

UNSOLVED PROBLEMS IN ARITHMETIC

It is the nature of arithmetic that it can pose more problems than it can solve. Indeed, one of the triumphs of mathematical logic is the demonstration that there are problems that can never be solved

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world provide systematic ways of distinguishing correct argument from incorrect. Mathematics, and specifically geometry, provided an ever stronger motive for an inquiry into logic; the Pythagoreans had discovered the existence of incommensurables. Commensurate means having a common measure, and it is obvious that the unit is the common measure of all natural numbers (1, 2, 3 and so on). A number was understood as being a multiple composed of units. The arithmetical unit was thought to be

There are arithmetic problems that a child of 10 can understand but that have nevertheless remained unsolved for tens, hundreds and even thousands of years. For a long time it was thought that if a problem of arithmetic could be accurately stated, only one method might exist for its solution, and that if a problem was unsolved, a solution would ultimately be found through the application of a new method or the more ingenious application of an old one. Metaphorically, an area of contemporary mathematical logic, it is worth that the continuing failure to solve some of the unsolved arithmetic problems may be due not to lack of ingenuity but rather to inherent limitations of the capabilities of both men and machines. This possibility gives metaphysical philosophy as well as mathematical logic.

It was through a refinement of some original Greek ideas that much natural logic came into being in the late 13th and the early 20th century. Hence our account begins with Aristotle, who introduced logic. He did so in response to twin demands, one philosophical and one mathematical. The philosophical demand arose in response to the extreme variety of arguments with which he had to deal. Thales had argued that the basic stuff of the world is water. Anaximander that it is not one thing but an indefinite matter—something or other. Heraclitus had argued that all things are in motion. Parmenides that no things are. Protagoras had argued that ethical judgments are relative. Socrates that they are not, and so on. Aristotle also had to be prepared to deal with paradoxes, such as Zeno's paradox of Achilles and the tortoise or quasilogical arguments such as the one alleged to have taken place between Protagoras and Euclides (see *Illustration on pages 52 and 53*). There was a need to lay down general principles that



PYTHAGORAS ARISTOTLE EUCLID ZENO OF ELEA

PHILOSOPHERS AND MATHEMATICIANS ponder insolubility in this drawing by Bernard Bryson. Aristotle (384-322 B.C.), founded logic in order to deal with such issues as those advanced by Pythagoras (c. 500-500 B.C.) and Zeno (c. 490-430 B.C.). Euclid (c. 300

indivisible, fractions such as $1/2$ or $2/3$ were understood not as being a part of a unit but always as being one unit out of two, two out of three and so on. On the basis of this arithmetic it was probably obvious that given any two lengths there had to be a geometric unit so small that either length would be a multiple of it. It would then follow that every two lengths would be in a definite fixed proportion to each other: first length is to second length as x is to y , where x is the number of geometric units in the first length and y is the number of units in the second.

Given the Pythagoreans' quasilogical beliefs in number as the underlying principle of arithmetic, geometry, cosmology and philosophy, their discovery of incommensurable lengths must have been a shock. The first of many clashes between science and religion in the West. The discovery came in the form of a proof that the side and the diagonal of a square have no common measure (see *bottom illustration on page 54*). Further

