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The Origins of Number Concepts

Experiments with children indicate that they first become aware of numbers in terms of ordered sequences and only later in terms of quantities. Such findings may give rise to a newer "new math"

by Charles J. Brainerd

How do we first learn concepts of number? The question has been a topic of debate among philosophers and mathematicians at least since the time of Pythagoras in the sixth century B.C. Today it continues to be important not only to scholars but also to society as a whole; witness the triumph of the "new math" in the elementary schools of America during the past decade. The fact remains that comprehensive studies of the origin of number concepts in young children are relatively rare. I have recently participated in investigations at the University of Alberta that have developed some new information on the subject. Before I describe these studies it will be useful to review the main theories of the origin of number concepts.

→ Pythagoras believed that what we call the positive integers or natural numbers (1, 2, 3 and so on) were god-given entities that formed the ultimate foundation both of mathematics and of the universe. The Pythagoreans' own discovery of such "incommensurable" quantities as the ratio between the diameter of a circle and its circumference ultimately dispelled the belief that the universe was built on natural numbers. That the natural numbers provided the foundation of mathematics, however, persisted as an article of faith among mathematicians until well into the 19th century.

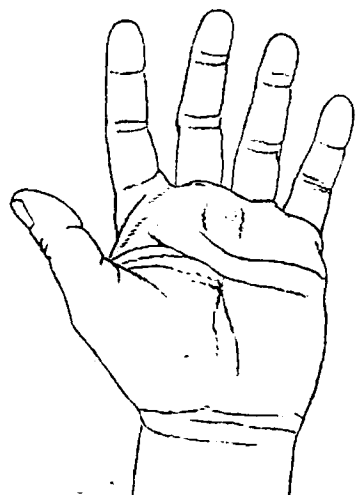
By then the attitude toward the natural numbers had begun to change. The centrality of natural numbers was no longer considered an accepted fact but was viewed as a conjecture that required rigorous proof. The proofs usually took the form of a stepwise derivation of such well-known number systems as rational, real and complex numbers from the natural numbers themselves. Two examples are the attempts of Karl Weierstrass and

Richard Dedekind to "arithmetize" mathematical analysis. Both scholars derived real numbers—the combined set of all rational and irrational numbers that is employed in most classical mathematics—from the natural numbers. A third example is the proposal of Leopold Kronecker to found all mathematics on the natural numbers. This Kronecker attempted to accomplish solely with "finitary" methods, that is, methods invoking neither nonfinite entities nor proofs involving more than a finite number of steps.

Still other mathematicians, in particular those who were conversant with contemporary advances in symbolic logic, put forward the suggestion that, far

from being god-given, natural numbers were constructions of the human mind. The three most famous propagators of this suggestion were Gottlob Frege, Giuseppe Peano and Bertrand Russell. Obviously a theory was needed that would trace the rise of the natural numbers from some more basic notion or notions, but how was such a theory to be constructed? If most or all of classical mathematics had evolved from the natural numbers, it was improbable that the required theory could be devised entirely within the bounds of classical mathematics.

First Frege, then Peano and finally Russell turned to symbolic logic as a potential source of the fundamental no-



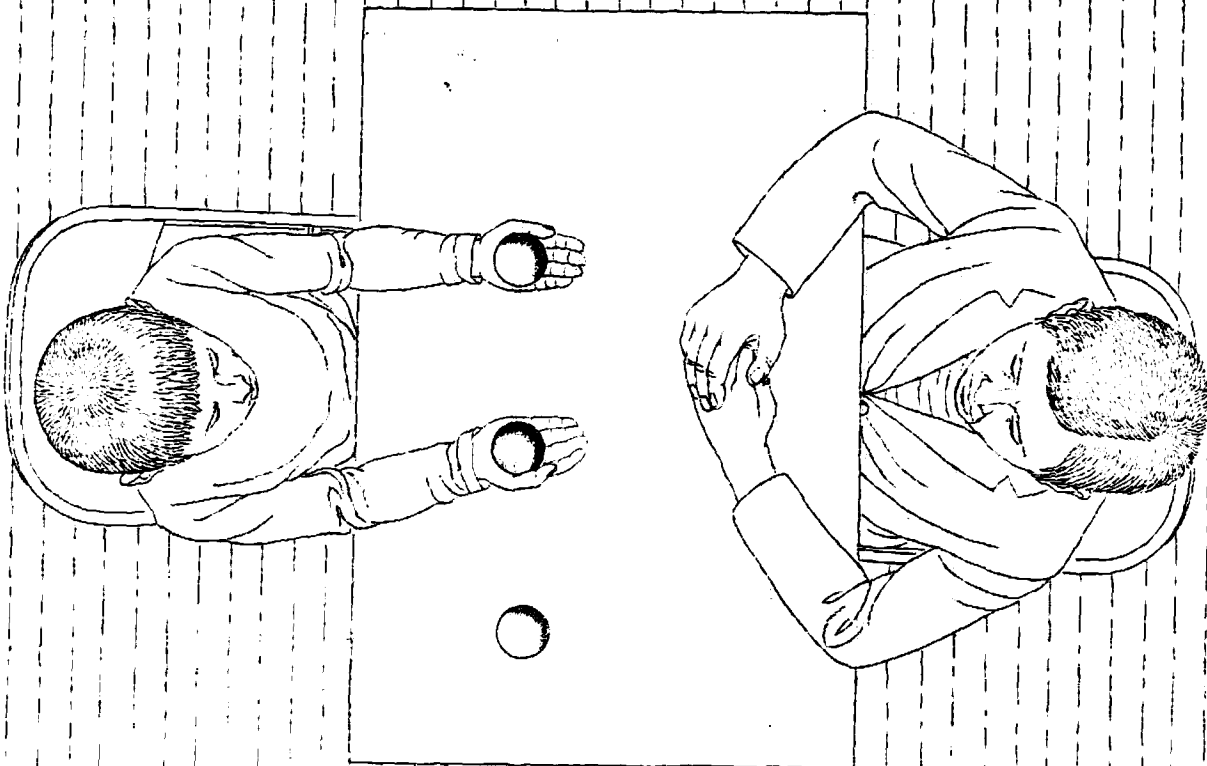
CARDINATION

CARDINATION determines the number of objects in a set by matching rather than by counting. In this example the objects correspond in number to the five fingers of a hand; thus the set's cardinal number is 5.



