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GREEK GEOMETRY FROM THALES TO EUCLID.*

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The only notice of his work which has come down to us is contained in the following passage of Pappus:—

‘For the quadrature of the circle a certain curve² was employed by Dinostratus, Nicomedes, and some other more recent geometers, which has received its name from the property that belongs to it; for it is called by them the quadratrix (*τετραγωνίζουσα*), and its generation is as follows:—

‘Let a square $\alpha\beta\gamma\delta$ be assumed, and about the centre γ let the quadrant³ $\beta\epsilon\delta$ be described, and let the line $\gamma\beta$ be

* The previous portions of this Paper have appeared in HERMATHENA, Vol. iii., No. v.; Vol. iv., No. vii.; and Vol. v., Nos. x. and xi.

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¹ See HERMATHENA, vol. v. p. 406 (a).

² *γραμμή*. The Greeks had no special name for ‘a curve.’

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IN HERMATHENA

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Pappus has, moreover, transmitted to us the property of the quadratrix, from which it received its name, together with the proof. It is as follows:—

‘If $\alpha\beta\gamma\delta$ be a square, and $\beta\epsilon\delta$ be the quadrant about the centre γ , and the line $\beta\eta\theta$ be the quadratrix described as in the manner given above; it is proved that: as the quadrant $\delta\epsilon\beta$ is to the straight line $\beta\gamma$, so is $\beta\gamma$ to the straight line $\gamma\theta$. For if it is not, the quadrant $\delta\epsilon\beta$ will be to the line $\beta\gamma$ as $\beta\gamma$ to a line greater than $\gamma\theta$, or to a lesser.

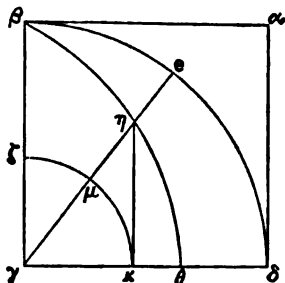
‘In the first place let it be, if possible [as $\beta\gamma$], to a greater line $\gamma\kappa$; and about the centre γ let the quadrant $\zeta\eta\kappa$ be described, cutting the curve at the point η ; let the perpendicular $\eta\lambda$ be drawn, and let the joining line $\gamma\eta$ be produced to the point ϵ . Since then: as the quadrant $\delta\epsilon\beta$ is to the straight line $\beta\gamma$, so is $\beta\gamma$ —that is $\gamma\delta$ —to the line $\gamma\kappa$, and as $\gamma\delta$ is to $\gamma\kappa$, so is the quadrant $\beta\epsilon\delta$ to the quadrant $\zeta\eta\kappa$ (for the circumferences of circles are to each other as their diameters),⁵ it is evident that the quadrant $\zeta\eta\kappa$ is equal to the straight line $\beta\gamma$. And since, on account of the property of the curve, there is: as the quadrant $\beta\epsilon\delta$ is to the arc $\epsilon\delta$, so is $\beta\gamma$ to $\eta\lambda$; and therefore: as the quadrant $\zeta\eta\kappa$ is to the arc $\eta\kappa$, so is the straight line $\beta\gamma$ to the line $\eta\lambda$. And it has been shown that the quadrant $\zeta\eta\kappa$ is equal to the straight line $\beta\gamma$; therefore the arc $\eta\kappa$ will be equal to the straight line $\eta\lambda$, which is absurd. Therefore it is not true that: as the quadrant $\beta\epsilon\delta$ is to the straight line $\beta\gamma$, so is $\beta\gamma$ to a line greater than $\gamma\theta$.’

‘Further, I say, that neither is it to a line less than $\gamma\theta$. For, if possible, let it be to $\gamma\kappa$, and about the centre γ let the quadrant $\zeta\mu\kappa$ be described, and let the line $\kappa\eta$ be drawn at right angles to the line $\gamma\delta$, cutting the quadratrix at the point η , and let the joining line $\gamma\eta$ be produced to the

⁵ ‘Hoc theorema extat v propos. 11 et VIII propos. 22; simul autem scriptor tacite efficit circulorum arcus quibus

aequales anguli insistent inter se esse ut radios.’ (*Ibid.* p. 257, n.)

point ϵ . In like manner then to what has been proved above, we show that the quadrant $\zeta\mu\kappa$ is equal to the straight line $\beta\gamma$, and that: as the quadrant $\beta\epsilon\delta$ is to the arc $\epsilon\delta$ —that is, as the quadrant $\zeta\mu\kappa$ to the arc $\mu\kappa$ —so is the



straight line $\beta\gamma$ to the line $\eta\kappa$. From which it is evident that the arc $\mu\kappa$ is equal to the straight line $\kappa\eta$, which is absurd. Therefore it is not true that: as the quadrant $\beta\epsilon\delta$ is to the straight line $\beta\gamma$, so is $\beta\gamma$ to a line less than $\gamma\theta$. Neither is it to a greater, as has been proved above; therefore it is to the line $\gamma\theta$ itself.*

Pappus continues—‘This also is evident, that if a third proportional be taken to the straight lines $\theta\gamma$, $\gamma\beta$, the straight line [thus found] will be equal to the quadrant $\beta\epsilon\delta$; and four times this line will be equal to the circumference of the whole circle. But the straight line, which is equal to the circumference of a circle, being found, it is evident that a square equal to the circle itself can be easily constructed: for the rectangle under the perimeter of a circle and its radius is double of the circle, as Archimedes proved.’

Pappus also relates that Sporus justly found fault with this curve, for two reasons:—

* *Ibid.* pp. 256, 258.

† ‘Paulo aliis verbis Pappus id theoremata enuntiat atque ipse Archimedes circuli dimens. propos. 1: πᾶς κύκλος

ἴσος ἐστὶ τριγώνῳ ὀρθογωνίῳ, οὗ ἡ μὲν ἐκ τοῦ κέντρου ἴση μὲν τῶν περὶ τὴν ὀρθήν, ἡ δὲ περίμετρος τῇ λοιπῇ. (*Ibid.* p. 259, n. 2.)