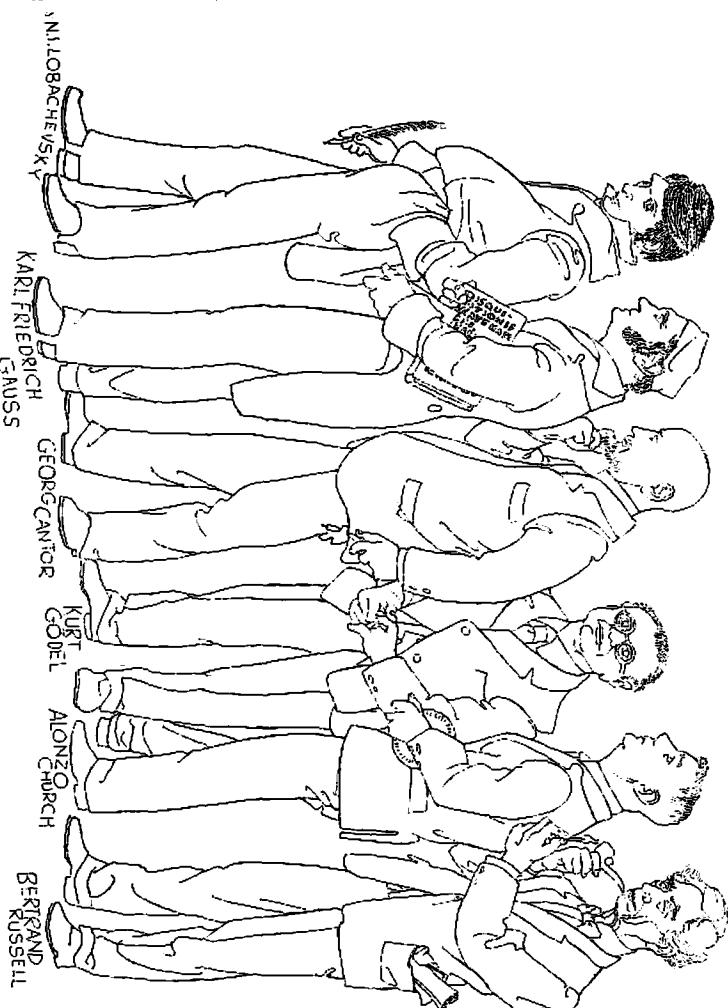
examples of proofs that clashed with strong beliefs held on other grounds—including some fallacious "proofs" such as the argument of Achilles and the tortoise—raised the question of the exact nature and reliability of proof.

Aristotelian Logic

Hence it was in response to both philosophical and mathematical considerations that Aristotle invented logic. He concentrated his attention on four general types of proposition: the universal affirmative, the universal negative, the particular affirmative and the particular negative. Examples of each in turn are: "Every man is white," "No man is white," "Some man is white" and "Some man is not white." Aristotle considered what arguments are valid if two such propositions sharing a term are assumed to be true. The result was the famous theory of the syllogism. For example, from "Every man is white" and "Some cancer victims are not white" there is

only one validly drawn conclusion: "Some cancer victims are not men." The three propositions logically constitute a syllogism. Logic was understood to be a study of the connection between premises and conclusions. That is, to assert that an argument is valid is to assert not that the premises are true but rather that if the premises are true, then the conclusion is also true. Aristotle went further and considered what he called a demonstration, a valid syllogism with necessarily true premises. He then gave an answer to the question of when something is proved in geometry. A proposition is proved in geometry if it is the conclusion of a demonstration. According to Aristotle, not all knowledge is demonstrative. One must start with self-evident truths (which are not demonstrable) and proceed syllogistically to conclusions. In this conception lay the germ of the traditional axiomatic method.

The classic example of this method is Euclid's *Elements*, which dates from shortly after Aristotle's time. Euclid had



NIKOLOBACHEVSKY KARL FRIEDRICH GAUSS GEORG CANTOR KURT GOEDEL ALONZO CHURCH BERNARD RUSSELL

(A.C.) applied Aristotle's axiomatic method to geometry. Karl Friedrich Gauss (1777-1855) and N. I. Lobachevsky (1792-1856) developed non-Euclidean geometry. Georg Cantor (1845-1918) founded set theory, within which Bernard Russell (1872-1970) discovered a paradox. Kurt Gödel (1906-) and Alonzo Church (1903-) put forward theorems showing some problems are unsolvable.

