is said to have resided in Egypt, and, on his return to Miletus in his old age, to have brought with him from that country the knowledge of geometry and

³ From these words we infer that the *History of Geometry* by Eudemus is most probably referred to, inasmuch as he lived at the time here indicated, and his history is elsewhere mentioned by Proclus.—Proclus, ed. G. Friedlein,

pp. 299, 333, 352, and 379.

⁴ Procli Diadochi *in primum Euclidis elementorum librum commentarii*. Ex recognitione G. Friedlein. Lipsiae, 1873, pp. 64–68.

astronomy. To the knowledge thus introduced he added the capital creation of the geometry of lines, which was essentially abstract in its character. The only geometry known to the Egyptian priests was that of surfaces, together with a sketch of that of solids, a geometry consisting of some simple quadratures and elementary cubatures, which they had obtained empirically; Thales, on the other hand, introduced *abstract* geometry, the object of which is to establish precise *relations* between the different parts of a figure, so that some of them could be found by means of others in a manner strictly rigorous. This was a phenomenon quite new in the world, and due, in fact, to the abstract spirit of the Greeks. In connection with the new impulse given to geometry, there arose with Thales, moreover, scientific astronomy, also an abstract science, and undoubtedly a Greek creation. The astronomy of the Greeks differs from that of the Orientals in this respect, that the astronomy of the latter, which is altogether concrete and empirical, consisted merely in determining the duration of some periods, or in indicating, by means of a mechanical process, the motions of the sun and planets, whilst the astronomy of the Greeks aimed at the discovery of the geometric laws of the motions of the heavenly bodies.*

* The importance, for the present research, of bearing in mind this abstract character of Greek science consists in this, that it furnishes a clue by means of which we can, in many cases, recognise theorems of purely Greek growth, and distinguish them from those of eastern extraction. The neglect of this consideration has led some recent writers on the early history of geometry greatly to exaggerate the obligations of the Greeks to the Orientals; whilst others have attributed to

the Greeks the discovery of truths which were known to the Egyptians. See, in relation to the distinction between abstract and concrete science, and its bearing on the history of Greek Mathematics, amongst many passages in the works of Auguste Comte, *Système de Politique Positive*, vol. III., ch. iv., p. 297, and *seq.*, vol. I., ch. i., pp. 424-437; and see, also, *Les Grands Types de l'Humanité*, par P. Laffitte, vol. II., Leçon 151ème, p. 280, and *seq.*—*Appréciation de la Science Antique*.

The following notices of the geometrical work of Thales have been preserved :—

(a). He is reported to have first demonstrated that the circle was bisected by its diameter ;⁶

(b). He is said first to have stated the theorem that the angles at the base of every isosceles triangle are equal, “or, as in archaic fashion he phrased it, *like* (ὁμοῖαι) ;”⁷

(c). Eudemus attributes to him the theorem that when two straight lines cut each other, the vertically opposite angles are equal ;⁸

(d). Pamphila⁹ relates that he, having learned geometry from the Egyptians, was the first person to describe a right-angled triangle in a circle ; others, however, of whom Apollodorus (ὁ λογιστικός) is one, say the same of Pythagoras ;¹⁰

(e). He never had any teacher except during the time when he went to Egypt and associated with the priests. Hieronymus also says that he measured the pyramids, making an observation on our shadows when they are of the same length as ourselves, and applying it to the pyramids.¹¹ To the same effect Pliny—“Mensuram altitudinis earum omniumque similium deprehendere invenit Thales Milesius, umbram metiendo, qua hora par esse corpori solet ;”¹²

(This is told in a different manner by Plutarch. Niloxenus is introduced as conversing with Thales concerning Amasis, King of Egypt.—“Although he [Amasis] admired you [Thales] for other things, yet he particularly liked the

⁶ Proclus, ed. Friedlein, p. 157.

⁷ *Ibid.*, p. 250.

⁸ *Ibid.*, p. 299.

⁹ Pamphila was a female historian who lived at the time of Nero ; an Epidaurian according to Suidas, an Egyptian according to Photius.

¹⁰ Diogenes Laertius, I., c. 1, n. 3,

ed. C. G. Cobet, p. 6.

¹¹ ὁ δὲ Ἱερώνυμος καὶ ἐκμετρήσαι φησὶ αὐτὸν τὰς πυραμίδας ἐκ τῆς σκιᾶς παρατηρήσαντα ὅτε ἡμῶν ἰσομεγέθεις εἰσὶ. Diog. Laert., I., c. 1, n. 6., ed. Cobet, p. 6.

¹² Plin. *Hist. Nat.*, xxxvi. 17.

manner by which you measured the height of the pyramid without any trouble or instrument ; for, by merely placing a staff at the extremity of the shadow which the pyramid casts, you formed two triangles by the contact of the sun-beams, and showed that the height of the pyramid was to the length of the staff in the same ratio as their respective shadows").¹³

(f). Proclus tells us that Thales measured the distance of vessels from the shore by a geometrical process, and that Eudemus, in his history of geometry, refers the theorem *Eucl.* i. 26 to Thales, for he says that it is necessary to use this theorem in determining the distance of ships at sea according to the method employed by Thales in this investigation ;¹⁴

(g). Proclus, or rather Eudemus, tells us in the passage quoted above *in extenso* that Thales brought the knowledge of geometry to Greece, and added many things, attempting some in a more abstract manner, and some in a more intuitional or sensible manner.¹⁵

Let us now examine what inferences as to the geometrical knowledge of Thales can be drawn from the preceding notices.

First inference.—Thales must have known the theorem that the sum of the three angles of a triangle is equal to two right angles.

Pamphila, in (d), refers to the discovery of the property of a circle that all triangles described on a diameter as base with their vertices on the circumference have their vertical angles right.¹⁶

¹³ *Plat. Sept. Sap. Conviv.* 2. vol. iii., p. 174, ed. Didot.

¹⁴ Proclus, ed. Friedlein, p. 352.

¹⁵ *Ibid.*, p. 65.

¹⁶ This is unquestionably the discovery referred to. The manner in

which it has been stated by Diogenes Laertius shows that he did not distinguish between a problem and a theorem ; and further, that he was ignorant of geometry. To this effect Proclus—
“When, therefore, anyone proposes to

Assuming, then, that this theorem was known to Thales, he must have known that the sum of the three angles of any right-angled triangle is equal to two right angles, for, if the vertex of any of these right-angled triangles be connected with the centre of the circle, the right-angled triangle will be resolved into two isosceles triangles, and since the angles at the base of an isosceles triangle are equal—a theorem attributed to Thales (*b*)—it follows that the sum of the angles at the base of the right-angled triangle is equal to the vertical angle, and that therefore the sum of the three angles of the right-angled triangle is equal to two right angles. Further, since any triangle can be resolved into two right-angled triangles, it follows immediately that the sum of the three angles of any triangle is equal to two right angles. If, then, we accept the evidence of Pamphila as satisfactory, we are forced to the conclusion that Thales must have known this theorem. No doubt the knowledge of this theorem (*Euclid* i., 32) is required in the proof given in the elements of Euclid of the property of the circle (iii., 31), the discovery of which is attributed to Thales by Pamphila, and some writers have inferred hence that Thales must have known the theorem (i., 32).¹⁷ Although I agree with this conclusion, for the reasons given

inscribe an equilateral triangle in a circle, he proposes a problem: for it is possible to inscribe one that is not equilateral. But when anyone asserts that the angles at the base of an isosceles triangle are equal, he must affirm that he proposes a theorem: for it is not possible that the angles at the base of an isosceles triangle should be unequal to each other. On which account if anyone, stating it as a problem, should say that he wishes to inscribe a right angle in a semicircle, he must be considered as ignorant of geometry, since

every angle in a semicircle is necessarily a right one."—Taylor's Proclus, vol. I., p. 110. Procl. ed. Friedlein, pp. 79, 80.

Sir G. C. Lewis has subjected himself to the same criticism when he says—"According to Pamphila, he first solved the problem of inscribing a right-angled triangle in a circle."—G. Cornwall Lewis, *Historical Survey of the Astronomy of the Ancients*, p. 83.

¹⁷ So F. A. Finger, *De Primordiis Geometriæ apud Græcos*, p. 20, Heidelberg, 1831.

above, yet I consider the inference founded on the demonstration given by Euclid to be inadmissible, for we are informed by Proclus, on the authority of Eudemus, that the theorem (*Euclid* i., 32) was first proved in a general way by the Pythagoreans, and their proof, which does not differ substantially from that given by Euclid, has been preserved by Proclus.¹⁸ Further, Geminus states that the ancient geometers observed the equality to two right angles in each species of triangle separately, first in equilateral, then in isosceles, and lastly in scalene triangles,¹⁹ and it is plain that the geometers older than the Pythagoreans can be no other than Thales and his successors in the Ionic school.

If I may be permitted to offer a conjecture, in conformity with the notice of Geminus, as to the manner in which the theorem was arrived at in the different species of triangles, I would suggest that Thales had been led by the concrete geometry of the Egyptians to contemplate floors covered with tiles in the form of equilateral triangles or regular hexagons,²⁰ and had observed that six equilateral triangles could be placed round a common vertex, from which he saw that six such angles made up four right angles, and that consequently the sum of the three angles of an equilateral triangle is equal to two right angles (*c*). The observation of a floor covered with square tiles would lead to a similar conclusion with respect to the isosceles right-angled triangle.²¹ Further, if a perpen-

¹⁸ Proclus, ed. Friedlein, p. 379.

¹⁹ Apollonii *Conica*, ed. Hallejus, p. 9, Oxon. 1710.

²⁰ Floors or walls covered with tiles of various colours were common in Egypt. See Wilkinson's "*Ancient Egyptians*," vol. ii., pp. 287 and 292.