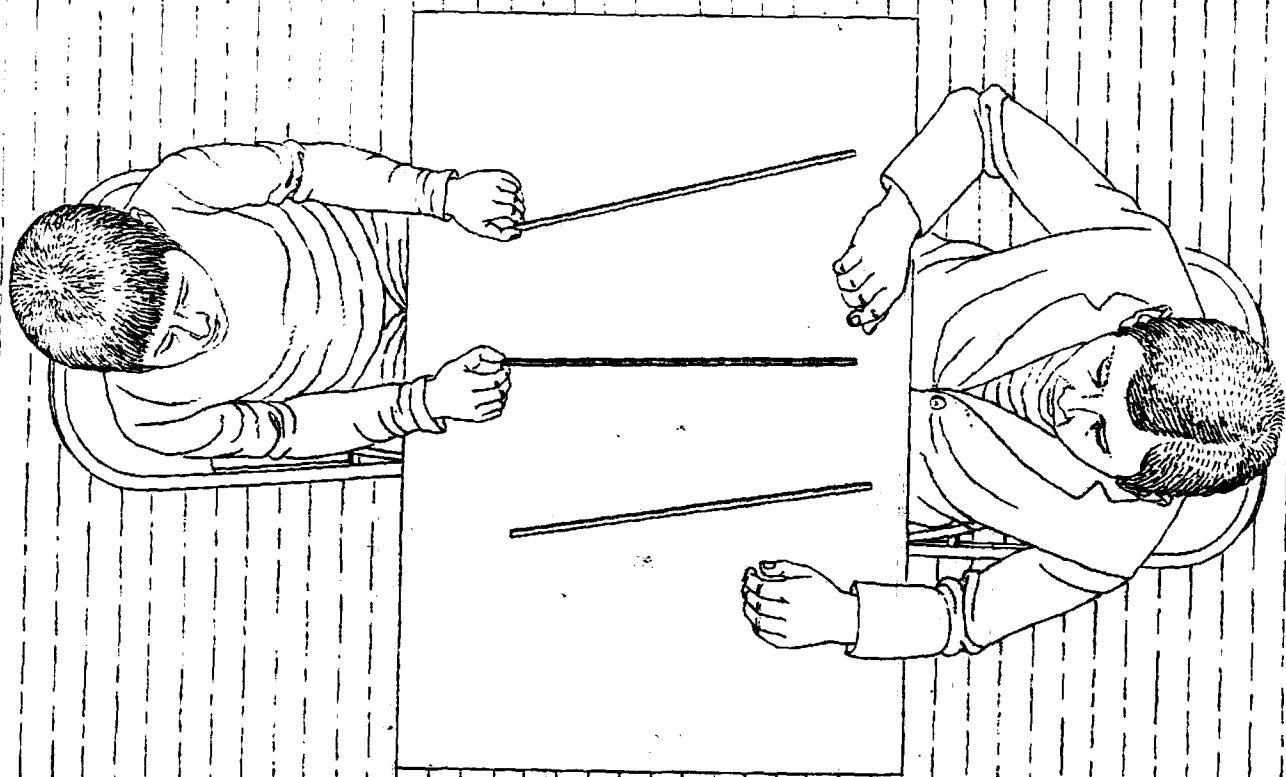
ORDINATION

ORDINATION determines the number of objects in a set by counting "One, two, three" and so on rather than by matching. In this set, as in all finite sets, the last ordinal number, 5, is the cardinal number of the set.



CONCEPT OF ORDINATION in terms of "heavier than/lighter than" was tested by asking the subjects to lift clay balls that were identical in size but different in weight. In some tests the heaviest

ball was at the right end of the row and the lightest at the left; in other tests the order was reversed. The comparisons in all the tests were with a third ball (middle), which was intermediate in weight.



SECOND ORDINATION TEST, involving the relation "longer than/shorter than," presented the subjects with three lengths of wooden dowel; the longest stick was half a centimeter longer and the shortest stick half a centimeter shorter than the third stick.

which was always placed between the other two. As with the weight test, the order of presentation was randomly left to right or right to left. Subjects compared long and short sticks with intermediate stick (color) but never directly compared long stick with short.

tions necessary for a theory of natural number. Frege was the first of the three to publish a specific theory. In *Die Grundlagen der Arithmetik*, which appeared in 1884, he proposed that the natural numbers could be reduced to the notion of "class" and the operation of "correspondence," by virtue of which classes are quantified. According to Frege, each natural number n was to be regarded as a "superordinate class" whose members, "subordinate classes," each contain precisely n elements. Given two subordinate classes, A and B , the two are said to be members of the same superordinate class, that is, instances of the same number, if and only if a one-to-one correspondence can be established between their respective elements. If instead the correspondence is many to one, then A and B are said to be instances of different numbers.

In essence Frege's theory states that the series of natural numbers presents a general problem of quantification, but that the general problem can be reduced to the more restricted notion of "cardination," or quantifying classes. The commonest example of cardination is the matching of things.

Frege's cardinal theory remained generally unknown until Russell rediscovered it in 1901. Russell subsequently published the cardinal theory, with full acknowledgement to Frege, in his *Principles of Mathematics* (1903), in his joint work with Alfred North Whitehead, *Principia Mathematica* (1910-1913), and in his *Introduction to Mathematical Philosophy* (1919).

Between the time Frege first published the cardinal theory and the time Russell rediscovered it Peano developed a second theory about the natural numbers. This theory first appeared in his work *Formulaire de Mathématiques* (1894) in the form of five axioms that I shall slightly reword here. First, 1 is a natural number. Second, any number that is the successor of a natural number is itself a natural number. Third, no two natural numbers have the same successor. Fourth, the natural number 1 is not the successor of any natural number. Fifth, if a series of natural numbers includes both the number 1 and the successor of every natural number, then the series contains all natural numbers.

In essence Peano's theory places the natural numbers in an ordinal relation or, in the language of symbolic logic, a "transitive, asymmetrical relation." If we are willing to stipulate that the relation R that obtains between every nonidenti-

cal pair of natural numbers be an ordinal relation, then the complete series of natural numbers can be constructed stepwise with the aid of the rule of mathematical induction. Like Frege's cardinal theory, Peano's states that the series of natural numbers presents a general problem of quantification. Unlike Frege's theory, however, Peano's ordinal theory reduces the general problem to the more restricted notion of quantifying transitive, asymmetrical relations, or ordination. The commonest example of ordination is the counting of things.

Just which of the two theories, the cardinal or the ordinal, is mathematically preferable is a question that has never been answered to everyone's satisfaction. Reasonable objections can be lodged against both. For example, the cardinal theory is subject to the celebrated paradox, discovered by Russell in 1901, concerning the class composed of all those classes that are not members of themselves. Briefly, if Y is the class of all those classes that are not members of themselves, then it can be shown that Y both is and is not a member of itself.

With respect to the ordinal theory, as Russell pointed out, whereas Peano's five axioms obviously are satisfied by the series of natural numbers, they are equally satisfied by other number systems. For example, the rational fractions ($1, 1/2, 1/3, 1/4$ and so on) satisfy the axioms, as will any series of mathematical or empirical entities that has a beginning, no repetitions and no end and is such that every entity can be reached in a finite number of steps. In short, the domain of application of the ordinal theory is much wider than the series of natural numbers.