1. 'It takes for granted the very thing for which the quadratrix is employed; for it is not possible to make one point move from β to γ along the straight line $\beta\gamma$ in the same time that another point moves along the quadrant $\beta\epsilon\delta$, unless the ratio of the straight line to the quadrant is first known, inasmuch as it is necessary that the rates of the motions should be to each other in the same ratio.'

2. 'The extremity of the curve which is employed for the quadrature of the circle—that is, the point in which the quadratrix cuts the straight line $\gamma\delta$ —is not found; for when the straight lines $\gamma\beta$, $\beta\alpha$, being moved, are brought simultaneously to the end of their motion, they coincide with the line $\gamma\delta$, and no longer cut one another—for the cutting ceases before the coincidence with the line $\alpha\delta$, which intersection on the other hand is taken as the extremity of the curve, in which it meets the straight line $\alpha\delta$: unless, perhaps, some one might say that the curve should be considered as produced—just as we suppose that straight lines are produced—as far as $\alpha\delta$; but this by no means follows from the principles laid down; but in order that this point θ may be assumed, the ratio of the quadrant to the straight line must be presupposed.'

He then adds, that 'unless this ratio is given, one should not—trusting to the authority of the inventors—accept a curve, which is rather of a mechanical kind ($\tau\eta\nu\ \gamma\rho\alpha\mu\mu\eta\nu\ \mu\eta\chi\alpha\nu\iota\kappa\omega\tau\acute{\epsilon}\rho\alpha\nu\ \pi\omega\varsigma\ \omicron\upsilon\delta\sigma\alpha\nu$).'⁸

Sporus was a mathematician whose solution of the Delian problem has been handed down by Eutocius in his Commentary on the treatise of Archimedes *On the Sphere and Cylinder*;⁹ this solution, he tells us, is the same as that of Pappus, which precedes it in Eutocius, and which is also given by Pappus himself in the third and eighth books of

⁸ *Ibid.* pp. 252, 254.

mentariis Eutocii, ed. Heiberg, vol. iii.

⁹ Archimedis, *Opera omnia cum com-*

pp. 90, 92.

his *Collections*.¹⁰ M. Paul Tannery thinks that Sporus was the teacher, or an elder fellow-pupil of Pappus, and places him towards the end of the third century of our era; and, further, he identifies him with Porus (Sporus) of Nicaea, the author of a collection entitled *Ἀριστοτελικά Κηρία* (see HERMATHENA, vol. iv. p. 188), which contained, according to M. Tannery, extracts from mathematical works relating to the quadrature of the circle and the duplication of the cube, as also a compilation in relation to the *Meteorologics* of Aristotle. M. Tannery is of opinion, moreover, that the historical works of Eudemus were driven out of the field at an early period by compilations from them, that the *History of Geometry* in particular did not survive the fourth century, and that this Collection of Sporus was the principal source from which Pappus, Simplicius, and Eutocius derived their information concerning these two famous geometrical problems.¹¹

In any case, it seems to me probable that a valuable fragment of the *History of Geometry* of Eudemus is preserved in the extracts from Pappus given above, whether they have been taken by Pappus from that *History*, or derived second-hand through Sporus [Porus].

On examining the demonstration of the property of the quadratrix given above, we see that the following theorems are required for it:—

(a). The circumferences of circles are to each other as their diameters.

(b). The arcs of two concentric circles, which subtend the same angle at their common centre, are to each other as the quadrants of those circles.

¹⁰ Pappi, *Op. cit.*, vol. i. p. 64, sq., vol. iii. p. 1070, sq.

¹¹ *Sur les fragments d'Eudème de Rhodes relatifs à l'histoire des mathématiques*; also, *Sur Sporus de Nicée*; *Annales de la Faculté des Lettres de*

Bordeaux, pp. 70-76, 257-261, 1882. Cf. *Pour l'histoire des lignes et surfaces courbes dans l'antiquité*. Bulletin des Sciences Mathém. et Astronom., 2^e série t. vii.

This theorem is an immediate consequence of Euclid, vi. 33 :—

(c). In equal circles, angles at the centre have the same ratio to each other as the arcs on which they stand.

We see, further, that the following assumptions are made in the proof:—

1°. An arc of a circle less than a quadrant is greater than the perpendicular let fall from one of its extremities on the radius drawn through the other ;

2°. And is less than the tangent drawn at one extremity of the arc to meet the radius produced through the other.

We notice, moreover, that the proof is indirect ; and it is, indeed, as Cantor has remarked, the first of the kind with which we meet.¹² We have seen, however, that Eudoxus must have been familiar with this method of reasoning (see HERMATHENA, vol. v., p. 224) ; and we know that Autolycus of Pitane, in Aeolis, who was a contemporary of Dinostratus, makes use of the argument :—*ὅπερ ἐστὶν ἄτοπον, or ἀδύνατον*, in many propositions of his book *Περὶ κινουμένης σφαίρας*.¹³

We see, too, that the investigation of Dinostratus, which gives a graphical solution of the determination of the ratio of the circumference of a circle to its diameter, is a complement to the work of Eudoxus, for the problem which was solved by means of the quadratrix arose naturally from the theorem that *circles are to each other as the squares on their diameters*.

It is to be observed, then, in the first place, that the problem which is solved above by means of the quadratrix is, in reality, the rectification of the quadrant, and that it

¹² Cantor, *Geschich. der Math.*, p. 213.

¹³ Autolyci, *Op. cit.*, pp. 12, 4 ; 14, 7 ; 24, 14 ; 32, 4 ; 8, 17 ; 22, 1.

is taken for granted that the quadrature of the circle—from which the name of the curve is derived—follows from its rectification. Secondly, we see that in order to make this inference the theorem—the area of a circle is equal to one-half the rectangle under the circumference, or four times the quadrant, and the radius—must be assumed. This theorem is equivalent to the first proposition of Archimedes, *Dimensio circuli*, referred to above. Lastly, it is noteworthy that the rectification of the quadrant is obtained by means of principles which are substantially the same as those assumed by Archimedes, and adopted by all geometers, ancient and modern.¹⁴

It seems to be a legitimate inference from this that these axioms must be referred back to Dinostratus, and most probably to Eudoxus.