The axioms of a formal system are allowed to contain only undefined symbols. Thus in one sense the axioms are not about anything; the interpretation

Suppose s and t have a common measure. Then

By the Pythagorean theorem,

$$d^2 = y^2 + z^2 = 25,$$

Hence, $y^2 = 2x$, if x is even.
that y is even. So x must be odd,
since x and y have no common divisor
if y is even, then $y = 2z$ and
 $y^2 = 4z^2 = 2x$, and so $x^2 = 2z^2$.
that x is even. Si-

INCOMENSURABILITY was discussed through a proof, given here in modern notation, that the side and diagonal of a square are incommensurate. The proof is

the Pythagorean theorem; the square of the hypotenuse of a right triangle is the sum of the squares of the two

consider a game in which there are nine playing cards—the ace and 2 through 9—lying face up on a table between two players. The players take turns picking

alternatively with X_2 or O_2 (see *mutation* on page 581). Then it is easy to see that for each card game there is a game of ticktackoe and vice versa.

A formal system has some analog with a natural language. Its symbols respond to letters of the alphabet, I

various operations any speaker can perform on the language, such as changing a sentence from active to passive or from affirmative to negative, are not identifiable

portant differences between national gauges and formal systems, but they are close enough so that

The exhibition of the application rules is a proof. More explicitly,

into mathematical logic. The formal axiomatic method applied to arithmetic developed out of the traditional axiomatic method of geometry, with which it is compared here. The elements of the two systems are described and examples of each element are cited.

Properties of Formal Systems

dependent of some formal system to answer the question. We give an interpretation to the system by assigning members of the

Consider now a formal system (call it

"complete," that is, every truth about the natural numbers expressible in the symbolism of the system is a theorem. Finally, it can be shown that the system

represents an infinite number of problems, they are essentially similar, so that anyone familiar with long division has an algorithm by which he can solve any

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show that there can be no algorithm to solve the predicate "Sentences x express a truth of arithmetic" in the way that there is a long-divison algorithm to solve the predicate " x is divisible by y ". Church proved his theorem by an argument that has similarities both to Gödel's argument and to Cantor's argument showing that there are more real numbers than natural numbers. Church's proof demonstrated that the assumption that there is a decision procedure for arithmetic leads to an absurdity. This means that there is no algorithm that will always work for finding arithmetical truth—not merely that we have not yet found such an algorithm. In other words, just as there is no method in ticktacktoe that will guarantee a win against all strategies, so there is no method in arithmetic that will guarantee a proof for all truths. It follows that there will always be arithmetical truths for which present

methods will not work and that the creation of new methods will forever be necessary.

The limitative theorems of Gödel and Church appear to have far-reaching philosophical consequences, and yet someone might object: "I deny that any philosophical conclusions can be drawn from these two theorems. Gödel and Church have shown that all the known formal systems for arithmetic are incomplete and lack a decision procedure, but this is hardly any more significant than the fact that one cannot square the circle or trisect an angle. It is perfectly possible to do these things, but not with merely a straightedge and compass. Similarly, all that the theorems of Gödel and Church come to is that given the means they have selected (just as Euclid selected a straightedge and compass), their conclusions follow. There is no philosophical import to their theorems, which

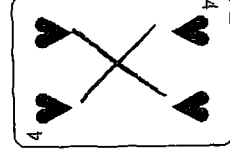
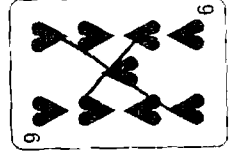
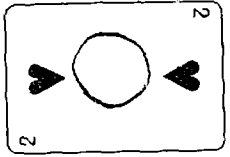
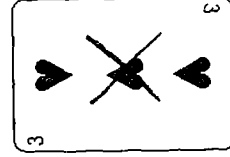
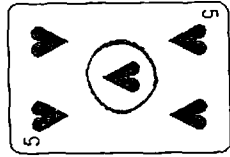
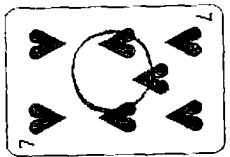
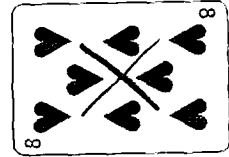
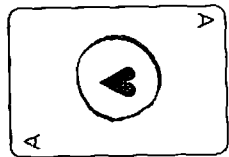
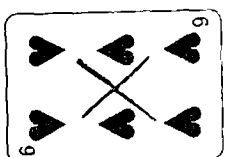
simply suggest that we must look for other means."