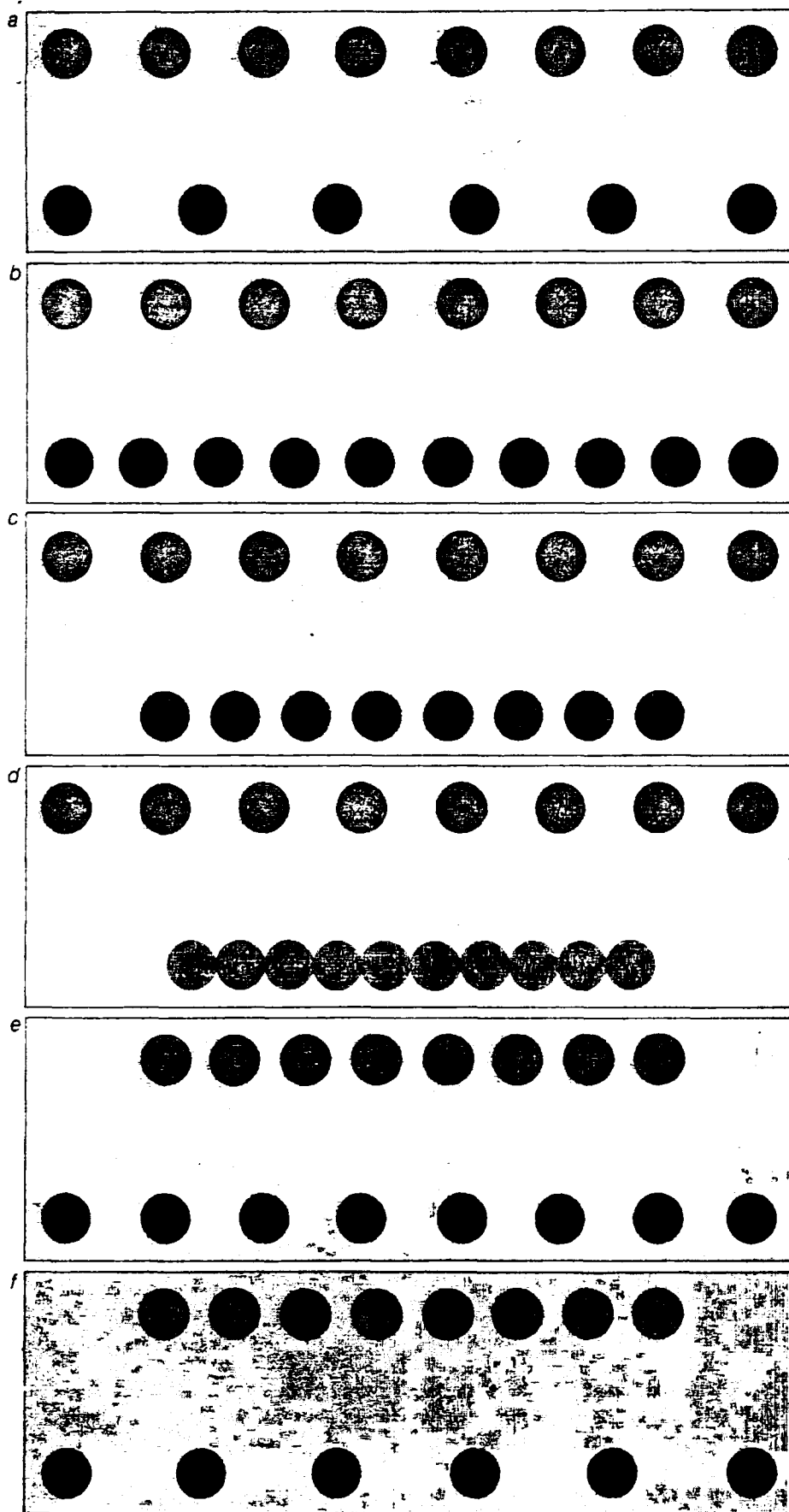
Because there is no universally accepted mathematical basis for choosing between the cardinal and the ordinal theories the choice becomes a subjective matter. Typically the choice is determined by one's degree of sympathy with one or another of three modern schools of mathematical thought: logicism, formalism and intuitionism. Those who lean toward logicism favor the cardinal theory, a choice that is natural enough when one considers that the codiscoverers of the theory, Frege and Russell, were the principal founders of logicism. Those whose sympathies are with formalism lean toward the ordinal theory; the fact that Peano's axioms seem to denude the number concept of innate "meaning" probably explains this preference. As for intuitionists, they have in effect returned to the Pythagorean position that the natural numbers must

be accepted without further analysis as the foundation of mathematics. They deny that the natural numbers are the invention of mathematical minds and offer a "psychological" thesis: The series of natural numbers is an innate intuition, present at birth in all normal members of the human species.

Nonmathematical scholars tend to view with profound indifference the tortures that mathematicians suffer over such basic issues as the nature of number. They have learned from centuries of hard experience that the mere fact that the foundations of some mathematical system or concept are not secure need not deter them from employing the system in their work. On the contrary, mathematical notions whose foundations have been matters of continuous debate have often yielded the most mileage; the notion of an infinitesimal is perhaps the best-known example.

An infinitesimal can be defined as a quantity that, although it is infinitely small, is nevertheless greater than zero. The calculus, as it was originally formulated by Newton and Leibniz, involved infinitesimals; Newton called them fluxions. Intense disagreement followed over the question of whether such a curious item as an infinitesimal could be viewed as a legitimate mathematical concept. By the latter half of the 19th century the debate seemed to have been finally silenced by Weierstrass' success in reformulating the calculus without infinitesimals. Within the past decade, however, Abraham Robinson's work on nonstandard analysis has resurrected the infinitesimal [see "Nonstandard Analysis," by Martin Davis and Reuben Hersh; *SCIENTIFIC AMERICAN*, June, 1972]. Throughout this long period the calculus of infinitesimals has been regularly applied in the physical and biological sciences without regard to its foggy mathematical status and without damage to the disciplines concerned.

Just because neither the sciences nor society has suffered much as a result of scholarly indifference toward (more precisely ignorance of) the debate over infinitesimals, it should not be assumed that the debate over the origins of the number concept can be ignored with equal safety. Unlike the infinitesimal, number is not the exclusive property, or even largely the personal property, of the mathematician. Number has been a concept of social importance since the dawn of recorded history. The significance to society of number and number-related skills has increased tremendous-



CONCEPT OF CARDINATION was tested by asking the subjects to decide, without counting, which of six matched sets of dots (a-f) were in one-to-one correspondence and which in many-to-one correspondence. The top and bottom row of dots in each set differed in color. If both rows contained the same number of dots, the dots were spaced so that the rows were not of equal length. Similarly, when one row contained more or fewer dots than the other, dots were spaced so that rows were of equal length or the row with fewer dots was longer.

ly with the rise of industrial civilization.