Pappus, no doubt, in two places—v., prop 11, and viii., prop. 22—proves that the circumferences of circles are to each other as their diameters,¹⁵ and, in each place, makes the proof depend on the theorem cited above. He adds, however, in the former proposition:—‘The same may be proved without assuming that the rectangle under the diameter of a circle and its periphery is four times the circle. For the similar polygons, which are inscribed in circles, or circumscribed about them, have perimeters which have the same ratio to each other as the radii of the

¹⁴ ‘Nous partirons, pour la solution de ce problème [de la rectification des courbes], du principe d’*Archimède*, adopté par tous les géomètres anciens et modernes, suivant lequel deux lignes courbes, ou composées de droites, ayant leurs concavités tournées du même côté et les mêmes extrémités, celle qui renferme l’autre est la plus longue. D’où il suit qu’un arc de

courbe tout concave du même côté, est plus grand que sa corde, et en même temps moindre que la somme des deux tangentes menées aux deux extrémités de l’arc, et comprises entre ces extrémités et leur point d’intersection.’—Lagrange, *Théorie des Fonctions Analytiques*, p. 218. Paris, 1813.

¹⁵ Pappi, *Op. cit.*, vol. i., pp. 334, 336; vol. iii., pp. 1104, 1106.

circles, so that also the circumferences of circles are to each other as their diameters.'

Bretschneider thinks that the criticisms of Sporus are not of much importance, and says that they only come to this:—'That the quadratrix cannot be constructed geometrically, but is obtained only mechanically by means of a series of points, which must then be joined by a steady stroke of the free hand.'¹⁶ It seems to me, however, that these criticisms are just; and that Sporus and Pappus are right in maintaining that the description of the curve assumes the very thing for which the quadratrix is employed.¹⁷

Bretschneider shows that the theorem from which the quadratrix derives its name can be easily obtained by the infinitesimal method, 'by means of the proportion $\beta\epsilon\delta : \gamma\delta :: \epsilon\delta : \eta\lambda$, from the observation that the nearer the radius $\gamma\epsilon$ approaches to $\gamma\delta$, the more nearly does the sector $\gamma\epsilon\delta$ approach to a triangle similar to the triangle $\gamma\lambda\eta$; and therefore, for the limiting case, where $\gamma\epsilon$ and $\gamma\delta$ coincide, the ratio $\epsilon\delta : \eta\lambda$ actually passes over into that of $\gamma\delta : \gamma\theta$.' He adds:—'Such considerations have often served the old geometers as means for their discoveries, but are never used as proofs. The latter are always given through the *reductio ad absurdum*, which, indeed, allows no trace of the way followed in the inquiry to be recognized.'¹⁸ This observation is both just and important.

The same remark has been made by M. P. Laffitte, who points out that, in the establishment of any truth, there are

¹⁶ Bretschneider, *Geom. v. Eukl.*, p. 96.

¹⁷ 'Various other modes might be found of making either of these curves [the quadratrix of Dinostratus and the quadratrix of Tschirnhausen] square the circle; but the fact is that the descrip-

tion of the curves themselves assumes the point which their use is to determine.'—*English Cyclopædia*, sub. v., *Quadratrix*.

¹⁸ Bretschneider, *Geom. v. Eukl.*, p. 154.

two parts (or operations) which, he says, have not been hitherto sufficiently distinguished :

1°. The invention or the discovery of the proposition.

2°. Its proof.

And he further observes, that, after the discovery has been arrived at, the proof is often furnished by the method *ex absurdo*.¹⁹

In a former part of this Paper (HERMATHENA, vol. iv. pp. 220, *sq.*), I gave reasons in support of Hankel's opinion that the Hippias referred to by Proclus, in connexion with the quadratrix, is not Hippias of Elis.²⁰ As I mentioned, however, in giving them, I had not then read Cantor's defence of the common opinion ; but, on reading it subsequently, I was much struck with the force of his arguments, and introduced them in a note—the only course then open to me. M. Paul Tannery, in a Paper, the first part of which was published in the *Bulletin des Sciences Mathématiques et Astronomiques*, Octobre, 1883, and entitled, 'Pour l'histoire des lignes et surfaces courbes dans

¹⁹ P. Laffitte, *Les Grands Types de l'Humanité*, vol. ii., pp. 308, *et sq.* ; p. 328, *et seq.*

²⁰ For convenience of reference I quote them here :—

1. Hippias of Elis is not one of those to whom the progress of Geometry is attributed in the summary of the history of geometry preserved by Proclus, although he is mentioned in it as an authority for the statement concerning Ameristus [or Mamercus]. The omission of his name would be strange if he were the inventor of the quadratrix.

2. Diogenes Laertius tells us that Archytas was the first to apply an

organic motion to a geometrical diagram ; and the description of the quadratrix requires such a motion.

3. Pappus tells us that : 'For the quadrature of a circle a certain line was assumed by Dinostratus, Nicomedes, and some other more recent geometers, which received its name from this property : it is called by them the quadratrix.'

4. With respect to the observation of Montucla, I may mention that there was a skilful mechanic and geometer named Hippias contemporary with Lucian, who describes a bath constructed by him.

l'antiquité,'²¹ has criticized the reasons advanced by me against the common opinion:—

With reference to argument 1°, he replies:—‘This omission is sufficiently explained by the discredit under which the sophists laboured in the eyes of Eudemus; and the list in question presents a much more remarkable one—that of Democritus.’