²¹ Although the theorem that "only three kinds of regular polygons—the equilateral triangle, the square and the hexagon—can be placed about a point

so as to fill a space," is attributed by Proclus to Pythagoras or his school (*ἵσται τὸ θεώρημα τοῦτο Πυθαγόρειον*. Proclus, ed. Friedlein, p. 305), yet it is difficult to conceive that the Egyptians—who erected the pyramids—had not a practical knowledge of the fact that tiles of the forms above mentioned could be placed so as to form a continuous plane surface.

dicular be drawn from a vertex of an equilateral triangle on the opposite side,²² the triangle is divided into two right-angled triangles, which are in every respect equal to each other, hence the sum of the three angles of each of these right-angled triangles is easily seen to be two right angles. If now we suppose that Thales was led to examine whether the property, which he had observed in two distinct kinds of right-angled triangles, held generally for all right-angled triangles, it seems to me that, by completing the rectangle and drawing the second diagonal, he could easily see that the diagonals are equal, that they bisect each other, and that the vertical angle of the right-angled triangle is equal to the sum of the base angles. Further, if he constructed several right-angled triangles on the same hypotenuse he could see that their vertices are all equally distant from the middle point of their common hypotenuse, and therefore lie on the circumference of a circle described on that line as diameter, which is the theorem in question. It may be noticed that this remarkable property of the circle, with which, in fact, abstract geometry was inaugurated, struck the imagination of Dante :—

“O se del mezzo cerchio far si puote
Triangol sì, ch’un retto non avesse.”

Par. c. xiii. 101.

Second inference.—The conception of geometrical loci is due to Thales.

We are informed by Eudemus (*f*) that Thales knew that a triangle is determined if its base and base angles are given; further, we have seen that Thales knew that,

²² Though we are informed by Proclus (ed. Friedlein, p. 283), that Oenopides of Chios first solved (*ἔζητησεν*) this problem, yet Thales, and indeed the Egyptians, who were furnished with

the square, could not be ignorant of its mechanical solution. Observe that we are expressly told by Proclus that Thales attempted some things in an intuitional or sensible manner.

if the base is given, and the base angles not given separately, but their sum known to be a right angle, then there could be described an unlimited number of triangles satisfying the conditions of the question, and that their vertices all lie on the circumference of a circle described on the base as diameter. Hence it is manifest that the important conception of *geometrical loci*, which is attributed by Montucla, and after him by Chasles and other writers on the History of Mathematics, to the School of Plato,²³ had been formed by Thales.

Third inference.—Thales discovered the theorem that the sides of equiangular triangles are proportional.

The knowledge of this theorem is distinctly attributed to Thales by Plutarch in a passage quoted above (e). On the other hand, Hieronymus of Rhodes, a pupil of Aristotle, according to the testimony of Diogenes Laertius,²⁴ says that Thales measured the height of the pyramids by watching when bodies cast shadows of their own length, and to the same effect Pliny in the passage quoted above (e). Bretschneider thinks that Plutarch has spun out the story told by Hieronymus, attributing to Thales the knowledge of his own times, denies to Thales the knowledge of the theorem in question, and says that there is no trace of any theorems concerning similarity before Pythagoras.²⁵ He says further, that the Egyptians were altogether ignorant of the doctrine of the similarity of figures, that we do not find amongst them any trace of the doctrine of proportion, and that Greek writers say that this part of their mathe-

²³ Montucla, *Histoire des Mathématiques*, Tome i., p. 183, Paris, 1758. Chasles, *Aperçu Historique des Méthodes en Géométrie*, p. 5, Bruxelles, 1837. Chasles in the history of geometry before Euclid copies Montucla, and we have a remarkable instance of this here, for Chasles, after Montucla, calls Plato

“ce chef du Lycée.”

²⁴ But we have seen that the account given by Diogenes Laertius of the discovery of Thales mentioned by Pamphila is unintelligible and evinces ignorance of geometry on his part.

²⁵ Bretsch. *Die Geometrie und Geometer vor Euklides*, pp. 45, 46.

matical knowledge was derived from the Babylonians or Chaldaeans.²⁶ Bretschneider also endeavours to show that Thales could have obtained the solution of the second practical problem—the determination of the distance of a ship from the shore—by geometrical construction, a method long before known to the Egyptians.²⁷ Now, as Bretschneider denies to the Egyptians and to Thales any knowledge of the doctrine of proportion, it was plainly necessary, on this supposition, that Thales should find a sufficient extent of free and level ground on which to construct a triangle of the same dimensions as that he wished to measure; and even if he could have found such ground, the great length of the sides would have rendered the operations very difficult.²⁸ It is much simpler to accept the testimony of Plutarch, and suppose that the method of superseding such operations by using similar triangles is due to Thales.

If Thales had employed a right-angled triangle,²⁹ he could have solved this problem by the same principle which, we are told by Plutarch, he used in measuring the height of the pyramid, the only difference being that the right-

²⁶ *Ibid.*, p. 18.

²⁷ *Ibid.*, pp. 43, 44.

²⁸ In reference to this I may quote the following passage from Clairaut, *Elémens de Géométrie*, pp. 34-35. Paris, 1741.

“La méthode qu'on vient de donner pour mesurer les terrains, dans lesquels on ne sauroit tirer de lignes, fait souvent naître de grandes difficultés dans la pratique. On trouve rarement un espace uni et libre, assez grand pour faire des triangles égaux à ceux du terrain dont on cherche la mesure. Et même quand on en trouveroit, la grande longueur des côtés des triangles pourroit rendre les opérations très-difficiles :

abaisser une perpendiculaire sur une ligne du point qui en est éloigné seulement de 500 toises, ce seroit un ouvrage extrêmement pénible, et peut-être impraticable. Il importe donc d'avoir un moyen qui supplée à ces grandes opérations. Ce moyen s'offre comme de lui-même. Il vient, &c.”

²⁹ Observe that the inventions of the square and level are attributed by Pliny (*Nat. Hist.*, vii., 57) to Theodorus of Samos, who was a contemporary of Thales. They were, however, known long before this period to the Egyptians; so that to Theodorus is due at most the honour of having introduced them into Greece.

angled triangle is in one case in a vertical, and in the other in a horizontal plane.

From what has been said it is plain that there is a natural connection between the several theorems attributed to Thales, and that the two practical applications which he made of his geometrical knowledge are also connected with each other.

Let us now proceed to consider the importance of the work of Thales :—

I. In a scientific point of view :—

(a). We see, in the first place, that by his two theorems he founded the geometry of lines, which has ever since remained the principal part of geometry.³⁰

Vainly do some recent writers refer these geometrical discoveries of Thales to the Egyptians ; in doing so they ignore the distinction between the geometry of lines, which we owe to the genius of the Greeks, and that of areas and volumes—the only geometry known, and that empirically, to the ancient priesthoods. This view is confirmed by an ancient papyrus, that of Rhind,³¹ which is now in the British Museum. It contains a complete applied mathematics, in which the measurement of figures and solids plays the principal part ; there are no theorems properly so called ; everything is stated in the form of problems, not in general terms but in distinct numbers, *e. g.*—to measure a rectangle the sides of which contain two and ten units of length ; to find the surface of a circular area whose diameter is six units ; to mark out in a field a right-angled triangle

³⁰ Auguste Comte, *Système de Politique Positive*, vol. iii., p. 297.

³¹ Birch, in Lepsius' *Zeitschrift für Aegyptische Sprache und Alterthumskunde* (year 1868, p. 108). Bretschneider, *Geometrie vor Euklides*, p. 16. F. Hofer, *Histoire des Ma-*

thematiques, p. 69. Since this Paper was sent to the press, Dr. August Eisenlohr, of Heidelberg, has published this papyrus with a translation and commentary under the title "*Ein Mathematisches Handbuch der alten Ägypter.*"

whose sides measure ten and four units; to describe a trapezium whose parallel sides are six and four units, and each of the other sides twenty units. We find also in it indications for the measurement of solids, particularly of pyramids, whole and truncated.

It appears from the above that the Egyptians had made great progress in practical geometry. Of their proficiency and skill in geometrical constructions we have also the direct testimony of the ancients; for example, Democritus says: "No one has ever excelled me in the construction of lines according to certain indications—not even the so-called Egyptian Harpedonaptae."³³

(*b*). Thales may, in the second place, be fairly considered to have laid the foundation of Algebra, for his first theorem establishes an equation in the true sense of the word, while the second institutes a proportion.³³

II. In a philosophic point of view:—

We see that in these two theorems of Thales the first type of a *natural law*—*i. e.*, the expression of a fixed dependence between different quantities, or, in another form, the disentanglement of constancy in the midst of variety—has decisively arisen.³⁴

III. Lastly, in a practical point of view:—

Thales furnished the first example of an application of theoretical geometry to practice,³⁵ and laid the foundation of an important branch of the same—the measurement of heights and distances.

I have now pointed out the importance of the geometrical discoveries of Thales, and attempted to appreciate his work. His successors of the Ionic School followed

³³ Mullach, *Fragmenta Philosophorum Graecorum*, p. 371, Democritus ap. Clem. Alex. *Strom.* I. p. 357, ed. Potter.

Pos. vol. iii., p. 300).

³⁴ P. Laffitte, *Les Grands Types de l'Humanité*, vol. ii., p. 292.

³⁵ *Ibid.*, p. 294.

³⁶ Auguste Comte (*Système de Pol.*

him in other lines of thought, and were, for the most part, occupied with physical theories on the nature of the universe—speculations which have their representatives at the present time—and added little or nothing to the development of science, except in astronomy. The further progress of geometry was certainly not due to them.

Without doubt Anaxagoras of Clazomenae, one of the latest representatives of this School, is said to have been occupied during his exile with the problem of the quadrature of the circle, but this was in his old age, and after the works of another School—to which the early progress of geometry was really due—had become the common property of the Hellenic race. I refer to the immortal School of Pythagoras.

II.

About the middle of the sixth century before the Christian era, a great change had taken place: Ionia, no longer free and prosperous, had fallen under the yoke, first of Lydia, then of Persia, and the very name Ionian—the name by which the Greeks were known in the whole East—had become a reproach, and was shunned by their kinsmen on the other side of the Aegean.³⁶ On the other hand, Athens and Sparta had not become pre-eminent; the days of Marathon and Salamis were yet to come. Meanwhile the glory of the Hellenic name was maintained chiefly by the Italic Greeks, who were then in the height of their prosperity, and had recently obtained for their territory the well-earned appellation of *ἡ μεγάλη Ἑλλάς*.³⁷ It should be noted, too, that at this period there was great commercial intercourse between the Hellenic cities of Italy and Asia; and further, that some of them, as Sybaris and Miletus on the one hand, and Tarentum and Cnidus on the other, were

³⁶ Herodotus, i. 143.

i., p. 141, 1844.