In most Western nations today children receive considerable exposure to number concepts soon after they begin to speak. More or less haphazard at first, the exposure becomes simultaneously more intense and more systematic with the onset of formal education. During the first few years of elementary school roughly 50 percent of the curriculum is normally given over to inculcating the natural numbers and methods of manipulating them. The child learns to add, subtract, multiply and divide the natural numbers and finally to derive other number systems from them. We expect our children to possess by pubescence a numerical competence much higher than that of an educated Greek or Roman adult of two millennia ago. We have even gone so far as to develop labels that imply mental turpitude on the part of those otherwise normal children who fail to attain the standards of numerical competence that we deem desirable, for example "learning disability" and "underachievement."

Considering the emphasis our society puts on numerical competence, it might be expected that extensive information would long ago have been collected on how the concept of number naturally unfolds in the course of a child's mental growth. The truth is that we possess very few hard facts about the emergence of numerical ideas in children's thinking. It is the paucity of such data that gives social relevance to the mathematical debate on the origins of the number concept I have just outlined. In what follows I shall describe some recent studies in the area of developmental psychology that were designed to investigate the relation, if any, between the actual emergence of numerical ideas in young children and the hypothetical patterns of emergence suggested by the two principal theories of the origin of number concepts.

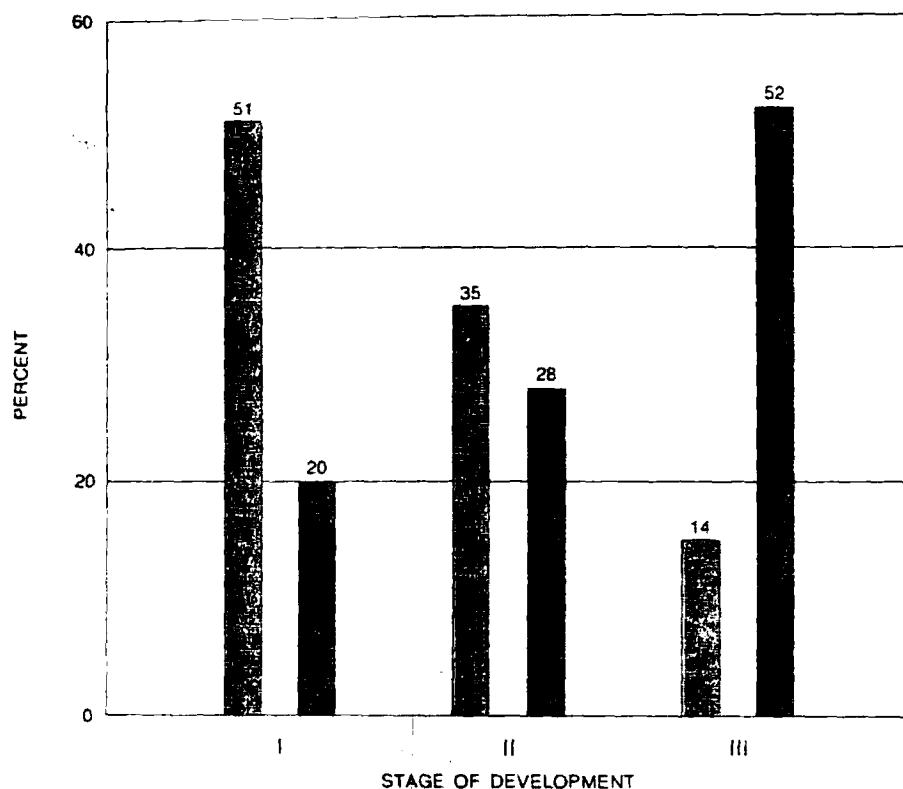
When the mathematician says that some concept *Y* (for example natural number) is "founded on" or "based on" some other concept *X* (for example cardination or ordination), he means that *X* is more fundamental in a logical sense than *Y*. When the developmental psychologist makes the same statement, he means that *X* plays a necessary part in the emergence of *Y* in the thinking of the growing child. From the psychologist's viewpoint such a statement leads to two major predictions that can be tested experimentally. The first prediction is that *X* will invariably appear

earlier than Y in the course of the child's mental development. The second is that improvements in Y will be a consequence of improvements in X.

Different research strategies are used to examine these two predictions. The "X before Y" prediction calls for normative, or average-performance, studies. The normative study consists of four steps. First, tests are devised that measure the presence of the concepts in question in the child's thinking. Then the children to be tested are drawn from the range of ages when the concepts in question are normally acquired. The third step is testing for the presence or absence of the concepts. The fourth is comparison of the test results to determine the presence or absence of inter-concept correlations of the predicted "X before Y" kind.

The "better X means better Y" prediction is tested by what is known as a transfer experiment, a method that is slightly more complicated than the normative method. Seven steps are involved; the first three are the same as the first three steps in a normative study. Taken together, they constitute the "pre-test" phase of the transfer experiment. The information collected in the pretest phase is then used to divide the subjects into experimental groups and control groups. In the fifth step the experimental groups receive training designed to improve their performance in one or another of the concepts in question. After that the tests administered in the pretest phase are given to the experimental and control groups. The seventh and final step compares the test results both to determine the degree of improvement that resulted from training and to find whether or not training in one specific concept tends to "transfer" and improve performance with respect to another concept for which no training was provided.

Both normative and transfer experiments appear to be simple, but a major pitfall awaits the unwary investigator. In constructing the tests whereby the subjects' grasp of various concepts will be assessed, one must guard against loosely defined methods of assessment that bear only a vague or intuitive resemblance to the concepts that are to be measured. For example, in the present instance it was essential that the tests concerning the concepts of cardination and ordination be explicitly deduced from the mathematical conceptions of these notions. If that had not been done, the findings of the experiments could only have been suggestive rather than



GRASP OF CONCEPTS, as revealed by the test scores, placed each of 180 subjects in one of three stages of development. The children at Stage I were totally incapable of either cardination (color) or ordination (gray). Those at Stage II were capable of many-to-one cardination (color) and of left-to-right ordination (gray) but failed to grasp one-to-one and right-to-left relations. Children at Stage III grasped cardination (color) and ordination (gray) fully. As the percent of children at each of the stages indicates, the concept of ordination appeared in the children's thinking long before the concept of cardination.