With reference to 2°, he says:—‘This observation is not accurate. An indefinite number of points of the quadratrix, as near as one wishes, may be obtained by the ruler and compass; and it is doubtful whether the ancients sought any other process for the construction of this curve.’ M. Tannery continues:—‘The authority of Diogenes Laertius is, moreover, so much the less acceptable, inasmuch as he speaks in express terms of the solution of the Delian problem by Archytas. Now, Eutocius (*Archimedes*, ed. Torelli, pp. 143–144) has preserved to us, on the one side, this solution, in which there is not any employment of an instrument; and, on the other side (p. 145), a letter, in which Eratosthenes states that, “if Archytas, Eudoxus, &c., were able to prove the accuracy of their solutions, they could not realise them manually and practically, except, to a certain extent, Menaechmus, but in a very troublesome way.”’²²

‘The *Mesolabe* of Eratosthenes is, in fact, the oldest instrument of which the employment for a geometrical construction is known. This text indicates that, before Menaechmus, people were not engrossed with the practical tracing of curves; whilst the inventor of the conic sections would have tried, more or less, to resolve this question for the lines which he had discovered.’

As to these observations of M. Tannery, I admit that

²¹ *Bulletin des Sc. Math. et Astron.*,
2^e série, vii. 1 (1883), pp. 279 sq.

²² See HERMATHENA, volume v.,
p. 195.

Diogenes Laertius is not a safe guide in mathematics, as indeed I noticed in the first part of my Paper (HERMATHENA, vol. iii., p. 167, n. 16). In quoting him, I certainly did not mean to convey that, in my opinion, Archytas had actually traced the curve, used in his solution of the Delian problem, by any mechanical means; and I agree with M. Tannery that the letter of Eratosthenes is quite decisive on that point. At the same time it is evident that the conception of a curve being traced by means of motion is contained in the solution of Archytas, to whom, along with Philolaus, his master, and Eudoxus, his pupil, the first notions of mechanics are attributed. And with respect to the quadratrix itself, although, as M. Tannery remarks, an indefinite number of points on the quadratrix, as near as one wishes, can be obtained with the ruler and compass, yet the conception of motion is no less involved in the nature and very definition of the curve.

In reply to my observation 3°, M. Tannery says:—‘The divergence of the accounts given by Proclus and by Pappus is easily explained by the difference of the sources from which they drew. All that the former says of curves is undoubtedly borrowed from Geminus, an author of the first century before the Christian era; and his language proves that Geminus was acquainted with a writing of Hippias on the quadratrix, and regarded him as the inventor of this curve, though he was aware that Nicomedes also was engaged with it.’ M. Tannery continues:—‘As to Pappus, he quotes Geminus only *apropos* of the works of Archimedes on mechanics. He does not appear to have borrowed anything from him for geometry, particularly in the part which is concerned with curved lines and surfaces;’ and adds:—‘One can scarcely doubt but that Sporus was the source from which Pappus has derived what he says on the quadratrix.’ We have noticed this above.

With reference to 4°, M. Tannery says:—‘The existence of the Hippias referred to in it is by no means proved, for the writing in question seems to be only a pure fancy; but in any case it is impossible to think of any geometer posterior to Geminus, or even, as it seems to me, to Nicomedes.’

The suggestion which I made concerning Hippias, the contemporary of Lucian, was thrown out by me without sufficient consideration in reply to the observation of Montucla. Later, I became aware of the ideal character of that writing, and that it was the work of a *pseudo-Lucian*.²³

The result of the whole discussion seems to be: that the quadratrix was invented, probably by Hippias of Elis, with the object of trisecting an angle, and was originally employed for that purpose; that subsequently Dinostratus used the curve for the quadrature of the circle, and that its name was thence derived. This seems to be Cantor's view of the matter.²⁴ M. Tannery tells us that he, too, had at first interpreted the passage of Pappus in the same way as Cantor; but that, on further consideration, he thinks that it is open to grave objections. He says:—‘In the first place, the text of Geminus in Proclus clearly supposes that the name of the curve had been given to it by its inventor, Hippias. On the other hand, it is evident that the practical use of the curve implies the construction of a model cut in a square, having the quadratrix in place of the hypotenuse, and which could be applied, like our *protractor*, to the figures under consideration. Consequently, the determination of the intersection of the curve with the axis at once becomes necessary; and the problem is not, in

²³ See Zeller, *History of Greek Philosophy from the earliest period to the time of Socrates*, vol. ii., p. 422, n. 2,

E.T.

²⁴ Cantor, *Geschichte der Mathematik*, pp. 167 and 212.

reality, so difficult that we should think that Hippias was incapable of perceiving its relation to the quadrature of the circle. Finally, the fame of this last problem was at the time sufficiently great to lead Hippias to borrow from it the name of his curve, rather than from the problem which he had, without any doubt, considered in the first place.²⁵

These views of M. Tannery seem to me to be quite inadmissible, and are indeed quite inconsistent with what we know of Greek geometry (see HERMATHENA, vol. iv., p. 221 *et seq.*; vol. v., p. 223 *et seq.*).²⁶ The problem solved by means of the quadratrix must, as stated above, be regarded as the natural complement of the work of Eudoxus; and it is significant, therefore, that the solution was effected by Dinostratus, who probably was his pupil. Nor does the finding of the point of intersection of the curve with the axis necessarily involve the determination of π ; for, as seems to be suggested by Pappus, the required point might be regarded as determined by the production of the curve. The nature of the proof, too, which is indirect, appears to me to be post-Eudoxian. Should it be said that the theorem required for the determination of π was obtained first by the infinitesimal method, I would reply that it was not likely that this was done by Hippias of Elis, who was a senior contemporary of Democritus. If, then, the text in Proclus supposes that the name of the curve had been given to it by its inventor, it follows, in my opinion, that this could not have been Hippias of Elis. I am, however, on the whole, disposed to accept Cantor's view as given above.

²⁵ *Bull. des Sc. Math. et Astron.*, 2^e serie, vii., 1. p. 281.