³⁷ Polybius, ii., 39; ed. Bekker, vol.

bound by ties of the most intimate character.³⁸ It is not surprising, then, that after the Persian conquest of Ionia, Pythagoras, Xenophanes, and others, left their native country, and, following the current of civilization, removed to Magna Graecia.

As the introduction of geometry into Greece is by common consent attributed to Thales, so all³⁹ are agreed that to Pythagoras of Samos, the second of the great philosophers of Greece, and founder of the Italic School, is due the honour of having raised mathematics to the rank of a science.

The statements of ancient writers concerning this great man are most conflicting, and all that relates to him personally is involved in obscurity; for example, the dates given for his birth vary within the limits of eighty-four years—43rd to 64th Olympiad.⁴⁰ It seems desirable, however, if for no other reason than to fix our ideas, that we should adopt some definite date for the birth of Pythagoras; and there is an additional reason for doing so, inasmuch as some writers, by neglecting this, have become confused, and fallen into inconsistencies in the notices which they have given of his life. Of the various dates which have been assigned for the birth of Pythagoras, the one which seems to me to harmonise best with the records of the most trustworthy writers is that given by Ritter, and adopted by Grote, Brandis, Ueberweg, and Hankel, namely, about 580 B. C. (49th Olymp.) This date would accord with the following statements:—

That Pythagoras had personal relations with Thales, then old, of whom he was regarded by all antiquity as the

³⁸ Herod., vi. 21, and iii. 138.

³⁹ Aristotle, Diogenes Laertius, Proclus, amongst others.

⁴⁰ See G. H. Lewes, *Biographical*

History of Philosophy, Book ii., c. ii., where the various dates given by scholars are cited.

successor, and by whom he was incited to visit Egypt,⁴¹—mother of all the civilization of the West ;

That he left his country being still a young man, and, on this supposition as to the date of his birth, in the early years of the reign of Croesus (560–546 B. C.), when Ionia was still free ;

That he resided in Egypt many years, so that he learned the Egyptian language, and became imbued with the philosophy of the priests of the country ;⁴²

That he probably visited Crete and Tyre, and may have even extended his journeys to Babylon, at that time Chaldaean and free ;

That on his return to Samos, finding his country under the tyranny of Polycrates,⁴³ and Ionia under the dominion of the Persians, he migrated to Italy in the early years of Tarquinius Superbus ;⁴⁴

And that he founded his Brotherhood at Crotona, where for the space of twenty years or more he lived and taught, being held in the highest estimation, and even looked on almost as divine by the population—native as well as Hellenic ; and then, soon after the destruction of Sybaris (510 B. C.), being banished by a democratic party under Cylon, he removed to Metapontum, where he died soon afterwards.

All who have treated of Pythagoras and the Pythagoreans have experienced great difficulties. These difficulties are due partly to the circumstance that the reports of the earlier and most reliable authorities have for the most part been lost, while those which have come down to us are not always consistent with each other. On the other hand, we have pretty full accounts from later writers, especially those

⁴¹ Iamblichus, *de Vita Pyth.*, c. ii., 12.

ap. Porphyry, *de Vita Pyth.*, 9.

⁴² Isocrates is the oldest authority for this, *Busiris*, c. 11.

⁴³ Cicero, *de Rep.* II., 15 ; *Tusc. Disp.*, I., xvi., 38.

⁴⁴ Diog. Laert., viii. 3 ; Aristoxenus,

of the Neo-Pythagorean School; but these notices, which are mixed up with fables, were written with a particular object in view, and are in general highly coloured; they are particularly to be suspected, as Zeller has remarked, because the notices are fuller and more circumstantial the greater the interval from Pythagoras. Some recent authors, therefore, even go to the length of omitting from their account of the Pythagoreans everything which depends solely on the evidence of the Neo-Pythagoreans. In doing so, these authors no doubt effect a simplification, but it seems to me that they are not justified in this proceeding, as the Neo-Pythagoreans had access to ancient and reliable authorities which have unfortunately been lost since.⁴³

Though the difficulties to which I refer have been felt chiefly by those who have treated of the Pythagorean *philosophy*, yet we cannot, in the present inquiry, altogether escape from them; for, in the first place, there was, in the whole period of which we treat, an intimate connection between the growth of philosophy and that of science, each re-acting on the other; and, further, this was particularly the case in the School of Pythagoras, owing to the fact, that whilst on the one hand he united the study of geometry with that of arithmetic, on the other he made numbers the base of his philosophical system, as well physical as metaphysical.

It is to be observed, too, that the early Pythagoreans published nothing, and that, moreover, with a noble self-denial, they referred back to their master all their discoveries. Hence, it is not possible to separate what was done by him from what was done by his early disciples, and we

⁴³ For example, the *History of Geometry*, by Eudemus of Rhodes, one of the principal pupils of Aristotle, is quoted by Theon of Smyrna, Proclus, Simplicius, and Eutocius, the last two

of whom lived in the reign of Justinian. Eudemus also wrote a *History of Astronomy*. Theophrastus, too, Aristotle's successor, wrote *Histories of Arithmetic, Geometry, and Astronomy*.

are under the necessity, therefore, of treating the work of the early Pythagorean School as a whole.⁴⁶

All agree, as was stated above, that Pythagoras first raised mathematics to the rank of a science, and that we owe to him two new branches—arithmetic and music.

We have the following statements on the subject:—

(a). In the age of these philosophers [the Eleats and Atomists], and even before them, lived those called Pythagoreans, who first applied themselves to mathematics, a science they improved: and, penetrated with it, they fancied that the principles of mathematics were the principles of all things; ⁴⁷

(b.) Eudemus informs us, in the passage quoted above *in extenso*, that Pythagoras changed geometry into the form of a liberal science, regarding its principles in a purely abstract manner, and investigated his theorems from the immaterial and intellectual point of view; and that he also discovered the theory of irrational qualities, and the construction of the mundane figures [the five regular solids]; ⁴⁸

(c.) It was Pythagoras, also, who carried geometry to perfection, after Moeris ⁴⁹ had first found out the principles of the elements of that science, as Anticlides tells us in the second book of his *History of Alexander*; and the part

⁴⁶ “Pythagoras and his earliest successors do not appear to have committed any of their doctrines to writing. According to Porphyrius (*de Vita Pyth.* p. 40), Lysis and Archippus collected in a written form some of the principal Pythagorean doctrines, which were handed down as heirlooms in their families, under strict injunctions that they should not be made public. But amid the different and inconsistent accounts of the matter, the first publication of the Pythagorean doctrines is pretty uniformly attributed to Philo-

laus.”—Smith’s *Dictionary*, in *v. Philolaus*. Philolaus was born at Crotona, or Tarentum, and was a contemporary of Socrates and Democritus. See Diog. Laert. in *Vita Pythag.*, viii., i., 15; in *Vita Empedoclis*, viii., ii., 2; and in *Vita Democriti*, ix., vii., 6. See also Iamblichus, *de Vita Pythag.*, c. 18, s. 88.

⁴⁷ Aristot. *Met.*, i., 5, 985, N. 23, ed. Bekker.

⁴⁸ Procl. *Comm.*, ed. Friedlein, p. 65.

⁴⁹ An ancient King of Egypt, who reigned 900 years before Herodotus.

of the science to which Pythagoras applied himself above all others was arithmetic ;⁵⁰

(d.) Pythagoras seems to have esteemed arithmetic above everything, and to have advanced it by diverting it from the service of commerce, and likening all things to numbers ;⁵¹

(e.) He was the first person who introduced measures and weights among the Greeks, as Aristoxenus the musician informs us ;⁵²

(f.) He discovered the numerical relations of the musical scale ;⁵³

(g.) The word mathematics originated with the Pythagoreans ;⁵⁴

(h.) The Pythagoreans made a four-fold division of mathematical science, attributing one of its parts to the how many, τὸ πυσόν, and the other to the how much, τὸ πηλικόν ; and they assigned to each of these parts a two-fold division. Discrete quantity, or the how many, either subsists by itself, or must be considered with relation to some other ; and continued quantity, or the how much, is either stable or in motion. Hence arithmetic contemplates that discrete quantity which subsists by itself, but music that which is related to another ; and geometry considers continued quantity so far as it is immovable ; but astronomy (τὴν σφαιρικὴν) contemplates continued quantity so far as it is of a self-motive nature ;⁵⁵

(i.) Favorinus says that he employed definitions on

⁵⁰ Diog. Laert., viii. 11, ed. Cobet, p. 207.

⁵¹ Aristoxenus, *Fragm.* ap. Stob. *Eclog. Phys.*, I., ii., 6 ; ed. Heeren, vol. I., p. 17.

⁵² Diog. Laert., viii., 13, ed. Cobet, p. 208.

⁵³ τὸν τε κανόνα τὸν ἐκ μιᾶς χορδῆς

εἶρεῖν. Diog. Laert., viii., 11, ed. Cobet, p. 207.

⁵⁴ Procli *Comm.*, Friedlein, p. 45.

⁵⁵ Procli *Comm.*, ed. Friedlein, p. 35. As to the distinction between τὸ πηλικόν, continuous, and τὸ πυσόν, discrete, quantity, see Iambl., in *Nic. G. Arithm. introd.* ed. Ten., p. 148.

account of the mathematical subjects to which he applied himself (ὅροις χρῆσασθαι διὰ τῆς μαθηματικῆς ὕλης).⁵⁶

As to the particular work done by this school in geometry, the following statements have been handed down to us :—

(a.) The Pythagoreans define a point as unity having position (μονάδα προσλαβοῦσαν θέσιν);⁵⁷

(b.) They considered a point as analogous to the monad, a line to the duad, a superficies to the triad, and a body to the tetrad;⁵⁸

(c.) The plane around a point is completely filled by six equilateral triangles, four squares, or three regular hexagons: this is a Pythagorean theorem;⁵⁹

(d.) The peripatetic Eudemus ascribes to the Pythagoreans the discovery of the theorem that the interior angles of a triangle are equal to two right angles (*Eucl.* i. 32), and states their method of proving it, which was substantially the same as that of Euclid;⁶⁰

(e.) Proclus informs us in his commentary on Euclid, i., 44, that Eudemus says that the problems concerning the application of areas—in which the term application is not to be taken in its restricted sense (παραβολή) in which it is used in this proposition, but also in its wider signification, embracing ὑπερβολή and ἔλλειψις, in which it is used in the 28th and 29th propositions of the Sixth Book,—are old, and inventions of the Pythagoreans;⁶¹

⁵⁶ Diog. Laert., viii., 25, ed. Cobet, p. 215.