This objection would have a great deal of sting if it were not for one circumstance: There do not appear to be any other means. This profound conclusion was arrived at through the following reasoning. Euclid had studied ideal triangles in an effort to understand the concept of space. In an analogous way Emil Post in the 1930's undertook an analysis of what an ideal human computer does when he computes, in an effort to understand the concept of a decision procedure. Independently and at about the same time A. M. Turing undertook a corresponding analysis for the same purpose, of an ideal computing machine. The two analyses were shown to be equivalent. Church suggested that any actual computation by men or machines could be duplicated by the ideal man or ideal machine. Church's thesis is an empirical one, but the evidence for it is overwhelming. If we accept it, the limitative aspects of Gödel's and Church's theorems can be made explicit (see *Illustration on page 601*). In fact, long ago H. G. Cantor has suggested that the limitative theorems of Gödel and Church are "the only known psychological laws comparable in exactitude with the laws of physics."

Unanswerable Questions

The central change that the limitative theorems required of all previous theories of the nature of mathematics was the recognition that there are unanswerable questions in the subject. Either it had been thought that if a question could be made precise, that question had an answer. Now it was seen that perhaps some precise questions do not have precise answers. By way of analogy, think of an object, say a light bulb. If you then ask, "Is it made partly of glass?" the answer will probably be yes; if you ask the question, "Is it made partly of cork?" the answer will probably be no. If, however, you ask, "Does it weigh exactly 3.1 ounces?" the question is probably unanswerable. The reality toward which the question is directed is indeterminate in some ways. Such indeterminateness is characteristic of products of the imagination, including artistic creations. (How often did Juliet sneeze during the year before she met Romeo?) In these areas it is pointless to ask questions about things that are not determined by evidence.

Compared with imaginative creations, physical reality is determinate, and yet the results of quantum theory suggest that physical reality is also indeterminate



LIAR PARADOX

This sentence is not true	This sentence is not provable.
Let L be the name of the above sentence.	Let P be the name of the above sentence.
Suppose L is true. Then what it says is correct and L says that it is not true. So if L is true, it is not true.	Suppose P is provable. Then, since it states that it is not provable, it is not true. So if P is provable, it is not true.

GÖDEL'S ARGUMENT (IN MORE DETAIL)

For any natural number x , proof x is not a proof in A of sentence x .	Let G be the name of the above sentence (G is a natural number so chosen that sentence n is G itself).
Suppose G is provable in the formal system A . Assuming A is correct, what G expresses is true. However, G expresses that G is not provable, so there is an inconsistency. Hence if A is consistent, G is not provable. This means that each of the following is true: Proof 1 is not a proof in A of sentence n . Proof 2 is not a proof in A of sentence n . Proof 3 is not a proof in A of sentence n .	It can be shown that each is also provable. But "For any natural number x , proof x is not a proof in A of sentence x " is not provable. Hence if A is consistent, each and every numerical instance of G is provable but G itself is not. Any formal system that contains a formal sentence such that each and every numerical instance is provable, but the general statement is called <i>ω</i> -incomplete. Hence if A is consistent, G is not provable and A is <i>ω</i> -incomplete.

Suppose L is not true. Then what L says is true. So if L is not true, it is true. So if L is not true, it is true.	Suppose P is not provable. Then, since it states that it is not provable, it is true. So if P is not provable, it is true.	Suppose not G is provable in the formal system A . (not: G = it is false that G = negation of G). Then, since we know that each and every numerical instance of G is provable, we have a system where there exists a formal sentence each and every instance of which is provable but the negation of which is also provable. Since each and every instance of G is true (provided that A is consistent), the proof of not G is a proof of a falsehood. Hence, if A is correct, not G is not provable.	Therefore, G is true of the natural numbers if and only if every numerical instance of G is provable in A but G is not provable in A .
Conclusion: The system in which L is expressible is inconsistent.	Conclusion: The system in which P is expressible is either incorrect (that is, proves falsehoods) or incomplete (that is, contains truths not provable in the system).	Conclusion: The system in which G is expressible that is, either incorrect (that is, proves falsehoods) or incomplete (that is, contains truths not provable in the system).	Conclusion: The system in which G is expressible that is, A is either incorrect (that is, proves falsehoods) or <i>ω</i> -incomplete (that is, contains truths not provable in the system).