²⁶ Cf. Heiberg, *Griechische und römische Mathematik*, Philologus, 1884, *Jahresberichte*, p. 474: 'Während Hankel p. 121 ff. die exhaustionsmethode auf Hippokrates zurückgehen

liess, und Cantor p. 209 die möglichkeit zugibt, hebt Allman, *Greek Geometry &c.* II. p. 221 ff. mit recht hervor, dass wir nicht berechtigt sind, diese methode für älter als Eudoxus zu halten.'

Pappus has preserved the name, and given some account of the work, of one other great geometer, who was a predecessor, and probably a senior contemporary of Euclid—Aristaeus the Elder. We have no details whatever of his life.

The passages in Pappus relating to him are as follows :—

(a) 'That which is called $\delta \text{ ἀναλυόμενος [τόπος]}$,²⁷ that is, the department of mathematics which treats of analysis, is, in short, a certain peculiar matter prepared for those who, having gone through the elements, wish to acquire the power of solving problems proposed to them in the construction of lines; and it is useful for this purpose only. It has been treated of by three men—Euclid, the author of the Elements, Apollonius of Perga, and Aristaeus the elder—and proceeds by the method of analysis and synthesis.'²⁸

Pappus, having defined analysis and synthesis, proceeds to give a complete list of the books, arranged in

²⁷ [τόπος] $\delta \text{ καλούμενος ἀναλυόμενος. τόπος, 'locus, i. e. quicquid aliqua mathematicarum parte comprehenditur: } \delta \text{ ἀστρονομούμενος τόπος, vi. 474, 3; } \delta \text{ ἀναλυόμενος τόπος, vii. 672, 4.}'$ *Index Graecitatis*, Pappi, *Op. cit.*, voluminis iii., tomus ii., p. 114. ' $\delta \text{ ἀναλ. τόπ., locus de resolutione, id est doctrina analytica.}'$ *Ibid.* sub voce, *ἀναλύνειν*, p. 5. Compare what Marinus says on the same subject in his Commentary on the *Data* of Euclid:

'What is the value of the treatise about *Data* ?'

'The *datum* having been divided in a general way, and as far as is sufficient for the present need, the next point is to state the utility of treatment of

the subject. This also is one of those things which have their result in relation to something else. For the knowledge of this is necessary in the highest degree for $\tauὸν \text{ ἀναλυόμενον τόπον}$ as it is called; and how much value $\delta \text{ ἀναλ. τόπ.}$ has in mathematical science, and the kindred science of optics and music, has been defined elsewhere, and that analysis is the discovery of a proof, and that it helps us to the discovery of things similar, and that it is more important to possess the analytical faculty than to have many proofs of particular things.' *Euclidis Data*, ed. Cl. Hardy, p. 13. Cf. Pappi, *Op. cit.*, Appendix, p. 1275.

²⁸ Pappi, *ibid.* vii., vol. ii. p. 634.

order, which are contained in the τόπ. ἀναλ. He enumerates thirty-three books in all, amongst which we find 'five books of Aristaeus on Solid loci' (Ἀρισταίου τόπων στερεῶν πέντε): the remaining books, with the exception of two by Eratosthenes concerning *means* (περὶ μεσοτήτων δύο), were written by Euclid and Apollonius.²⁹

(b) ' [These plane problems then, are found in the τόπ. ἀναλ., and are set out first, with the exception of the *means* of Eratosthenes; for these come last. Next to plane problems order requires the consideration of solid problems. Now, they call solid problems, not only those which are proposed in solid figures, but also those which, not being capable of solution by plane loci, are solved by means of the three conic lines, and so it is necessary to write first concerning these. Five books of the *Elements of Conics* were first published by the elder Aristaeus, which were written in a compendious manner, inasmuch as those who took up the study of them were now able to follow him].³⁰

(c) 'Apollonius, completing Euclid's four books of conics, and adding four others, published eight volumes of conics. But Aristaeus, who wrote the five volumes of solid loci, which have come down to the present time, in continuation of the conics (Ἀρισταῖος δὲ, δς γέγραφε τὰ μέχρι τοῦ νῦν ἀναδιδόμενα στερεῶν τόπων τεύχη ἑ συνεχῇ τοῖς κωνικοῖς), called [as also did those before Apollonius] the first of the three conic lines, the section of the acute-angled cone; the second, the section of the right-angled cone; the third, the section of the obtuse-angled cone. But since in each of these three cones, according to the way in which it is cut, these three lines exist, Apollonius, as it appears, felt a difficulty as to why at all his predecessors distinguished

²⁹ *Ibid.*, p. 636.

³⁰ *Ibid.*, p. 672. 'τὰ μὲν—γεγραμμένα, interpolatori tribuit Hultsch.'

The spaced words are supplied in translation.

by name the section of an acute-angled cone, which might also be that of the right-angled and obtuse-angled cone; and, again, the section of the right-angled cone, which might also be that of the acute-angled and obtuse-angled cone; and the section of the obtuse-angled cone, which might also be that of the acute-angled and the right-angled cone. Wherefore, changing the names, he called that which had been named the section of the acute-angled cone, the ellipse; the section of the right-angled cone, the parabola; and the section of the obtuse-angled cone, the hyperbola—each from a certain peculiar property. For the rectangle applied to a certain straight line in the section of the acute-angled cone is deficient (ἐλλείπει) by a square; in the section of the obtuse-angled cone it is excessive (ὑπερβάλλει) by a square; finally, in the section of the right-angled cone the rectangle applied (παρὰβαλλόμενον) is neither deficient nor excessive.