⁵⁷ Procli *Comm.* ed. Friedlein, p. 95.

⁵⁸ *Ibid.*, p. 97.

⁵⁹ *Ibid.*, p. 305.

⁶⁰ *Ibid.*, p. 379.

⁶¹ *Ibid.*, p. 419. The words of Proclus are interesting :—

“According to Eudemus, the inventions respecting the *application*, *excess*,

and *defect* of areas are ancient, and are due to the Pythagoreans. Moderns borrowing these names transferred them to the so-called conic lines—the parabola, the hyperbola, the ellipse; as the older school in their nomenclature concerning the description of areas *in plano* on a finite right line regarded the terms thus :—

“An area is said to be *applied* (παρα-

(f.) This is to some extent confirmed by Plutarch, who says that Pythagoras sacrificed an ox on account of the geometrical diagram, as Apollodotus [-rus] says :—

Ἦνίκα Πυθαγόρης τὸ περικλεῆς εὔρετο γράμμα,
Κεῖν' ἐφ' ὅτῳ λαμπρὴν ἤγετο βουθυσίην,

either the one relating to the hypotenuse—namely, that the square on it is equal to the sum of the squares on the sides—or that relating to the problem concerning the application of areas (εἴτε πρόβλημα περὶ τοῦ χωρίου τῆς παραβολῆς) ;⁶²

(g.) One of the most elegant (γεωμετρικωτάτοις) theorems, or rather problems, is to construct a figure equal to one and similar to another given figure, for the solution of which also they say that Pythagoras offered a sacrifice : and indeed it is finer and more elegant than the theorem which shows that the square on the hypotenuse is equal to the sum of the squares on the sides ;⁶³

(h.) Eudemus, in the passage already quoted from Proclus, says Pythagoras discovered the construction of the regular solids ;⁶⁴

βάλλειν) to a given right line when an area equal in content to some given one is described thereon ; but when the base of the area is greater than the given line, then the area is said to be in *excess* (ὑπερβάλλειν) ; but when the base is less, so that some part of the given line lies without the described area, then the area is said to be in *defect* (ἐλλεῖν). Euclid uses in this way, in his Sixth Book, the terms *excess* and *defect*. . . . The term *application* (παραβάλλειν), which we owe to the Pythagoreans, has this signification."

⁶² Plutarch, *non posse suaviter vivi sec. Epicurum*, c. xi. ; Plut., *Opera*, ed. Didot, vol. iv., p. 1338. Some authors,

rendering περὶ τοῦ χωρίου τῆς παραβολῆς "concerning the area of the parabola," have ascribed to Pythagoras the quadrature of the parabola—which was in fact one of the great discoveries of Archimedes ; and this, though Archimedes himself tells us that no one before him had considered the question ; and though further he gives in his letter to Dositheus the history of his discovery, which, as is well known, was first obtained from mechanical considerations, and then by geometrical reasonings.

⁶³ Plutarch, *Symp.*, viii., *Quaestio* 2, c. 4. Plut. *Opera*, ed. Didot, vol. iv., p. 877.

⁶⁴ Procl. *Comm.*, ed. Friedlein, p. 65.

(z.) But particularly as to Hippasus, who was a Pythagorean, they say that he perished in the sea on account of his impiety, inasmuch as he boasted that he first divulged the knowledge of the sphere with the twelve pentagons [the ordinate dodecahedron inscribed in the sphere]: Hippasus assumed the glory of the discovery to himself, whereas everything belonged to Him—for thus they designate Pythagoras, and do not call him by name;⁶⁵

(j.) The triple interwoven triangle or Pentagram—star-shaped regular pentagon—was used as a symbol or sign of recognition by the Pythagoreans, and was called by them Health (*ὑγίεια*);⁶⁶

(k.) The discovery of the law of the three squares (*Eucl.* I., 47), commonly called the *Theorem of Pythagoras*, is attributed to him by—amongst others—Vitruvius,⁶⁷ Diogenes Laertius,⁶⁸ Proclus,⁶⁹ and Plutarch (*f*). Plutarch, however, attributes to the Egyptians the knowledge of this theorem in the particular case where the sides are 3, 4, and 5;⁷⁰

(l.) One of the methods of finding right-angled triangles whose sides can be expressed in numbers—that

⁶⁵ Iambl., *de Vit. Pyth.*, c. 18, s. 88.

⁶⁶ Scholiast on Aristophanes, *Nub.* 611; also Lucian, *pro Lapsu in Salut.*, s. 5. That the Pythagoreans used such symbols we learn from Iamblichus (*de Vit. Pyth.*, c. 33, ss. 237 and 238). This figure is referred to Pythagoras himself, and in the middle ages was called *Pythagorae figura*. It is said to have obtained its special name from his having written the letters ν , γ , ι , θ ($= \epsilon\iota$), α , at its prominent vertices. We learn from Kepler (*Opera Omnia*, ed. Frisch, vol. v., p. 122) that even so late as Pa-

racelsus it was regarded by him as the symbol of health. See Chasles, *Histoire de Géométrie*, pp. 477 et seqq.

⁶⁷ *De Arch.*, ix., Praef. 5, 6, and 7.

⁶⁸ Where the same couplet from Apollodorus as that in (*f*) is found, except that *κλειρην ἡγάγε* occurs in place of *λαμπρην ἡγετο*. Diog. Laert., viii., 11, p. 207, ed. Cobet.

⁶⁹ Procli *Comm.*, p. 426, ed. Friedlein.

⁷⁰ *De Is. et Osir.*, c. 56. Plut. *Op.*, vol. iii., p. 457, Didot.

setting out from the odd numbers—is attributed to Pythagoras;⁷¹

(*m.*) The discovery of irrational quantities is ascribed to Pythagoras by Eudemus in the passage quoted above from Proclus;⁷²

(*n.*) The three proportions—arithmetical, geometrical, and harmonical, were known to Pythagoras;⁷³

(*o.*) Formerly, in the time of Pythagoras and the mathematicians under him, there were three means only—the arithmetical, the geometrical, and the third in order which was known by the name *ὑπεραντία*, but which Archytas and Hippasus designated the harmonical, since it appeared to include the ratios concerning harmony and melody (*μετακληθεῖσα οὗτοι κατὰ τὸ ἀρμολογούμενον καὶ ἐμμελὲς ἐφαίνετο λόγους περιέχουσα*);⁷⁴

(*p.*) With reference to the means corresponding to these proportions, Iamblichus says:⁷⁵—We must now speak of the most perfect proportion, consisting of four terms, and properly called the musical, for it clearly contains the musical ratios of harmonical symphonies. It is said to be an invention of the Babylonians, and to have been brought first into Greece by Pythagoras;⁷⁶

⁷¹ Procli *Comm.*, ed. Friedlein, p. 428; Heronis Alex., *Geom. et Ster. Rel.*, ed. F. Hultsch, pp. 56, 146.

⁷² Procli *Comm.*, ed. Friedlein, p. 65.

⁷³ Nicom. G. *Introd. Ar.* c. xxii., ed. R. Hoche, p. 122.

⁷⁴ Iamblichus in *Nicomachi Arithmetica* a S. Tennulio, p. 141.

⁷⁵ *Ibid.*, p. 168.

⁷⁶ *Ibid.*, p. 168. As an example of this proportion, Nicomachus gives the numbers 6, 8, 9, 12, the harmonical and arithmetical means between two numbers forming a geometrical proportion

with the numbers themselves. (Nicom. *Instit. Arithm.* ed. Ast. p. 153, and *Animad.*, p. 329; see, also, Iambl., in *Nicom. Arithm.* ed. Ten., pp. 172 et seq.)

Hankel, commenting on this passage of Iamblichus, says: "What we are to do with the report, that this proportion was known to the Babylonians, and only brought into Greece by Pythagoras, must be left to the judgment of the reader."—*Geschichte der Mathematik*, p. 105. In another part of his book, however, after refer-

(g.) The doctrine of arithmetical progressions is attributed to Pythagoras;⁷⁷

(r.) It would appear that he had considered the special case of *triangular* numbers. Thus Lucian:—ΠΥΘ. Εἰς' ἐπὶ τραπεζίῃσιν ἀριθμεῖν. ΑΓ. Οἶδα καὶ νῦν ἀριθμεῖν. ΠΥΘ. Πῶς ἀριθμεῖς; ΑΓ. Ἐν, δύο, τρία, τέτταρα. ΠΥΘ. Ὅρας; ἃ σὺ δοκέεις τέτταρα, ταῦτα δέκα ἐστὶ καὶ τρίγωνον ἐντελὲς καὶ ἡμέτερον δοκίον.⁷⁸

(s.) Another of his doctrines was, that of all solid figures the sphere was the most beautiful; and of all plane figures, the circle.⁷⁹

(t.) Also Iamblichus, in his commentary on the Categories of Aristotle, says that Aristotle may perhaps not have squared the circle; but that the Pythagoreans had done so, as is evident, he adds, from the demonstrations of the Pythagorean Sextos who had got by tradition the manner of proof.⁸⁰

On examining the purely geometrical work of Pythagoras and his early disciples, we observe that it is much concerned with the geometry of areas, and we are indeed struck with its Egyptian character. This appears in the theorem (c) concerning the filling up a plane by regular polygons, as already noted; in the construction of the regular solids (h)—for some of them are found in the Egyptian architecture; in the problems concerning the application of areas (e); and lastly, in the law of the three

ring to two authentic documents of the Babylonians which have come down to us, he says: "We cannot, therefore, doubt that the Babylonians occupied themselves with such progressions [arithmetical and geometrical]; and a Greek notice that they knew proportions, nay, even invented the so-called perfect or musical proportion, gains thereby in value."—*Ibid.*, p. 67.

⁷⁷ *Theologumena Arithmetica*, p. 153, ed. F. Ast, Lipsiæ, 1817.

⁷⁸ Lucian, *Βίων πρᾶσις*, 4, vol. i., p. 317, ed. C. Jacobitz.

⁷⁹ Καὶ τῶν σχημάτων τὸ κάλλιστον σφαῖραν εἶναι τῶν στερεῶν κύκλον, Diog. Laert., *in Vita Pyth.*, viii., 19.