GÖDEL'S PROOF of his incompleteness theorem is suggested by an analogy with the "Liar paradox" of the ancient Greeks. Gödel dealt

with provability rather than with truth. What he showed was that a formal system for arithmetic must be incorrect or incomplete,

in certain ways, just as indeterminate-ness, previously considered peculiar to imaginative creations, was found in the physical world with the discovery of the quantum theory, so indeterminateness was also found in mathematics with the discovery of the limitative theorems. The impact of this discovery is comparable to that of the Pythagoreans' discovery of the incommensurability of the side and the diagonal of a square. It upsets fundamental philosophical conceptions.

The philosophical impact can be indicated by another comparison with Euclidean geometry. With assumptions limited to Euclid's it is not possible to prove the existence of a square whose area is equal to that of a given circle. Within these conceptual limitations the existence of such a square is accidental; it is neither necessary nor impossible.

The indeterminateness is due to a refusal to make additional assumptions. The indeterminateness in arithmetic, in contrast, is due to a systematic inability to enlarge our assumptions so that they are adequate to prove all the truths of arithmetic. From the point of view of any particular human being there must exist accidental truths of arithmetic, that is, statements that are true but are such that nothing he assumes would enable him to prove them to be true; they would be neither necessary nor impossible. What the limitative theorems represent, then, is the discovery of an abstract structure for which it is impossible for any human being to make systematically complete and correct assumptions, just as one cannot extend but not eliminate by certain techniques (for example by a method

scope), so our abstract conceptions are limited and the methods of mathematics intended to extend them (for example, mathematical induction) are at best partial in their effect. Our powers of conceptual discrimination have limits just as our powers of perceptual discrimination do.

There does not seem to be any way around these conclusions; indeed, the results reach still further. In order to perceive this a distinction must be drawn between the undecidable and the unsolvable. A sentence is undecidable in a given system if neither it nor its negation is provable in the system. The concept of undecidable is relative to a system; what is undecidable in one system may be decidable in another. For example, G is undecidable in A but is, as we have seen, decidable (and true) in the larger system

STRATEGY for a card game in which the object is to be the first to pick up three cards adding to 15 is seen to be identical with the strategy for ticktacktoe. Similarly, mathematicians look for identical structures in areas of mathematics that are apparently different.

THEOREM	LANGUAGE OF PSYCHOLOGY	LANGUAGE OF PHYSICS
GÖDEL'S INCOMPLETENESS THEOREM	There is no consistent human computer capable of formulating a proposition that, if carried out, would produce all the true and only the true sentences of arithmetic.	There is no consistent computing machine that can be programmed to produce all the true and only the true sentences of arithmetic.
CHURCH'S THEOREM	There exists a set of problems of arithmetic that no consistent human computer can solve.	There exists a set of problems of arithmetic that no consistent computing machine can be programmed to solve.

LAUTMAN'S ASPECT of the theorems of Gödel and Church becomes explicit, as shown here, in the light of Church's thesis, which is to the effect that any actual computation by a man or a machine can be duplicated by an ideal man or an ideal machine.