‘[But this happened to Aristaeus, since he did not perceive that, according to a peculiar position of the plane cutting the cone, the three curves exist in each of the cones, which curves he named from the peculiarity of the cone. For if the cutting plane be drawn parallel to one side of the cone, one only of the three curves is generated, and that one always the same, which Aristaeus named the section of that so cut cone.]’³¹

(d) ‘But as to what he [Apollonius] says in the third book, that the locus with three or four lines has not been completed by Euclid—for neither he himself, nor anyone else, could [solve that locus] by those conical [theorems] only which had been proved up to the time of Euclid, as also he himself testifies, saying that it was not possible to complete it without those things which he was compelled to discuss

³¹ *Ibid.*, p. 672, l. 18–p. 674, l. 19.
‘l. 12. τοῦτο δ’ ἐπαθεν (scil. δ’ Ἀρισταίος)—l. 19. τομήν interpolator tri-

buit Hultsch.’ Cf. Procli, *Comm.*, ed. Friedlein, pp. 419, 420. See also HERMATHENA, vol. v. p. 417.

before-hand—[as to this, Euclid, approving of Aristaeus as a worthy mathematician on account of the conics which he had handed down, and not being in haste, nor wishing to lay down anew the same treatment of these subjects (ὁ δὲ Εὐκλείδης ἀποδεχόμενος τὸν Ἀρισταῖον ἄξιον ὄντα ἐφ' οἷς ἤδη παραδεδώκει κωνικοῖς, καὶ μὴ φθάσας ἢ μὴ θελήσας ἐπικαταβάλλεσθαι τούτων τὴν αὐτὴν πραγματείαν)—for he was most kind and friendly to all those who were able to advance mathematics to any extent, as is right, and by no means disposed to cavil, but accurate, and no boaster like this man Apollonius—wrote as much as could be proved by his conics: sc. those of Aristaeus concerning that locus—not attributing any finality to his demonstration, for then it would be necessary to blame him, but, as it is, not at all; since Apollonius also himself, who left many things in his conics unfinished, is not brought to task for it. But he Apollonius has been able to add to that locus (τῷ τόπῳ) what was wanting, having been furnished with the ideas by the books already written by Euclid on the same locus (περὶ τοῦ τόπου), and having been for a long time a fellow-pupil of the disciples of Euclid in Alexandria, from which source he derived his habit of thought, which is not unscientific. Such is this locus with three or four lines, on which he plumes himself greatly, adding, that he knew that he owed thanks to him who first wrote about it.]³²

(e) We learn from Hypsicles that Aristaeus wrote a book on the *Comparison of the five regular solids*, and that it contained the theorem: 'The same circle circumscribes the pentagon of the dodecahedron and the triangle of the

³² *Ibid.*, p. 676, l. 19—p. 678, l. 15. 'l. 25. δὲ Εὐκλείδης—p. 678, l. 15, τοιοῦτός ἐστιν, scholiastae cuidam historiae quidem veterum mathematicorum non imperito, sed qui dicendi genere languido et inconcinno usus sit,

tribuit Hultsch,' *Ibid.* p. 677. As Hultsch says, 'the writer of this passage has employed a feeble and awkward manner of expression'; and it is difficult to see the exact meaning of it. The spaced words are supplied in translation.

icosahedron, these solids being inscribed in the same sphere'. Hypsicles says, further, that 'this theorem is also given by Apollonius in the second edition of his *Comparison of the dodecahedron with the icosahedron*,³³ which is: The surface of the dodecahedron is to the surface of the icosahedron as the dodecahedron itself is to the icosahedron; since the perpendiculars from the centre of the sphere to the pentagon of the dodecahedron and to the triangle of the icosahedron are the same'.³⁴

The foregoing extracts lead us to form a high opinion of Aristaeus, and to see that he was one of the most important geometers before Euclid. We have, therefore, great reason to regret the total loss of his writings.

In the passage (a) Aristaeus, Euclid, and Apollonius are named as the three authors on the doctrine of analysis. This passage shows, further, the value that was attached by the ancients to the five books of Aristaeus on *solid loci*, which was one of the works—indeed one of the higher works—included in the τόπ. ἀναλ. From the passage (b) it would appear that Aristaeus published also a work on the *elements of conics* in five books—an abridgment introductory to the study of solid loci. Of his work on *solid loci* it is, moreover, stated in (c): 'Αρισταῖος δέ, ὅς γέγραφε τὰ μέχρι τοῦ νῦν ἀναδιδόμενα στερεῶν τύπων τεύχη ἐσυνεχῇ τοῖς κωνικοῖς. This passage admits of several interpretations:—

1. That the work on solid loci was intended as an extension of the theory of conics;

2. Aristaeus first wrote the τύποι στερεοί in five books, and then, to facilitate the study of them, he wrote the κωνικὰ στοιχεῖα—an epitome—also in five books;

3. τοῖς κωνικοῖς might possibly refer to the conics of Euclid.

³³ πέπτε σχημάτων σύγκρισις.

book is in reality the work of Hyp-

³⁴ Euclid, Book xiv., Prop. 2. This

sicles.

We learn further from (c) that Aristaeus gave to the conic sections their original names, those by which they were known before Apollonius.³⁵ From (d) we learn that Euclid praised the conics of Aristaeus, whom he valued highly, and from the words ἐφ' οἷς ἤδη παραδεδώκει κωνικοῖς, and φθάσας, it has been concluded that he was a predecessor, and probably a senior contemporary of Euclid.³⁶

We have seen that the passage (b) is regarded by Hultsch as an interpolation. In this Heiberg agrees, and infers thence that Aristaeus wrote only one work on the conic sections—τόποι στερεοί in five books—and holds that the generally received opinion that Aristaeus, besides the five books τόποι στερεοί, had written five more books κωνικὰ στοιχεῖα is not sufficiently well founded. He says: 'The only passage which can be adduced for it, Pappus vii., p. 672, 11: ἦν μὲν οὖν ἀναδεδομένα κωνικῶν στοιχείων πρότερον Ἀρισταίου τοῦ πρεσβυτέρου ἐ τεύχη, ὡς ἂν ἤδη δυνατοῖς οὖσι τοῖς ταῦτα παραλαμβάνουσιν ἐπιτομώτερον γεγραμμένα, is rightly rejected by Hultsch as not genuine,' and continues, 'It occurs in a perfectly wrong place where Apollonius περὶ νεύσεων is referred to, is objectionable in many respects in point of language, and contains nothing but what a reader of Pappus already would find in him; I believe, therefore, that we, in the words p. 672, 4-14, have a scholium which originally stood in the margin after p. 672, 16, and later fell into the text in a wrong place: the scholiast has then called the five books τόποι στερεοί, here incorrectly στοιχεῖα κωνικά. And even were the passage genuine (and only misplaced) the probability would be then that Pappus here by στοιχεῖα κωνικά had meant the τόποι'.³⁶

With this conclusion of Heiberg I cannot agree. In the first place, it should be observed that the passages of Pappus enclosed by Hultsch in [] are to be considered

³⁵ Cf. HERMATHENA, v., pp. 416, 417.