⁸⁰ Simplicius, *Comment.*, &c., ap. Bretsch., *Die Geometrie vor Euklides*, p. 108.

squares (k), coupled with the rule given by Pythagoras for the construction of right-angled triangles in numbers (l).

According to Plutarch, the Egyptians knew that a triangle whose sides consist of 3, 4, and 5 parts, must be right-angled. "The Egyptians may perhaps have imagined the nature of the universe like the most beautiful triangle, as also Plato appears to have made use of it in his work on the State, where he sketches the picture of matrimony. That triangle contains one of the perpendiculars of 3, the base of 4, and the hypotenuse of 5 parts, the square of which is equal to those of the containing sides. The perpendicular may be regarded as the male, the base as the female, the hypotenuse as the offspring of both, and thus Osiris as the originating principle ($\acute{\alpha}\rho\chi\eta$), Isis as the receptive principle ($\acute{\upsilon}\pi\omicron\delta\omicron\chi\eta$), and Horus as the product ($\acute{\alpha}\pi\omicron\tau\acute{\epsilon}\lambda\epsilon\sigma\mu\alpha$)."⁸⁰

This passage is remarkable, and seems to indicate the way in which the knowledge of the useful geometrical fact enunciated in it may have been arrived at by the Egyptians. The contemplation of a draught-board, or of a floor covered with square tiles, or of a wall ruled with squares,⁸¹ would at once show that the square constructed on the diagonal of a square is equal to the sum of the squares constructed on the sides—each containing four of the right-angled isosceles triangles into which one of the squares is divided by its diagonal.

Although this observation would not serve them for practical uses, on account of the impossibility of presenting it arithmetically, yet it must have shown the possibility of

⁸⁰ Plutarch, *De Is. et Osir.* c. 56, vol. iii., p. 457, ed. Didot.

⁸¹ It was the custom of the Egyptians, where a subject was to be drawn, to rule the walls of the building accu-

rately with squares before the figures were introduced. See Wilkinson's *Ancient Egyptians*, vol. ii., pp. 265, 267.

constructing a square which would be the sum of two squares, and encouraged them to attempt the solution of the problem numerically. Now, the Egyptians, with whom speculations concerning generation were in vogue, could scarcely fail to have perceived, from the observation of a chequered board, that the element in the successive formation of squares is the gnomon (γνώμων),⁸² or common carpenter's square, which was known to them.⁸³ It remained then for them only to examine whether some particular gnomon might not be metamorphosed into a square, and, therefore, *vice versâ*. The solution would then be easy, being furnished at once from the contemplation of a floor or board composed of squares.

Each gnomon consists of an odd number of squares, and the successive gnomons correspond to the successive

⁸² Γνώμων means that by which anything is known, or criterion; its oldest concrete signification seems to be the carpenter's square (*norma*), by which a right angle is known. Hence, it came to denote a perpendicular, of which, indeed, it was the archaic name, as we learn from Proclus on Euclid, i., 12:—Τοῦτο τὸ πρόβλημα πρῶτον Οἰνοπίδης ἐξήγησεν χρήσιμον αὐτὸ πρὸς ἀστρολογίαν οἰόμενος· ὀνομάζει δὲ τὴν κἀθεν ἀρχαϊκῶς κατὰ γνόμενα, διότι καὶ ὁ γνώμων πρὸς ὁρθὰς ἐστὶ τῇ ὁρίζοντι (Procli Comm., ed. Friedlein, p. 283). Gnomon is also an instrument for measuring altitudes, by means of which the meridian can be found; it denotes, further, the index or style of a sundial, the shadow of which points out the hours.

In geometry it means the square or rectangle about the diagonal of a square or rectangle, together with the two

complements, on account of the resemblance of the figure to a carpenter's square; and then, more generally, the similar figure with regard to any parallelogram, as defined by Euclid, ii., Def. 2. Again, in a still more general signification, it means the figure which, being added to any figure, preserves the original form. See Hero, *Definitiones* (59).

When gnomons are added successively in this manner to a square monad, the first gnomon may be regarded as that consisting of three square monads, and is indeed the constituent of a simple Greek fret; the second, of five square monads, &c.; hence we have the *gnomonic* numbers, which were also looked on as male, or generating.

⁸³ Wilkinson's *Ancient Egyptians*, vol. ii., p. 111.

odd numbers,⁸⁴ and include, therefore, all odd squares. Suppose, now, two squares are given, one consisting of 16 and the other of 9 unit squares, and that it is proposed to form another square out of them. It is plain that the square consisting of 9 unit squares can take the form of the fourth gnomon, which, being placed round the former square, will generate a new square containing 25 unit squares. Similarly, it may have been observed that the 12th gnomon, consisting of 25 unit squares, could be transformed into a square, each of whose sides contain 5 units, and thus it may have been seen conversely that the latter square, by taking the gnomonic, or generating, form with respect to the square on 12 units as base, would produce the square of 13 units, and so on.

This, then, is my attempt to interpret what Plutarch has told us concerning Isis, Osiris, and Horus, bearing in mind that the odd, or gnomonic, numbers were regarded by Pythagoras as male, or generating.⁸⁵

⁸⁴ It may be observed here that we first count with *counters*, as is indicated by the Greek *ψηφίστew* and the Latin *calculare*. The counters might be equal squares, as well as any other like objects. There is an indication that the odd numbers were first regarded in this manner in the name *gnomonic* numbers, which the Pythagoreans applied to them, and that term was used in the same signification by Aristotle, and by subsequent writers, even up to Kepler. See Arist. *Phys.*, lib. iii., ed. Bekker, vol. i. p. 203; Stob., *Eclog.*, ab Heeren, vol. i., p. 24, and note; Kepleri *Opera Omnia*, ed. Ch. Frisch, vol. viii., *Mathematica*, pp. 164 et seq.

⁸⁵ This seems to me to throw light on some of the oppositions which are found

in the table of principles attributed by Aristotle to certain Pythagoreans (*Metaph.*, i., 5, 986 a, ed. Bekker).

The odd—or *gnomonic*—numbers are finite; the even, infinite. Odd numbers were regarded also as male, or generating. Further, by the addition of successive gnomons—consisting, as we have seen, each of an odd number of units—to the original unit square or monad, the square form is preserved. On the other hand, if we start from the simplest oblong (*ἑτερομήκης*), consisting of two unit squares, or monads, in juxtaposition, and place about it, after the manner of a gnomon—and gnomon, as we have seen, was used in this more extended sense also at a later period—4 unit squares, and

It is another matter to see that the triangle formed by 3, 4, and 5 units is right-angled, and this I think the

then in succession in like manner 6, 8, . . . unit squares, the oblong form *ἑτερομήκες* will be preserved. The elements, then, which generate a square are odd, while those of which the oblong is made up are even. The limited, the odd, the male, and the square, occur on one side of the table: while the unlimited, the even, the female, and the oblong, are met with on the other side.

The correctness of this view is confirmed by the following passage preserved by Stobaeus:—"Ἐπὶ δὲ τῇ μονάδι τῶν ἐφεξῆς περισσῶν γνωμόνων περιτιθεμένων, ὁ γινόμενος ἔστι τετράγωνός ἐστι. τῶν δὲ ἀρτίων ὁμοίως περιτιθέμενων, ἑτερομήκης καὶ ἄριστοι πάντες ἀποβαίνουσιν. ἴσον δὲ ἰσάκεις οὐδέστι.

"Explicanda haec sunt ex antiqua Pythagoricorum terminologia. Γνώμονες nempe de quibus hic loquitur auctor, vocabantur apud eos omnes numeri impares, *Joh. Philop. ad Aristot. Phys.*, l. iii., p. 131: Καὶ οἱ ἀριθμητικοὶ δὲ γνώμονας καλοῦσι πάντας τοὺς περιττοὺς ἀριθμούς. Causam adjicit Simplicius ad eundem locum, Γνώμονας δὲ ἐκάλουν τοὺς περιττοὺς οἱ Πυθαγόρειοι διότι προστιθέμενοι τοῖς τετραγώνοις, τὸ αὐτὸ σχῆμα φυλάττουσι, ὥσπερ καὶ οἱ ἐν γεωμετρικῇ γνώμονες. Quae nostro loco leguntur jam satis clara erunt. Vult nempe auctor, monade addita ad primum gnomonem, ad sequentes autem summam, quam proxime antecedentes numeri efficiunt, semper prodire numeros quadratos, v. c. positus gnomonibus 3, 5, 7, 9 primum $1 + 3 = 2^2$, tunc porro $1 + 3$ (i. e. 4) $+ 5 = 3^2$, $9 + 7 = 4^2$, $16 + 9 = 5^2$, et sic porro, cf. *Tiedem.*

Geist der Speculat. Philos., pp. 107, 108. Reliqua expedita sunt." Stob. *Eclog.* ab Heeren, lib. 1., p. 24 and note.

The passage of Aristotle referred to is—σημεῖον δ' εἶναι τοῦτον τὸ συμβαῖνον ἐπὶ τῶν ἀριθμῶν. περιτιθεμένων γὰρ τῶν γνωμόνων περὶ τὸ ἐν καὶ χωρὶς ὅτῃ μὲν ἄλλο ἔστι γίνεσθαι τὸ εἶδος. *Phys.*, iii., 4, p. 203^a, 14.

Compare, ἀλλ' ἐστὶ τίνα ἀξανάμενα ἃ οὐκ ἀλλοιοῦνται, οἷον τὸ τετράγωνον γνόμενος περιτεθέντος ἤδηται μὲν, ἀλλοιότερον δὲ οὐδὲν γαγένηται. *Cat.* 14, 15^a, 30, Arist., ed. Bekker.

Hankel gives a different explanation of the opposition between the square and oblong—

"When the Pythagoreans discovered the theory of the Irrational, and recognised its importance, it must, as will be at once admitted, appear most striking that the oppositions, which present themselves so naturally, of Rational and Irrational have no place in their table. Should they not be contained under the image of square and rectangle, which, in the extraction of the square root, have led precisely to those ideas?" *Geschichte der Mathematik*, p. 110, note.

Hankel also says—"Upon what the comparison of the odd with the limited may have been based, and whether upon the theory of the gnomons, can scarcely be made out now." *Ibid.* p. 109, note.