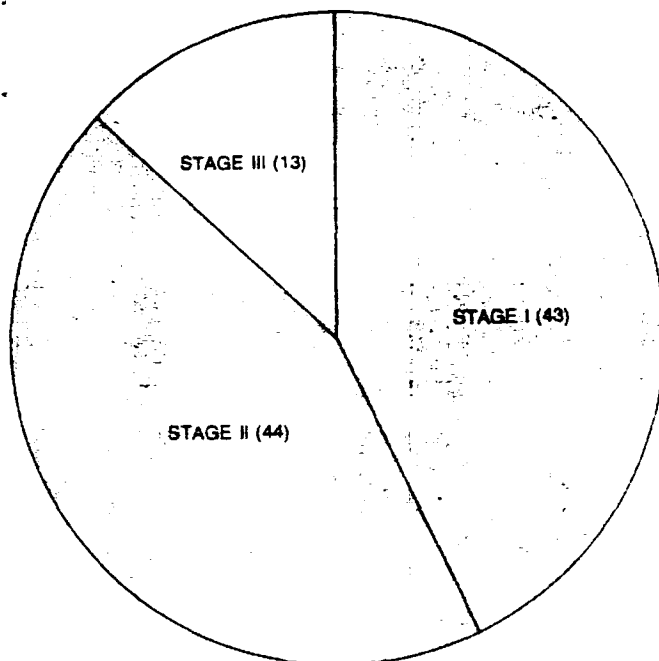
conclusive. Although the pitfall may seem obvious, it was not avoided in most earlier investigations of the growth of children's number concepts, not excluding Jean Piaget's historic studies in the late 1930's and early 1940's.

At the University of Alberta we recently undertook two large-scale studies of the growth of three arithmetic elements in the thinking of young children: the concept of cardination (the "set-matching" of Frege and Russell), the concept of ordination (the sequential quantification of Peano) and the children's competence in manipulating natural numbers. The studies were designed to answer three related questions of the "X before Y" kind: Which of the two comes first, cardination or ordination? Which comes first, ordination or natural-number competence? Which comes first, cardination or natural-number competence?

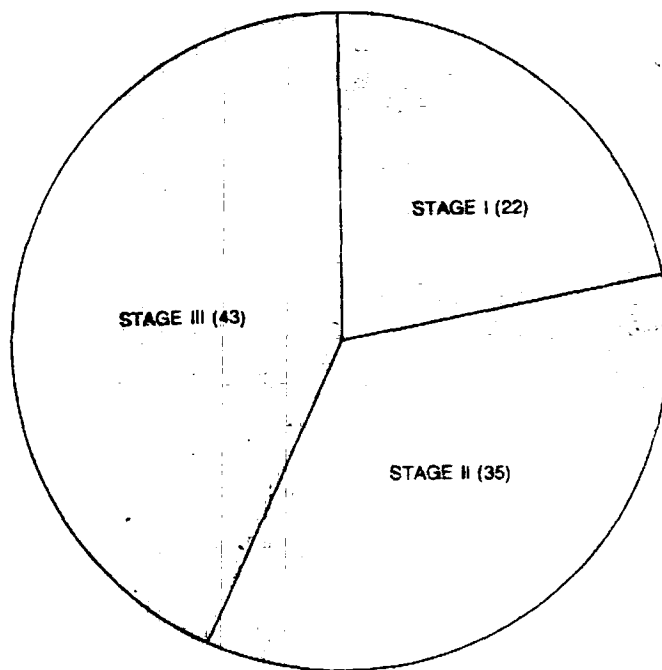
The first of our studies was concerned exclusively with the question of whether cardination or ordination came first. The subjects were 180 Canadian children between five and seven, the range of ages when the average child first acquires the

concepts of cardination and ordination and develops some competence in manipulating natural numbers. The tests for ordination assessed the children's capacity to quantify two familiar transitive, asymmetrical relations: "heavier than/lighter than" and "longer than/shorter than."

In the first instance each child was shown three clay balls in a row on a tabletop. All three balls were the same size; the balls at each end of the row were the same color [see top illustration on page 102]. Although the balls were identical in size, they were different in weight. In some tests the heaviest ball would be at the left end of the row and the lightest at the right end; in other tests the order was reversed. In each case the child was asked to compare the weights of the left and middle balls and then the weights of the middle and right balls. When that had been done, the child was questioned about the quantitative relation between the balls at each end of the row. The child had been given no information about this relation, so that a grasp of the ordination con-



STAGE III ORDINATION



STAGE I CARDINATION

SHARP CONTRAST between the elementary grasp of cardination and of ordination is apparent in this pair of charts. Of 93 children with a top-level grasp of ordination (left), only 13 percent (color)

simultaneously possessed a top-level grasp of cardination. Ninety-two children (right) had no comprehension at all of cardination; 43 percent (color) nonetheless had a top-level grasp of ordination.

cept was necessary if the questions were to be answered correctly.

In the second instance three lengths of quarter-inch dowel were arranged in a row from left to right in front of the child. The three sticks appeared to be identical in length, but actually the sticks at each end of the row, which were painted the same color, were respectively half a centimeter longer and half a centimeter shorter than the middle stick. As with the balls, the short stick was sometimes placed at the left and sometimes at the right [see bottom illustration on page 102]. The child was asked to compare the lengths of the left and middle sticks and the lengths of the middle and right sticks and was then questioned about the quantitative relation between the sticks at the ends of the row. Again, if the questions were to be answered correctly, ordination was necessary.

The two ordination tests were administered to the 180 children. It was at once apparent that the group comprised children at three distinctive stages of development. The children at what I shall call Stage I were totally incapable of ordination. Those at Stage II were capable of spatial ordering but not of true quantitative ordination. This is to say that they answered correctly when the order of lightest to heaviest or shortest to longest was from left to right but

could not do so when the order was reversed. The children at Stage III were capable of true quantitative ordination and answered correctly regardless of the order of presentation of the objects.

Because cardination is the quantification of classes on the basis of the correspondence between their respective classes, the cardination test was designed to assess the children's capacity to quantify pairs of classes that displayed either one-to-one or many-to-one correspondence. The test consisted of six problems [see illustration on page 104]. In each problem the child was shown two parallel rows of dots; the dots in the top row were red and the dots in the bottom row were blue. The minimum number of dots in a row was six and the maximum number 10. When each row had the same number of dots as the other (one-to-one correspondence), the dots were spaced so that the rows were nonetheless of unequal length. Similarly, when one row had more or fewer dots than the other (many-to-one correspondence), the dots were spaced so that the rows were the same length or the row with fewer dots was the longer. This arrangement guarded against the child's making his judgment on the basis of gross perceptual cues. To further ensure that the six pairs were quantified only by correspondence the child was directed not to count the

dots. (Counting, of course, is an ordinal operation.)

All 180 children were given the cardination test. As with ordination, three stages of development were found. The children at Stage I were totally incapable of cardination. The children at Stage II were capable of quantifying the many-to-one correspondences but not the one-to-one correspondences. The children at Stage III were capable of cardination on the basis of both kinds of correspondence.