³⁶ J. L. Heiberg, *Studien über Euclid*, p. 85.

as interpolations for reasons of style, not of substance. The passage referred to was either written by Pappus himself (as Cantor and others assume), or it originated with an experienced commentator (scholiast), whose statements in other passages also are acknowledged as correct—or, to doubt which there is no occasion; or else these scholia contain remnants of the tradition of the mathematical school of Alexandria, and this tradition must be considered on the whole as correct, so long as the contrary is not proved.³⁷

In the next place, Heiberg is not correct in saying that 'it is the only passage which can be adduced for it.' The same statement is made expressly in the text of Pappus himself, a few lines lower down, in the passage quoted above: 'Αρισταῖος δέ, ὃς γέγραφε τὰ μέχρι τοῦ νῦν ἀναδιδόμενα στερεῶν τύπων τεύχη ἐσυνεχῇ τοῖς κωνικοῖς (p. 672, l. 20). Heiberg tries to obviate this objection by interpreting *συνεχῇ* as meaning: 'which stands in connexion with the doctrine of the conic sections—depends on it'.³⁸ In passage (d), moreover, the conics of Aristaeus are, I think, directly referred to in the words: διὰ τῶν ἐκείνου [Ἀρισταίου] κωνικῶν. Heiberg, further, says that the interpolation, or scholium, occurs in a perfectly wrong place; but, as he shows, it has to be placed only two lines lower. My view of the matter is that given above, p. 123, 2:—Aristaeus first wrote the *τόποι στερεοί* in five books, and then, to facilitate the study of them, he wrote the elements of Conics—an epitome—also in five books.

³⁷ It is certain that Pappus had a school. It may, therefore, be assumed that one—or perhaps several—of his pupils had taken notes of his lectures; and that these notes, arising thus from the oral exposition of Pappus himself, were worked out further by his pupils, and formed Commentaries, which were

then written on the margin, and subsequently received into the text, of the work which has come down to us as Πάππου συναγωγή. These Commentaries are easily recognized by their style, but as to their contents, they must be considered to be of almost equal authority with the undoubted text of Pappus.

The *Conics* of Aristaeus, no doubt, do not appear in the list of books contained in the so-called τόπος ἀναλυόμενος; neither do those of Euclid: they were both replaced by the *Conics* of Apollonius in eight books.

We have seen that Aristaeus wrote a work on the comparison of the five regular solids, and that it contained the theorem: The same circle circumscribes the pentagon of the dodecahedron and the triangle of the icosahedron, these solids being inscribed in the same sphere (*e*).

If we examine the proof of this theorem as given by Hypsicles, we see that it depends on the following theorems:—

1. If a regular pentagon be inscribed in a circle, the square on a side, together with the square on the line subtending two sides of the pentagon, is five times the square on the radius of the circle;

2. If the line subtending two sides of a regular pentagon be cut in extreme and mean ratio, the greater segment is the side of the pentagon. Euclid, xiii. 8;

3. The side of a regular decagon inscribed in a circle is the greater segment of the radius cut in extreme and mean ratio;

4. The square on the side of a regular pentagon inscribed in a circle is equal to the sum of the squares on the sides of the regular hexagon and decagon inscribed in the same circle. Euclid, xiii. 10;

5. If an equilateral triangle be inscribed in a circle, the square on the side is three times the square on the radius. Euclid, xiii. 12;

6. The square on the diameter of a sphere is three times the square on the side of the inscribed cube. Euclid, xiii. 15;

7. The line subtending two sides of the pentagon of a dodecahedron inscribed in a sphere is the side of the cube inscribed in the same sphere;

This follows from (2) taken with the corollary of xiii. 17:

If the side of the cube be cut in extreme and mean ratio, the greater segment is the side of the dodecahedron ;

8. The square on the diameter of a sphere is five times the square on the radius of the circle by means of which the icosahedron is described—*i. e.* the circle circumscribing the pentagon which forms the base of the five equilateral triangles having for common vertex any vertex of the icosahedron. Euclid, xiii. 16, and Corollary.

From the fact that 'the work of Aristaeus on the *Comparison of the regular solids* is the newest and last that treated, *before* Euclid, of this subject,' Bretschneider infers that 'the contents of the thirteenth book of the Elements is a recapitulation, at least partial, of the work of Aristaeus'.³⁸ This supposition of Bretschneider receives, I think, great confirmation from the above examination, which shows that the principal propositions in Book xiii. of the Elements are required for the demonstration, as given by Hypsicles, of the theorem of Aristaeus. This theorem, moreover, goes beyond what is contained in the Elements on this subject.

Further, one of the four problems treated of by Pappus in the third book of his *Collection* is the inscription in the sphere of the five regular polyhedra. M. Paul Tannery has thrown out the suggestion that it is probably taken from the *Comparison of the five figures* by Aristaeus the elder, but has given no reasons for his opinion.³⁹ In support of this conjecture I would put forward that :—

1. Pappus concludes his treatment of the subject by saying that 'from the construction it is evident that the same circle circumscribes the triangle of the icosahedron and the pentagon of the dodecahedron inscribed in the same sphere,'⁴⁰ which is the theorem of Aristaeus, and ex-

³⁸ *Geom. v. Eukl.*, p. 171.

³⁹ *L'Arithmétique des Grecs dans Pappus*, Mémoires de la Société des

Sciences Phys. et Nat. de Bourdeaux, 2^e Série. Tome iii., p. 351, 1880.