that includes A and also the argument that C is undecidable in A . In contrast, the concept of unsolvability is absolute. A predicate of natural numbers is recursively solvable if it is solvable by the ideal human computer (or ideal computing machine) mentioned above; otherwise it is recursively unsolvable. For instance, the numerical predicate " x is divisible by 9" is recursively solvable. Sentence x expresses a truth of arithmetic, however, is not recursively solvable. This does not mean that for some given value of the predicate we cannot solve the problem. For example, if " $1 + 1 = 2$ " is sentence 2,467 in A , then "Sentence 2,467 expresses a truth of arithmetic" is both true and provable. What unsolvability means is that there is no technique whatever that will always work, as long as division always works. The question naturally arises of whether or not some well-known problem of elementary arithmetic is recursively unsolvable. There seems to be no reason why that could not be the case. For example, the predicate " x is a divisor of a perfect number" might very well be recursively unsolvable and therefore, on the basis of Church's thesis, unsolvable in the ordinary sense, that is, unsolvable by any man or computer.

Practical Problems

As if this were not enough, there are further possible difficulties. I have been describing what might be called the theoretically undecidable and the theoretically unsolvable. There also exist the phenomena of practical undecidability and practical unsolvability. A formal sentence is practically undecidable if it is theoretically decidable, but the proof is so long as to be physically impossible to carry out. For example, it might be that Goldbach's conjecture is true but that the shortest proof in a given system would require more periods of ink than there are atoms in the universe. Or if the

conjecture is false, the fact even number that is not the sum of two primes may be so large that the chance of discovering it is virtually nil. Actually the conjecture has been checked into the millions without being shown to be false, but it may nevertheless be false. Or again, a predicate is practically unsolvable if it is theoretically solvable but the techniques of solution are physically impossible to carry out. For example, the techniques for solution of the predicate " x is a divisor of a perfect number" may be so complex that they would be beyond the capacity of any conceivable human being or computer. The ability of both men and machines to deal with complications is limited, whereas arithmetic presents problems whose solutions are arbitrarily complicated. We have as yet no way of being sure whether or not some of the well-known problems of arithmetic have solutions of only the most extreme complexity.

It might be asked: "Grant that we cannot prove the completeness of formal systems of arithmetic and that there is no decision procedure. Grant further that there may be serious problems of practical undecidability and practical unsolvability for arithmetic. Can we at least prove that these systems of arithmetic are consistent?" So far the answer is yes and no. There are proofs for consistency based on methods so complicated as to be more in doubt than the consistency of the systems. On the other hand, if we limit ourselves to methods no more complicated than arithmetic itself, then we can indeed prove a formal sentence expressing consistency. In order to be sure that the formal sentence expresses consistency, however, we have to assume that arithmetic is consistent! As Gödel noted, the question of consistency turns on our ability "to replace the actions of classical mathematics about abstract entities of an objective Platonic realm by insights about the given operations of our mind."

Arithmetical truth thus has an "ideal" or "prospective" nature. (The terms have been suggested by John Myhill.) The analogy of a game is useful in explaining this notion. It often happens that a game is invented and rules are laid down that define it but that later a circumstance arises for which the rules give no guidance. At this point a decision has to be made as to what will henceforth be the rule governing that circumstance. The decision might be made on the ground of fairness or of what procedure makes for a better spectator sport or increases the danger or has some other effect. It cannot be made on the basis of the rules of the game, however, because the rules are incompletely defined. Now part of the impact of the intuitive theorems is that the rules by which we define and discover arithmetical truth not only are but must be incompletely defined. We are therefore forced to define the notion of arithmetical truth historically; it cannot be explicated once and for all but must be redefined continually. We have seen how both Gauss and Lobachevsky came to the conclusion that the problems of truth in non-Euclidean geometry required them to go beyond the data of pure geometry. In an analogous way we must go beyond the data of arithmetic in order to define arithmetical truth. Man has invented a game of arithmetic that is incomplete, apparently because of the incompleteness of man's ideas and his abilities.

Yet the temptation is strong to say that although arithmetic may be a game, it is a game in which we learn about reality—abstract reality. There is, then, an incommensurability between abstract reality and our ability to understand it completely. If the predicate "Sentence x expresses a truth of arithmetic" were solvable, arithmetic would become theoretically boring. It is not solvable, and so arithmetic will continue to absorb our interest precisely because its investigation demands human creativity.