Our most important finding came from comparing the results of the two tests. The difference in the numbers of children at each of the three stages of development had already suggested that ordination appears in a child's thinking long before cardination. For example, in ordination 93 of the 180 children were functioning at Stage III and 24 at Stage I, whereas in cardination only 15 children were at Stage III and 92 at Stage I [see illustration on preceding page]. When the same child's performances in both tests were compared, the "X before Y" position of ordination became even more apparent. Only 12 of the 93 children functioning at Stage III of ordination were also at Stage III of cardination. Of the 92 children functioning at Stage I of cardination, however, 40 were functioning at Stage III and 32 more at

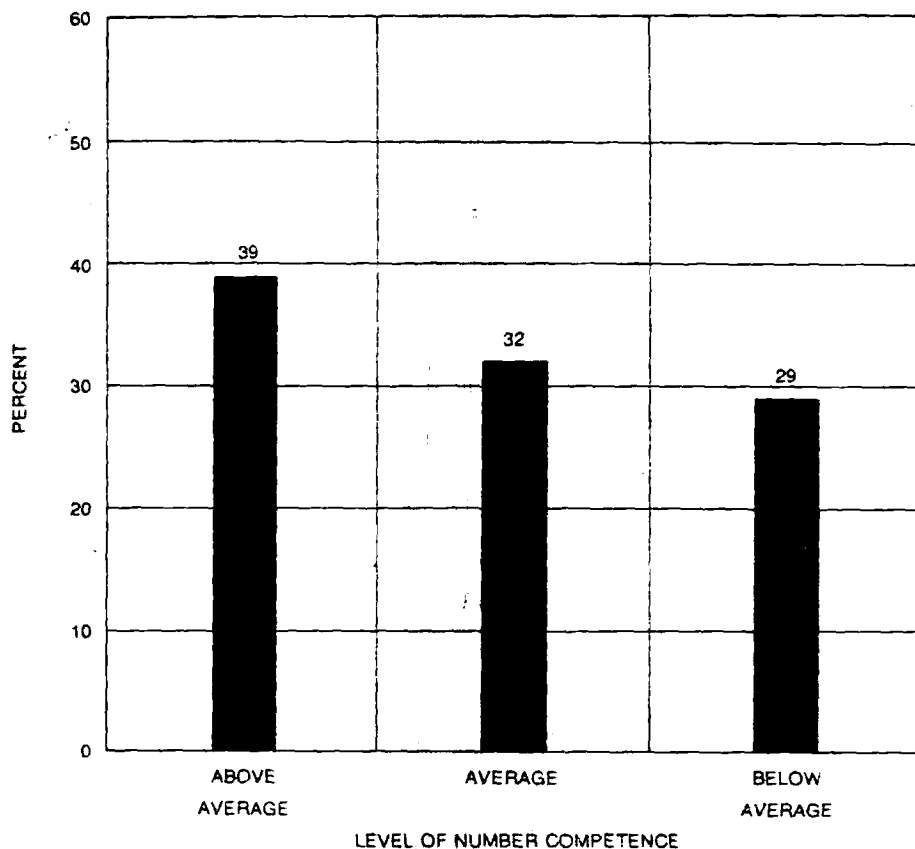
Stage II of ordination [see illustration on opposite page]. All things considered, one could not ask for a clearer answer to the question of whether ordination or cardination is the first concept to appear in a child's thinking.

For our next normative test we selected a second group of 180 schoolchildren, divided into equal numbers of kindergarten pupils (five to six years old) and first-graders (six to seven years old). The children were given the ordination and cardination tests described above and in addition were given a two-part test of natural-number competence. The first part of the test assessed the children's capacity to add the first four natural numbers; it consisted of the 16 problems $1 + 1 = ?$, $1 + 2 = ?$ through $4 + 4 = ?$. The second part of the test assessed the capacity to subtract natural numbers of 8 or less whose difference corresponds to one of the first four natural numbers. It too consisted of 16 problems: $2 - 1 = ?$, $3 - 1 = ?$ through $8 - 4 = ?$. The 90 six-to-seven-year-olds were given both parts of the test but only the first part was given to the 90 children aged five to six, who had not yet been introduced to subtraction in kindergarten.

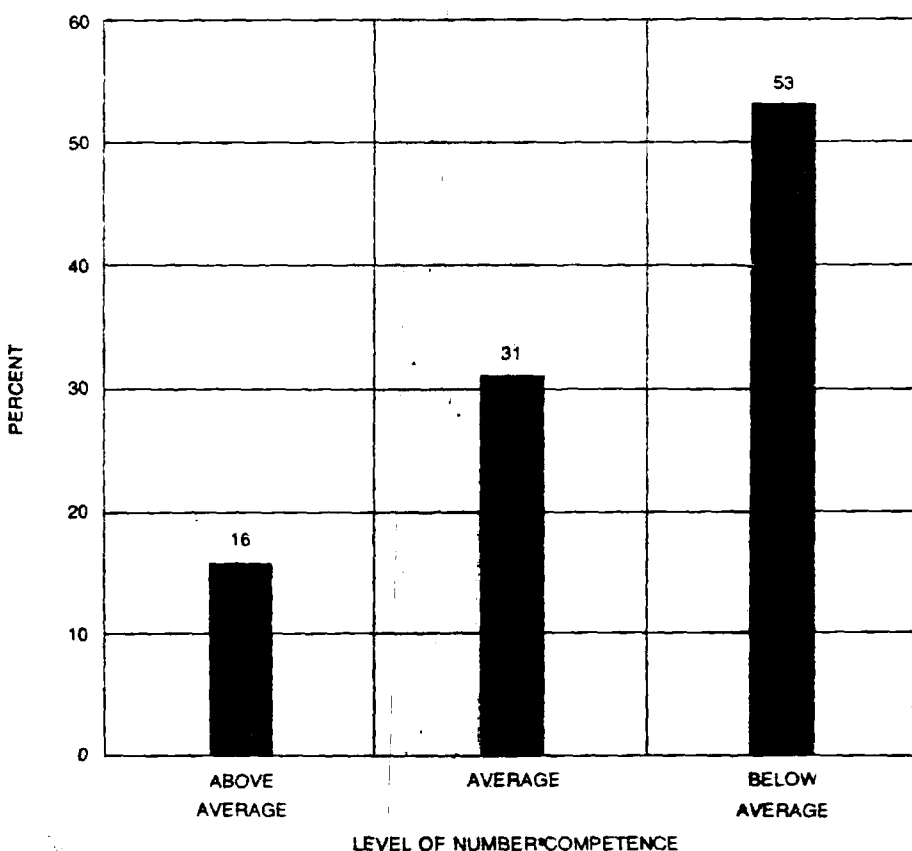
The number-test items were presented simultaneously in verbal and written form. For example, for the first problem the child was given a sheet of paper with $1 + 1 = ?$ printed on it and was simultaneously asked, "How many apples are one apple and one apple?" If the child wrote or spoke the correct answer, or did both, the problem was scored as having been solved.