May not the gnomon be looked on as *framing*, as it were, or limiting the squares?

Egyptians may have first arrived at by an induction founded on direct measurement, the opportunity for which was furnished to them by their pavements, or chequered plane surfaces.

The method given above for the formation of the square constructed on 5 units as the sum of those constructed on 4 units and on 3 units, and of that constructed on 13 units as the sum of those constructed on 12 units and 5 units, required only to be generalized in order to enable Pythagoras to arrive at his rule for finding right-angled triangles, which we are told sets out from the odd numbers.

The two rules of Pythagoras and of Plato are given by Proclus:—"But there are delivered certain methods of finding triangles of this kind [*sc.*, right-angled triangles whose sides can be expressed by numbers], one of which they refer to Plato, but the other to Pythagoras, as originating from odd numbers. For Pythagoras places a given odd number as the lesser of the sides about the right angle, and when he has taken the square constructed on it, and diminished it by unity, he places half the remainder as the greater of the sides about the right angle; and when he has added unity to this, he gets the hypotenuse. Thus, for example, when he has taken 3, and has formed from it a square number, and from this number 9 has taken unity, he takes the half of 8, that is 4, and to this again he adds unity, and makes 5; and thus obtains a right-angled triangle, having one of its sides of 3, the other of 4, and the hypotenuse of 5 units. But the Platonic method originates from even numbers. For when he has taken a given even number, he places it as one of the sides about the right angle, and when he has divided this into half, and squared the half, by adding unity to this square he gets the hypotenuse, but by subtracting unity from the square he forms the remaining side about the right angle. Thus, for example, taking 4, and squaring its half, 2, and thus getting

4, then subtracting 1 he gets 3, and by adding 1 he gets 5; and he obtains the same triangle as by the former method." ⁸⁶ It should be observed, however, that this is not necessarily the case; for example, we may obtain by the method of Plato a triangle whose sides are 8, 15, and 17 units, which cannot be got by the Pythagorean method.

The n^{th} square together with the n^{th} gnomon is the $(n + 1)^{\text{th}}$ square; if the n^{th} gnomon contains m^2 unit squares m being an *odd* number, we have $2n + 1 = m^2$, $\therefore n = \frac{m^2 - 1}{2}$;

hence the rule of Pythagoras. Similarly the sum of two successive gnomons contains an even number of unit squares, and may therefore consist of m^2 unit squares, where m is an *even* number; we have then $(2n - 1) + (2n + 1) = m^2$, or $n = \left(\frac{m}{2}\right)^2$: hence the rule ascribed to Plato by

Proclus.⁸⁷ This passage of Proclus, which is correctly interpreted by Hoefer,⁸⁸ was understood by Kepler,⁸⁹ who, indeed, was familiar with this work of Proclus, and often quotes it in his *Harmonia Mundi*.

Let us now examine how Pythagoras proved the theorem of the three squares. Though he could have discovered it as a consequence of the theorem concerning the proportionality of the sides of equiangular triangles, attributed above to Thales, yet there is no indication whatever of his having arrived at it in that deductive manner. On the

⁸⁶ Procli *Comm.*, ed. Friedlein, p. 428. Hero, *Geom.*, ed. Hultsch, pp. 56, 57.

⁸⁷ This rule is ascribed to Architas [no doubt, Archytas of Tarentum] by Boetius, *Geom.*, ed. Friedlein, p. 408.

⁸⁸ Hoefer, *Histoire des Math.*, p. 112.

⁸⁹ Kepleri *Opera Omnia*, ed. Frisch, vol. viii., pp. 163 et seq. It may be observed that this method is

capable of further extension, *e. g.*: the sum of 9 (an odd square number) successive gnomons may contain an odd number (say 49×9) of square units; hence we obtain a right-angled triangle in numbers, whose hypotenuse exceeds one side by 9 units—the three sides being 20, 21, and 29. Plato's method may be extended in like manner.

other hand the proof given in the Elements of Euclid clearly points to such an origin, for it depends on the theorem that the square on a side of a right-angled triangle is equal to the rectangle under the hypotenuse and its adjacent segment made by the perpendicular on it from the right angle—a theorem which follows at once from the similarity of each of the partial triangles, into which the original right-angled triangle is broken up by the perpendicular, with the whole. That the proof in the Elements is not the way in which the theorem was discovered is indeed stated directly by Proclus, who says:—

“If we attend to those who wish to investigate antiquity, we shall find them referring the present theorem to Pythagoras, . . . For my own part, I admire those who first investigated the truth of this theorem: but I admire still more the author of the Elements, because he has not only secured it by evident demonstration, but because he reduced it into a more general theorem in his sixth book by strict reasoning [Euclid, vi., 31].”⁹⁰

The simplest and most natural way of arriving at the theorem is the following, as suggested by Bretschneider⁹¹:—

A square can be dissected into the sum of two squares and two equal rectangles, as in Euclid, ii., 4; these two rectangles can, by drawing their diagonals, be decomposed into four equal right-angled triangles, the sum of the sides of each being the side of the square: again, these four right-angled triangles can be placed so that a vertex of each shall be in one of the corners of the square in such a way that a greater and less side are in continuation. The original square is thus dissected into the four triangles as

⁹⁰ Procli *Comm.* ed. Friedlein, p. 426.

Camerer, *Euclidis Element.*, vol. i., p.

⁹¹ Bretsch., *Die Geometrie vor Eu-
klides*, p. 82. This proof is old: see

444, and references given there.

before and the figure within, which is the square on the hypotenuse. This square then must be equal to the sum of the squares on the sides of the right-angled triangle. Hankel, in quoting this proof from Bretschneider, says that it may be objected that it bears by no means a specifically Greek colouring, but reminds us of the Indian method. This hypothesis as to the oriental origin of the theorem seems to me to be well founded. I would, however, attribute the discovery to the Egyptians, inasmuch as the theorem concerns the geometry of areas, and as the method used is that of the dissection of figures, for which the Egyptians were famous, as we have already seen. Moreover, the theorem concerning the areas connected with two lines and their sum (Euclid, ii., 4), which admits also of arithmetical interpretation, was certainly within their reach. The gnomon by which any square exceeds another breaks up naturally into a square and two equal rectangles.

I think also that the Egyptians knew that the difference between the squares on two lines is equal to the rectangle under their sum and difference—though they would not have stated it in that abstract manner. The two squares may be placed with a common vertex and adjacent sides coinciding in direction, so that their difference is a gnomon. This gnomon can, on account of the equality of the two complements,* be transformed into a rectangle which can be constructed by producing the side of the greater square so that it shall be equal to itself, and then we have the figure of Euclid, ii., 5, or to the side of the lesser square, in which case we have the figure of Euclid, ii., 6. Indeed I have little hesitation in attributing to the Egyptians the contents

* This theorem (Euclid, i. 43) Bretschneider says was called the "theorem of the gnomon." I do not know of any authority for this statement. If the theorem were so called, the word

gnomon was not used in it either as defined by Euclid (ii., *Def.* 2), or in the more general signification in Hero (*Def.* 58).

of the first ten propositions of the second book of Euclid. In the demonstrations of propositions 5, 6, 7, and 8, use is made of the gnomon, and propositions 9 and 10 also can be proved similarly without the aid of Euclid, i., 47.

It is well known that the Pythagoreans were much occupied with the construction of regular polygons and solids, which in their cosmology played an essential part as the fundamental forms of the elements of the universe.⁸⁸

We can trace the origin of these mathematical speculations in the theorem (c) that "the plane around a point is completely filled by six equilateral triangles or four squares, or three regular hexagons," a theorem attributed to the Pythagoreans, but which must have been known as a fact to the Egyptians. Plato also makes the Pythagorean Timaeus explain—"Each straight-lined figure consists of triangles, but all triangles can be dissected into rectangular ones which are either isosceles or scalene. Among the latter the most beautiful is that out of the doubling of which an equilateral arises, or in which the square of the greater perpendicular is three times that of the smaller, or in which the smaller perpendicular is half the hypotenuse. But two or four right-angled isosceles triangles, properly put together, form the square; two or six of the most beautiful scalene right-angled triangles form the equilateral triangle; and out of these two figures arise the solids which correspond with the four elements of the real world, the tetrahedron, octahedron, icosahedron, and the cube."⁸⁹

This dissection of figures into right-angled triangles may be fairly referred to Pythagoras, and indeed may have been derived by him from the Egyptians.

⁸⁸ Hankel says it cannot be ascertained with precision how far the Pythagoreans had penetrated into this theory, namely, whether the construction of the regular pentagon and ordi-

nate dodecahedron was known to them. Hankel, *Geschichte der Mathematik*, p. 95, note.

⁸⁹ Plato, *Tim.*, c. 20, s. 107.

The construction of the regular solids is distinctly ascribed to Pythagoras himself by Eudemus, in the passage in which he briefly states the principal services of Pythagoras to geometry. Of the five regular solids, three—the tetrahedron, the cube, and the octahedron—were certainly known to the Egyptians, and are to be found in their architecture. There remain, then, the icosahedron and the dodecahedron. Let us now examine what is required for the construction of these two solids.

In the formation of the tetrahedron, three, and in that of the octahedron, four, equal equilateral triangles had been placed with a common vertex and adjacent sides coincident, and it was known too that if six such triangles were placed round a common vertex with their adjacent sides coincident, they would lie in a plane, and that, therefore, no solid could be formed in that manner from them. It remained then to try whether five such equilateral triangles could be placed at a common vertex in like manner: on trial it would be found that they could be so placed, and that their bases would form a regular pentagon. The existence of a regular pentagon would thus be known. It was also known from the formation of the cube that three squares could be placed in a similar way with a common vertex, and that, further, if three equal and regular hexagons were placed round a point as common vertex with adjacent sides coincident, they would form a plane. It remained then only to try whether three equal regular pentagons could be placed with a common vertex, and in a similar way; this on trial would be found possible, and would lead to the construction of the regular dodecahedron, which was the regular solid last arrived at.⁹⁰

We see then that the construction of the regular pentagon is required for the formation of each of these two

⁹⁰ The four elements had been represented by the four other regular solids; the dodecahedron was then taken symbolically for the universe.

regular solids, and that therefore it must have been a discovery of Pythagoras. We have now to examine what knowledge of geometry was required for the solution of this problem.