⁴⁰ Pappus, *Op. cit.*, vol. i., p. 162.

expressed, moreover, in nearly the same words as in Hypsicles ;

2. Pappus says in Book vii., as we have seen, p. 119, that the works in the τόπος ἀναλυόμενος—of which the τόποι στερεοί of Aristaeus is one—proceed by the method of analysis and synthesis ; and it is to be observed that the investigation in Pappus of the problem, ‘to inscribe the regular solids,’ is made by the analytical method ;⁴¹

3. Pappus, moreover, in Book v., treats of ‘the comparison of the five figures having equal surface, viz. the pyramid, cube, octahedron, dodecahedron and icosahedron,’ and says that he will do so, ‘not by the so-called analytic method, by which some of the ancients (τῶν παλαιῶν) found their proofs, but by the synthetic method arranged by him in a more perspicuous and shorter manner’—ἐξῆς δὲ τούτοις γράψομεν, ὡς ὑπεσχόμεθα, τὰς συγκρίσεις τῶν ἴσην ἐπιφάνειαν ἔχόντων πέντε σχημάτων, πυραμίδος τε καὶ κύβου καὶ ὀκταέδρου δωδεκαέδρου τε καὶ εἰκοσαέδρου, οὐ διὰ τῆς ἀναλυτικῆς λεγομένης θεωρίας, δι’ ἧς ἔνιοι τῶν παλαιῶν ἐποιοῦντο τὰς ἀποδείξεις, ἀλλὰ διὰ τῆς κατὰ σύνθεσιν ἀγωγῆς ἐπὶ τὸ σαφέστερον καὶ συντομώτερον ὑπ’ ἐμοῦ διεσκευασμένης.⁴²

The *theorem of Aristaeus* can be proved in the following simple manner :—

If a regular dodecahedron be inscribed in a sphere, the poles of its faces will be the vertices of a regular icosahedron inscribed in the same sphere ; and, conversely, the vertices of the dodecahedron will be the poles of the faces of the icosahedron. Now let *R* be the pole of the circle circumscribing the pentagon *ABCDE* of the dodecahedron, and let *S* and *T* be the poles of the circles circumscribing the two other pentagons of the dodecahedron which have the vertex *A* in common : then *A* will be the pole of the circle circumscribing the triangle *RST* of the icosahedron.

⁴¹ *Ibid.*, pp. 142–162.

⁴² *Ibid.*, pp. 410, 412.

Now, if the points R and A be joined to O , the centre of the sphere, the lines OR , OA so drawn will be at right angles to the planes $ABCDE$, and RST respectively: let them intersect these planes at the points P and Q respectively. Then the two right-angled triangles ORQ , OAP —having equal hypotenuses OR , OA , and common angle ROA —will be equal in every respect; therefore $OP = OQ$ and $AP = BQ$. But AP and BQ are the radii of the circles circumscribing the pentagon of the dodecahedron and the triangle of the icosahedron, and OP , OQ are the perpendiculars drawn from the centre to these two planes.

In the first part of this Paper (HERMATHENA, vol. iii., pp. 194 sq.), we saw that ‘the Pythagoreans were much occupied with the construction of regular polygons and solids, which in their cosmology played an essential part as the fundamental forms of the elements of the universe’:⁴³ and in the second part (HERMATHENA, vol. iv., pp. 213 sq.),

⁴³ These Pythagorean ideas—which were adopted by Plato Πλάτων δὲ καὶ ἐν τοῖς πύθαγορίσι (see HERMATHENA, vol. iv., p. 213, n. 75)—played such an important part in antiquity that they gave rise to the belief, related by Proclus, that Euclid ‘proposed to himself the construction of the so-called Platonic bodies [the regular solids] as the final aim of his systematization of the Elements’. (See HERMATHENA, vol. iii., p. 164). This has been noticed by P. Ramus, who says: ‘Nihil in antiqua geometria speciosius visum est quinque corporibus ordinatis, eorumque gratia geometriam ut ex Proclo initio dictum est, inventam esse veteres illi crediderunt’; but he adds: ‘At in totis elementis nihil est istis argutiis ineptius et inutilius’.*

It may be interesting to some of the readers of this Paper to know that

William Allman, M.D., Professor of Botany in the University of Dublin (1809–1844), and father of the writer, in a Memoir entitled: An attempt to Illustrate a Mathematical Connexion between the Parts of Vegetables (read before the Royal Society of London in the year 1811), put forward the hypothesis that the minute cells in the young shoots of vegetables are of the dodecahedral form in Dicotyledonous plants; and of the icosahedral form in Monocotyledonous plants; and that by means of this hypothesis he accounted for the prevalence of the number 5, and the exogenous growth in the former, and of the number 3, and the endogenous growth in the latter.

* (Petri Rami Scholarum Mathematicarum, Libri unus et triginta. Francofurti, 1599, p. 306.)

I pointed out a problem of high philosophical importance to the Pythagoreans, which, in my judgment, naturally arose from their cosmological speculations, and which required for its solution a knowledge of stereometry, and also the solution of the famous problem: *to find two mean proportionals between two given lines*. In the same part (p. 215) I indicated the men who first solved this problem, and laid the foundation of stereometry; in the two following parts (HERMATHENA, vol. v., pp. 190 *sq.*, pp. 212 *sq.*, and pp. 403 *sq.*) I examined their work; and finally in this portion we have seen that Aristaeus wrote works on the conic sections and on the regular solids, and, further, that he is specially mentioned as one of those who cultivated the analytic method—the method by the aid of which these discoveries were made, as stated in HERMATHENA, vol. iv., p. 215. Aristaeus may, therefore, be regarded as having continued and summed up the work, which, arising from the speculations of Philolaus, was carried on by his successors—Archytas, Eudoxus, and Menaechmus. These men were related to one another in succession as master and pupil, and it seemed to me important that the continuity of their work should not be broken in its presentation.

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