After the test scores had been collected we consulted with the superintendents, principals and staffs of several elementary schools to determine what they considered to be below average, average and superior numerical skill in children of this age level. On the basis of these consultations we established three levels of performance. Twelve to 16 correct answers were scored as superior, six to 11 correct answers average, and zero to five correct answers below average. We sorted our subjects accordingly into three levels of numerical skill.

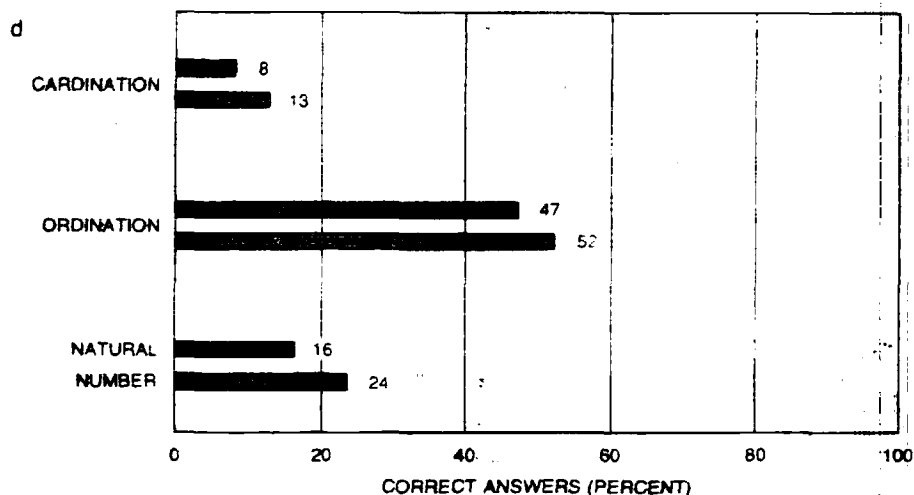
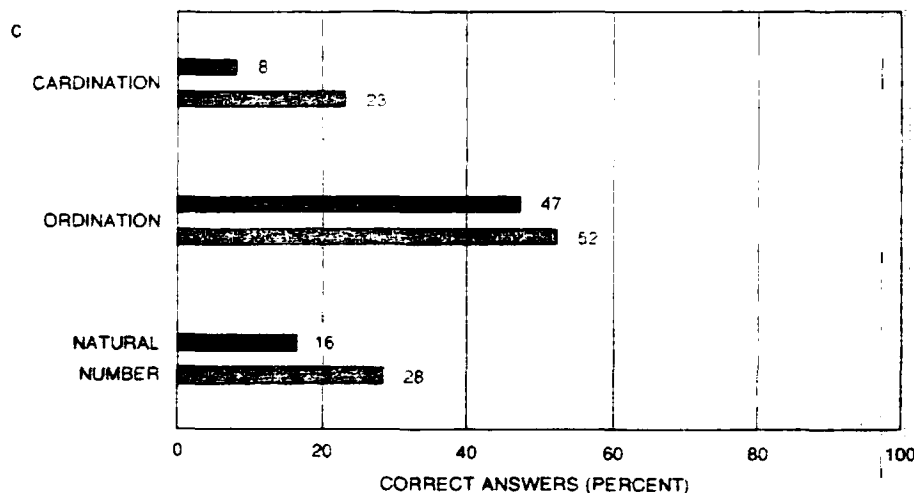
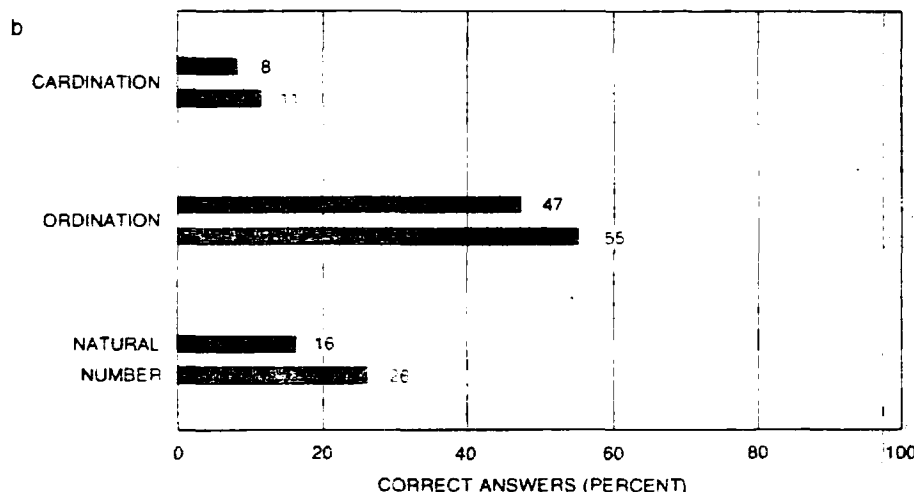
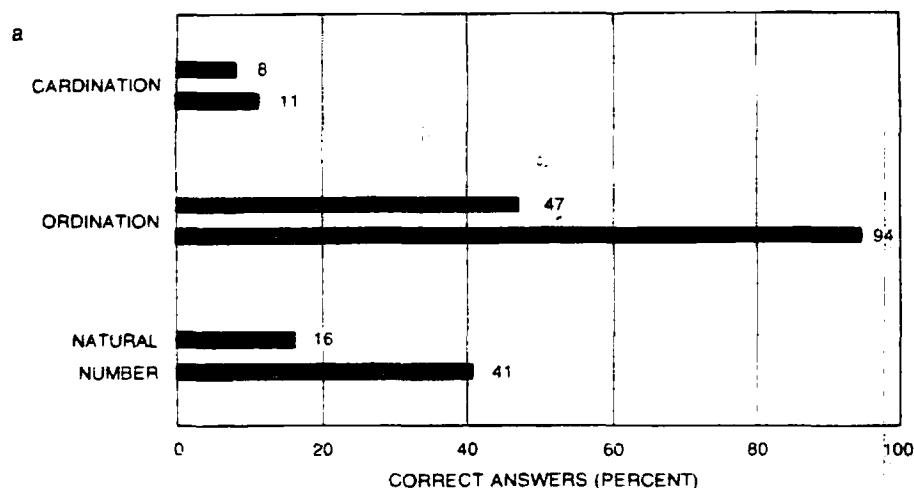
Analysis of the study showed first of all that the test results replicated our earlier findings about ordination and cardination. Among the second group of 180 children ordination emerged in the same three stages as it had among the first 180. Cardination emerged in the same three stages as before, and ordination emerged long before cardination. The next question was whether ordina-



GRASP OF ORDINATION proved to be associated with superior number competence among 180 additional elementary-school subjects. Of 119 children with a top-level grasp of ordination, nearly three-quarters showed above-average or average number competence.