If any vertex of a regular pentagon be connected with the two remote ones, an isosceles triangle will be formed having each of the base angles double the vertical angle. The construction of the regular pentagon depends, therefore, on the description of such a triangle (Euclid, iv., 10). Now, if either base angle of such a triangle be bisected, the isosceles triangle will be decomposed into two triangles, which are evidently also both isosceles. It is also evident that the one of which the base of the proposed is a side is equiangular with it. From a comparison of the sides of these two triangles it will appear at once by the second theorem, attributed above to Thales, that the problem is reduced to cutting a straight line so that one segment shall be a mean proportional between the whole line and the other segment (Euclid, vi., 30), or so that the rectangle under the whole line and one part shall be equal to the square on the other part (Euclid, ii., 11). To effect this, let us suppose the square on the greater segment to be constructed on one side of the line, and the rectangle under the whole line and the lesser segment on the other side. It is evident that by adding to both the rectangle under the whole line and the greater segment, the problem is reduced to the following:—To produce a given straight line so that the rectangle under the whole line thus produced and the part produced shall be equal to the square on the given line, or, in the language of the ancients, to apply to a given straight line a rectangle which shall be equal to a given area—in this case the square on the given line—and which shall be *excessive* by a square. Now it is to be observed that the problem is solved in this manner by Euclid (vi., 30, 1st method), and that we learn from

Eudemus that the problems concerning the *application* of areas and their *excess* and *defect* are old, and inventions of the Pythagoreans (*e*).⁹¹

The statements, then, of Iamblichus concerning Hip-
pasus (*z*)—that he divulged the sphere with the twelve
pentagons; and of Lucian and the scholiast on Aristo-
phanes (*y*)—that the pentagram was used as a symbol of
recognition amongst the Pythagoreans, become of greater
importance. We learn too from Iamblichus that the Py-
thagoreans made use of signs for that purpose.⁹²

Further, the discovery of irrational magnitudes is
ascribed to Pythagoras in the same passage of Eude-

⁹¹ It may be objected that this reason-
ing presupposes a knowledge, on the
part of Pythagoras, of the method of
geometrical analysis, which was in-
vented by Plato more than a century
later.

While admitting that it contains the
germ of that method, I reply in the
first place, that this manner of reason-
ing was not only natural and sponta-
neous, but that in fact in the solution
of problems there was no other way of
proceeding. And, to anticipate a little,
we shall see, secondly, that the oldest
fragment of Greek geometry extant—
that namely by Hippocrates of Chios—
contains traces of an analytical method,
and that, moreover, Proclus ascribes
to Hippocrates, who, it will appear,
was taught by the Pythagoreans the
method of reduction (*ἀπαγωγή*), a sys-
tematization, as it seems to me, of the
manner of reasoning that was sponta-
neous with Pythagoras. Proclus de-
fines *ἀπαγωγή* to be “a transition from
one problem or theorem to another,
which being known or determined, the
thing proposed is also plain. For ex-
ample: when the duplication of the
cube is investigated, geometers reduce

the question to another to which this
is consequent, *i. e.* the finding of two
mean proportionals, and afterwards
they inquire how between two given
straight lines two mean proportionals
may be found. But Hippocrates of
Chios is reported to have been the first
inventor of geometrical reduction (*ἀπα-
γωγή*): who also squared the lunule,
and made many other discoveries in
geometry, and who was excelled by no
geometer in his powers of construc-
tion.”—Proclus, ed. Friedlein, p. 212.
Lastly, we shall find that the passages
in Diogenes Laertius and Proclus,
which are relied on in support of the
statement that Plato invented this meth-
od, prove nothing more than that
Plato communicated it to Leodamas
of Thasos. For my part, I am con-
vinced that the gradual elaboration
of this famous method—by which ma-
thematics rose above the elements—is
due to the Pythagorean philosophers
from the founder to Theodorus of
Cyrene and Archytas of Tarentum,
who were Plato's masters in mathema-
tics.

⁹² Iambl. *de Pyth. Vita*, cxxxiii.,
p. 77, ed. Didot.

mus (m), and this discovery has been ever regarded as one of the greatest of antiquity. It is commonly assumed that Pythagoras was led to this theory from the consideration of the isosceles right-angled triangle. It seems to me, however, more probable that the discovery of incommensurable magnitudes was rather owing to the problem—To cut a line in extreme and mean ratio. From the solution of this problem it follows at once that, if on the greater segment of a line so cut a part be taken equal to the less, the greater segment, regarded as a new line, will be cut in a similar manner; and this process can be continued without end. On the other hand, if a similar method be adopted in the case of any two lines which are capable of numerical representation, the process would end. Hence would arise the distinction between commensurable and incommensurable quantities.

A reference to Euclid, x., 2, will show that the method above is the one used to prove that two magnitudes are incommensurable. And in Euclid, x., 3, it will be seen that the greatest common measure of two commensurable magnitudes is found by this process of continued subtraction.

It seems probable that Pythagoras, to whom is attributed one of the rules for representing the sides of right-angled triangles in numbers, tried to find the sides of an isosceles right-angled triangle numerically, and that, failing in the attempt, he suspected that the hypotenuse and a side had no common measure. He may have demonstrated the incommensurability of the side of a square and its diagonal. The nature of the old proof—which consisted of a *reductio ad absurdum*, showing that if the diagonal be commensurable with the side, it would follow that the same number would be odd and even⁹²—makes it more probable, however, that this was accomplished by his successors.

⁹² Aristoteles, *Analyt. Prior.*, I., c. 23, 41, a, 26, and c. 44, 50, a, 37, ed. Bekker.

Euclid has preserved this proof, x., 117. Hankel thinks he did so probably

for its historical interest only, since the irrationality follows self-evidently from x., 9; and x., 117, is merely an appendix.—Hankel, *Geschichte der Math.*, p. 102, note.

The existence of the irrational, as well as that of the regular dodecahedron, appears to have been regarded also by the school as one of their chief discoveries, and to have been preserved as a secret; it is remarkable, too, that a story similar to that told by Iamblichus of Hippasus is narrated of the person who first published the idea of the irrational, namely, that he suffered shipwreck, &c.⁹¹

Eudemus ascribes the problems concerning the application of figures to the Pythagoreans. The simplest cases of the problems (Euclid, vi., 28, 29)—those, namely, in which the given parallelogram is a square—correspond to the problem: To cut a straight line internally, or externally, so that the rectangle under the segments shall be equal to a given rectilineal figure. On examination it will be found that the solution of these problems depends on the problem Euclid, ii., 14, and the theorems Euclid, ii., 5 and 6, which we have seen were probably known to the Egyptians, together with the law of the three squares (Euclid, i., 47).

The finding of a mean proportional between two given lines, or the construction of a square which shall be equal to a given rectangle, must be referred, I have no doubt, to Pythagoras. The rectangle can be easily thrown into the form of a gnomon, and then exhibited as the difference between two squares, and therefore as a square by means of the law of the three squares.

Lastly, the solution of the problem to construct a rectilineal figure which shall be equal to one and similar to another given rectilineal figure is attributed by Plutarch to Pythagoras. The solution of this problem depends on the application of areas, and requires a knowledge of the theorems:—that similar rectilineal figures are to each other as the squares on their homologous sides; that if three

⁹¹ *Untersuchungen über die neu aufgefundenen Scholien des Proklus Diadochus zu Euclid's Elementen*, von

Dr. Joachim Heinrich Knoche, Herford, 1865, pp. 20 and 23.

lines be in geometrical proportion, the first is to the third as the square on the first is to the square on the second; and also on the solution of the problem, to find a mean proportional between two given straight lines. Now, we shall see later that Hippocrates of Chios—who was instructed in geometry by the Pythagoreans—must have known these theorems and the solution of this problem. We are justified, therefore, in ascribing this theorem also, if not with Plutarch to Pythagoras, at least to his early successors.

The theorem that similar polygons are to each other in the duplicate ratio of their homologous sides involves a first sketch, at least, of the doctrine of proportion.

That we owe the foundation and development of the doctrine of proportion to Pythagoras and his disciples is confirmed by the testimony of Nicomachus (π) and Iamblichus (σ and ρ).

From these passages it appears that the early Pythagoreans were acquainted not only with the arithmetical and geometrical means between two magnitudes, but also with their harmonical mean, which was then called $\acute{\upsilon}\pi\epsilon\rho\alpha\nu\tau\iota\lambda\alpha$.

When two quantities are compared, it may be considered *how much* the one is greater than the other, what is their *difference*; or it may be considered *how many times* the one is contained in the other, what is their *quotient*. The former relation of the two quantities is called their *arithmetical ratio*; the latter their *geometrical ratio*.

Let now three magnitudes, lines or numbers, a , b , c , be taken. If $a - b = b - c$, the three magnitudes are in arithmetical proportion; but if $a : b :: b : c$, they are in geometrical proportion.⁹⁵ In the latter case, it follows at once, from the

⁹⁵ In *lines* we may have $c = a - b$, or $a : b :: a - b$. This particular case, in which the geometrical and arithmetical ratios both occur in the same proportion, is worth noticing. The line a is

then the sum of the other two lines, and is said to be cut in extreme and mean ratio. This section, as we have seen, has arisen out of the construction of the regular pentagon, and we learn

second theorem of Thales (Euclid, vi., 4), that $a - b : b - c :: a : b$, whereas in the former case we have plainly $a - b : b - c :: a : a$. This might have suggested the consideration of three magnitudes, so taken that $a - b : b - c :: a : c$; three such magnitudes are in harmonical proportion.

The probability of the correctness of this view is indicated by the consideration of the three later proportions—

$$\left. \begin{array}{l} a : c :: b - c : a - b \\ b : c :: b - c : a - b \\ a : b :: b - c : a - b \end{array} \right\} \begin{array}{l} \dots \text{the contrary of the harmonical;} \\ \dots \text{the contrary of the geometrical.} \end{array}$$

The discovery of these proportions is attributed⁹⁸ to Hip-
pasus, Archytas, and Eudoxus.

We have seen also (*p*) that a knowledge of the so-called most perfect or musical proportion, which comprehends in it all the former ratios, is attributed by Iamblichus to Pythagoras—

$$a : \frac{a+b}{2} :: \frac{2ab}{a+b} : b.$$

We have also seen (*q*) that a knowledge of the doctrine of arithmetical progressions is attributed to Pythagoras. This much at least seems certain, that he was acquainted with the summation of the natural numbers, the odd numbers, and the even numbers, all of which are capable of geometrical representation.