GRASP OF CARDINATION proved not to be associated with competence in using numbers on the part of the same 180 subjects. Of the 95 children with no comprehension of cardination, nearly half nonetheless showed an above-average or average number competence.



tion preceded or followed natural-number competence in order of emergence. To find the answer we compared each child's stage of development in ordination (stages I, II or III) with the child's degree of natural-number competence (below average, average or superior). Of the 180 children 119 were functioning at Stage III, the top level of ordination, 39 were at Stage II and 22 were at Stage I. Of the 119 Stage III children 46 displayed the highest level of number competence, 38 displayed average competence and 35 were below average. The comparison demonstrated to our satisfaction that ordination precedes natural-number competence in order of emergence.

A third question remained: What is the first to emerge, natural-number competence or cardination? We compared each child's level of cardination with the child's degree of natural-number competence. Of the 180 children only 33 were functioning at Stage III, the top level of cardination. Thirty-eight were functioning at Stage II and 95 at Stage I. Of these 95, 15 displayed superior number competence and 30 average competence. The remaining 50 children displayed the lowest level of natural-number competence. The comparison satisfied us that natural-number competence emerges before cardination.

Taken together, our two normative studies showed an invariant sequence in the growth of the children's concept of number. Ordination was the first to emerge, followed by natural-number competence and then by cardination. The sequence strongly suggests that natural-number competence is founded on

TRANSFER EXPERIMENT (left) tested whether training in cardination and ordination would affect a child's competence in using natural numbers. A test group of 240 children was divided into four sets of 60, each set equally competent in cardination, ordination and use of numbers (gray bars). The sets were then paired; the children in sets *a* and *c* were specially trained in ordination and cardination respectively and the children in sets *b* and *d*, as controls, received only routine testing. After eight training or testing sessions all four sets were given a final test (colored bars). The training in ordination nearly doubled the scores of the children in set *a*; a transfer effect is also apparent in the improvement in natural-number scores for set *a*. The training in cardination nearly tripled the scores of the children in set *c*. The children's scores were nonetheless low and their training had no significant effect on natural-number scores.

through the correspondence of elements. The Pythagorean assumption was well entrenched in educational circles, however, and Russell's argument was ignored for nearly a quarter of a century. In the past decade the change that Russell called for has begun to take place. Most parents, however unaware they may be of the details, remember the rallying cry of the period: "New math." In the "new math" the logicist theory that natural numbers are derived from the quantification of classes was at last put into educational practice. Indeed, it would be more accurate to refer to the "new math" as the cardinal approach to early mathematics instruction, whereby the child is first introduced to the central idea of the logic of classes and the natural numbers are introduced later as by-products of the process of quantifying classes.

Today, although pockets of resistance remain, the cardinal approach dominates educational circles in North America almost as completely as the Pythagorean approach did when Russell spoke against it. Most major publishers of elementary textbooks either have adopted or are in the process of adopting some version of the cardinal approach, and continued resistance to the change has become very difficult.

Taking into account the implications of our normative and transfer experiments, it seems dangerous to test content with either the old Pythagorean approach or the new cardinal approach. The order of emergence of the various concepts in children's thinking is ordinal first, number second and cardinal third. Moreover, ordinal training tends to improve the number competence of young children, whereas cardinal training does not. The conclusion that a new "math" emphasizing ordinal notions is called for seems inescapable.

How might such an approach be instituted? The first concepts to be introduced should be ordination and ordinal relations. Later the natural numbers could be introduced as by-products of the quantification of transitive, asymmetrical relations. Still later cardinalization could be introduced as a generalization of the natural numbers. Such an approach would provide a much better fit with what is now known about the emergence of number concepts in the young child's mind than either the Pythagorean or the cardinal approach.

In the second experimental group was also superior to that of the control group, showing that cardinalization too improves as a result of training.

Our third finding was that the average improvement in ordinal performance was much greater than the average improvement in cardinalization, indicating that cardinalization is more difficult to acquire than ordination. Our last two findings concerned number performance; we found that the children in the experimental group trained in ordination were superior in number performance to the children in the matching control group. The number performance of the children in the experimental group trained in cardinalization, however, was not significantly superior to that of their control group. These findings confirmed what had been suggested by our normative studies: The conception of natural number derives from a prior understanding of ordination and not from a prior understanding of cardinalization.

It is instructive to consider the question of how children might best be taught numerical skills in the light of these findings. Whether we like it or not, number competence is socially prescribed in modern civilization. One might even say that number competence has come to have significant survival value. For example, some of the most important decisions made by the average person during a lifetime involve number-related ideas such as money. Yet it is obvious that if we do not design mathematical instruction to agree with the hard facts of mental growth, we run the risk of making early mathematics education unduly difficult for children. Even worse, we run the risk of discounting a large number of the minds we are attempting to instruct.

During the 19th century and most of the 20th it was standard practice in the nations of the West to begin the formal mathematical education of young children with the natural numbers. It was this practice that Russell decried in his introduction to the revision of *Principles of Mathematics* published in 1937. Russell argued that the practice was derived from the discredited Pythagorean assumption that the natural numbers are unanalyzable entities that we simply must accept as being given. It is safe to assume that he would have preferred a curriculum wherein the introduction of natural numbers is postponed until the child has been taught the rudiments of the logic of classes and, in particular, taught the quantification of classes.

a prior understanding of ordination and not on a prior understanding of cardinalization. It is possible to interpret the sequence established by our normative tests in other ways. For example, it could be asserted that ordination, natural-number competence and cardinalization are acquired in relative isolation from one another and that the observed sequence does not necessarily imply any cognitive dependence among these concepts. It was for this reason that we went on to our third experiment: the transfer experiment that would test the validity of our "better X means better Y" prediction. Our subjects were 240 children between the ages of five and six; each child was seen once a week over a 10-week period. During the first meeting the child was given the three tests employed in the earlier normative studies. The children were then divided into four groups of 60 each. The groups were carefully matched with respect to the children's test performances; this matching ensured that any differences in performance that might later emerge would not be due to initial differences among the groups.

Two of the groups were now selected as an experimental group and a control group for training in cardinalization. Over the next eight weeks both experimental groups were given a weekly test of the kind employed in the normative studies. The training was reinforced by simple feedback. This is to say that each child was told whether or not the answers to the test questions were correct; there were roughly 15 minutes of feedback training in each of the eight sessions. The control groups were given the same kind of weekly test but without feedback. No number tests were given during the eight weeks.

During the 10th and final testing session all four groups were given the same tests for ordination, number performance and cardinalization that had been administered during the first session, and the results of the final tests were subjected to statistical analysis. The analysis revealed five major findings [see illustration on opposite page]. First, the ordinalization performance of the 60 children in the first experimental group was clearly superior to that of the 60 children in the matching control group, demonstrating that ordinalization definitely improves as a result of feedback training. Second, the cardinalization performance of the children

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