Montucla says that Pythagoras laid the foundation of the doctrine of *perimetry* by proving that of all figures having the same perimeter the circle is the greatest, and

from Kepler that it was called by the moderns, on account of its many wonderful properties, *sectio divina*, et *proportio divina*. He sees in it a fine image of generation, since the addition to the line of its greater part produces a new line cut similarly, and so on. See Kepleri *Opera Omnia*, ed. Frisch,

vol. v., pp. 90 and 187 (*Harmonia Mundi*); also vol. i. p. 377 (*Litterae de Rebus Astrologicis*). The pentagram might be taken as the image of all this, as each of its sides and part of a side are cut in this *divine* proportion.

⁹⁸ Iambl. in Nic. *Arith.*, pp. 142, 159, 163. See above, p. 163.

that of all solids having the same surface the sphere is the greatest."

There is no evidence to support this assertion, though it is repeated by Chasles, Arneth, and others; it rests merely on an erroneous interpretation of the passage (s) in Diogenes Laertius, which says only that "of all solid figures the sphere is the most beautiful; and of all plane figures, the circle." Pythagoras attributes perfection and beauty to the sphere and circle on account of their regularity and uniformity. That this is the true signification of the passage is confirmed by Plato in the *Timaeus*,⁹⁸ when speaking of the Pythagorean cosmogony.⁹⁹

We must also deny to Pythagoras and his school a knowledge of the conic sections, and, in particular, of the quadrature of the parabola, attributed to him by some authors, and we have already noticed the misconception which gave rise to this erroneous conclusion.¹⁰⁰

Let us now see what conclusions can be drawn from the foregoing examination of the mathematical work of Pythagoras and his school, and thus form an estimate of the state of geometry about 480 B. C.:—

First, then, as to *matter*:—

It forms the bulk of the first two books of Euclid, and includes, further, a sketch of the doctrine of proportion—which was probably limited to commensurable magnitudes—together with some of the contents of the sixth book. It contains, too, the discovery of the irrational (*ἄλογον*), and the construction of the regular solids; the

⁹⁷ "Suivant Diogène, dont le texte est ici fort corrompu, et probablement transposé, il ébaucha aussi la doctrine des Isopérimètres, en démontrant que de toutes les figures de même contour, parmi les figures planes, c'est le cercle qui est la plus grande, et parmi les solides, la sphère."—Montucla,

Histoire des Mathématiques, tom. I., p. 113.

⁹⁸ *Timaeus*, 33, B., vol. vii., ed. Stallbaum, p. 129.

⁹⁹ See Bretschneider, *Die Geometrie vor Euklides*, pp. 89, 90.

¹⁰⁰ See above, p. 182, note.

latter requiring the description of certain regular polygons—the foundation, in fact, of the fourth book of Euclid.

The properties of the circle were not much known at this period, as may be inferred from the fact that not one remarkable theorem on this subject is mentioned; and we shall see later that Hippocrates of Chios did not know the theorem—that the angles in the same segment of a circle are equal to each other. Though this be so, there is, as we have seen, a tradition (*f*) that the problem of the quadrature of the circle also engaged the attention of the Pythagorean school—a problem which they probably derived from the Egyptians.¹⁰¹

Second, as to *form* :—

The Pythagoreans first severed geometry from the needs of practical life, and treated it as a liberal science, giving definitions, and introducing the manner of proof which has ever since been in use. Further, they distinguished between *discrete* and *continuous* quantities, and regarded geometry as a branch of mathematics, of which they made the fourfold division that lasted to the Middle Ages—the *quadrivium* (fourfold way to knowledge) of Boetius and the scholastic philosophy. And it may be observed, too, that the name of mathematics, as well as that of philosophy, is ascribed to them.

Third, as to *method* :—

One chief characteristic of the mathematical work of Pythagoras was the combination of arithmetic with geo-

¹⁰¹ This problem is considered in the Papyrus Rhind, pp. 97, 98, 117. The point of view from which it was regarded by the Egyptians was different from that of Archimedes. Whilst he made it to depend on the determination of the ratio of the circumference to the diameter, they sought to find from the

diameter the side of a square whose area should be equal to that of the circle. Their approximation was as follows :—The diameter being divided into nine equal parts, the side of the equivalent square was taken by them to consist of eight of those parts.

metry. The notions of an equation and a proportion—which are common to both, and contain the first germ of algebra—were, as we have seen, introduced amongst the Greeks by Thales. These notions, especially the latter, were elaborated by Pythagoras and his school, so that they reached the rank of a true scientific method in their Theory of Proportion. To Pythagoras, then, is due the honour of having supplied a method which is common to all branches of mathematics, and in this respect he is fully comparable to Descartes, to whom we owe the decisive combination of algebra with geometry.

It is necessary to dwell on this at some length, as modern writers are in the habit of looking on proportion as a branch of arithmetic¹⁰²—no doubt on account of the arithmetical point of view having finally prevailed in it—whereas for a long period it bore much more the marks of its geometrical origin.¹⁰³

That proportion was not thus regarded by the ancients, merely as a branch of arithmetic, is perfectly plain. We learn from Proclus that “Eratosthenes looked on proportion as the bond (σύνδεσμον) of mathematics.”¹⁰⁴

We are told, too, in an anonymous scholium on the Elements of Euclid, which Knoche attributes to Proclus, that the fifth book, which treats of proportion, is common to geometry, arithmetic, music, and, in a word, to all mathematical science.¹⁰⁵

And Kepler, who lived near enough to the ancients to reflect the spirit of their methods, says that one part of

¹⁰² Bretschneider (*Die Geometrie vor Euklides*, p. 74) and Hankel (*Geschichte der Mathematik*, p. 104) do so, although they are treating of the history of Greek geometry, which is clearly a mistake.

¹⁰³ On this see A. Comte, *Politique Positive*, vol. iii., ch. iv., p. 300.

¹⁰⁴ Procl. *Comm.*, ed. Freidlein, p. 43.

¹⁰⁵ Euclidis *Elem.* Graece ed. ab E. F. August, pars ii., p. 328, Berolini, 1829. *Untersuchungen über die neu aufgefundenen Scholien des Proklus zu Euclid's Elementen*, von Dr. J. H. Knoche, p. 10, Herford, 1865.

geometry is concerned with the comparison of figures and quantities, whence proportion arises ("unde proportio existit"). He also adds that arithmetic and geometry afford mutual aid to each other, and that they cannot be separated.¹⁰⁶

And since Pythagoras they have never been separated. On the contrary, the union between them, and indeed between the various branches of mathematics, first instituted by Pythagoras and his school, has ever since become more intimate and profound. We are plainly in presence of not merely a great mathematician, but of a great philosopher. It has been ever so—the greatest steps in the development of mathematics have been made by philosophers.

Modern writers are surprised that Thales, and indeed all the principal Greek philosophers prior to Pythagoras, are named as his masters. They are surprised, too, at the extent of the travels attributed to him. Yet there is no cause to wonder that he was believed by the ancients to have had these philosophers as his teachers, and to have extended his travels so widely in Greece, Egypt, and the East, in search of knowledge, for—like the geometrical figures on whose properties he loved to meditate—his philosophy was many-sided, and had points of contact with all these:—

He introduced the knowledge of arithmetic from the Phoenicians, and the doctrine of proportion from the Babylonians;

Like Moses, he was learned in all the wisdom of the

¹⁰⁶ "Et quidem geometriæ theoreti-
cæ initio hujus tractatus duas fecimus
partes, unam de magnitudinibus, qua-
tenus fiunt figurae, alteram de compara-
tione figurarum et quantitatum, unde
proportio existit.

"Hæ duæ scientiæ, arithmetica et

geometria speculativa, mutuas tradunt
operas nec ab invicem separari possunt,
quamvis et arithmetica sit principium
cognitionis."—Kepleri *Opera Omnia*,
ed. Dr. Ch. Frisch, vol. viii., p. 160,
Francofurti, 1870.

Egyptians, and carried their geometry and philosophy into Greece.

He continued the work commenced by Thales in abstract science, and invested geometry with the form which it has preserved to the present day.

In establishing the existence of the regular solids he showed his deductive power; in investigating the elementary laws of sound he proved his capacity for induction; and in combining arithmetic with geometry, and thereby instituting the theory of proportion, he gave an instance of his philosophic power.

These services, though great, do not form, however, the chief title of this Sage to the gratitude of mankind. He resolved that the knowledge which he had acquired with so great labour, and the doctrine which he had taken such pains to elaborate, should not be lost; and, as a husbandman selects good ground, and is careful to prepare it for the reception of the seed, which he trusts will produce fruit in due season, so Pythagoras devoted himself to the formation of a society of *élite*, which would be fit for the reception and transmission of his science and philosophy, and thus became one of the chief benefactors of humanity, and earned the gratitude of countless generations.

His disciples proved themselves worthy of their high mission. We have had already occasion to notice their noble self-renunciation, which they inherited from their master.

The moral dignity of these men is, further, shown by their admirable maxim—a maxim conceived in the spirit of true social philosophers—a *figure and a step*; but not a *figure and three oboli* (σχᾶμα καὶ βᾶμα, ἀλλ' οὐ σχᾶμα καὶ τριῶβολον).¹⁰⁷

¹⁰⁷ Procli *Comm.*, ed. Friedlein, p. 84. Taylor's *Commentaries of Proclus*, vol. i., p. 113. Taylor, in a note on this passage, says—"I do not find this aenigma among the Pythagoric symbols

which are extant, so that it is probably nowhere mentioned but in the present work."

Taylor is not correct in this statement. This symbol occurs in Iambli-

Such, then, were the men by whom the first steps in mathematics—the first steps ever the most difficult—were made.

In the continuation of the present paper we shall notice the events which led to the publication, through Hellas, of the results arrived at by this immortal School.

chus. See Iambl., *Adhortatio ad Philosophiam*, ed. Kiessling, *Symb.* p. 374. Τὸ δὲ πρότερον τὸ σχῆμα καὶ βῆμα τοῦ σχῆμα καὶ τριώβολον.
xxxvi., cap. xxi., p. 317; also *Expl.*

GEORGE J. ALLMAN.