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Demonstrating the Pythagorean Intervals

EVE BROWNING COLE

University of Minnesota, Duluth

One of the discoveries of the early Pythagorean school which seems to have proven tremendously exciting to its initiates concerned the relationship between musical harmonic intervals and the first four natural numbers. The basic harmonic intervals can be understood in terms of ratios, and the numbers involved in those ratios display interesting features. The octave, achieved by stopping a string at the half-way point along its vibrating length, thus consists of the ratio 2:1; the fifth, 3:2; the fourth, 4:3. These numbers added ($1 + 2 + 3 + 4$) equal 10. The number 10, figured in the pyramid-of-dots shape by which it would have been represented, therefore itself *contains* (in several senses) the numbers composing the harmonies. Or, a Pythagorean might say, the number 10 is composed of those harmonies. On this thought at any rate a music-constituted cosmology could be constructed. The pyramid that is 10, interlocking with other pyramid-10s, can be imagined to construct physical reality. The number 10, referred to by the Pythagoreans as the mystic "tetractys," was the perfect number, the number which "contains the fount and root of ever-flowing nature."¹

Its method of "containment" is explained somewhat obscurely (by Sextus) as follows:

[T]he whole universe is arranged according to attunement [*kata harmonian*], and the attunement is the system of three concords, the fourth, the fifth, and the octave, and of three concords the proportions are found in the four numbers just mentioned, in one, two, three, and four.²

The Pythagoreans did not themselves *discover* these basic harmonic intervals. They were already ubiquitously present in the domain of music practice. The Pythagoreans' original contribution was intellectual intoxication with the intervals' mathematical significance. As Walter Burkert writes,

[S]omething which may well have lost its interest for professional musicians came to be prized among (the Pythagoreans) as a fundamental insight into the nature of reality. The wondrous potency of music, which moves the world and compels the spirit, captured in the net of number—this was a cardinal element of the secret of the universe revealed to the wise Pythagoras.³

It is a powerful thought, by which the apparently disorderly face of the experienced world is soothed to serenity by means of a deep-level mathematicization of reality. But it is highly abstract, and can easily appear to students as mere numerological mumbo-jumbo. Hence many instructors avoid dwelling on this aspect of early Greek philosophy, preferring to stress the more mundane and credible Pythagorean theorem. This seems a pity, for at least two reasons. First, the notion of a musical/mathematical universe has a charm and grace about it which can be quite attractive to the imagination. Second, this general aspect of early Greek thought, in which the force and potential of mathematical demonstrations are apparently being felt and explored for the first time in history of western culture, deserves particular emphasis. For its implications are very rich. It could hardly be denied that a perception of the constitutive character of number itself plays a profound role in Plato's thought, and that, for example, the *Republic's* final definition of justice in the soul as a music-like harmony, paired with the *Phaedo's* earlier proposed definition of the soul in terms of harmony, is testimony to Pythagorean influence of some interesting sort.⁴

But more broadly, the musical/mathematical universe of the Pythagoreans displays kinship to any truly *rationalist* cosmology. For the rationalist believes in a universe whose comprehensibility is thoroughly predicated upon, and is guaranteed by, its accessibility to mathematical (or mathematical-like) explanation. Deductive certainty appears, in such a system, to be what is desired of a satisfactory epistemology. Mathematical results possess paradigmatic certainty, and therefore function as an ideal for philosophical and scientific explanations.

This constellation of ideas is therefore of undeniable importance, not only in Greek philosophy courses, but also in courses on early modern philosophy, in humanities courses dealing with Greek Culture or the rise of modern science,⁵ and in historically-oriented music theory courses as well.

The Pythagorean intervals are susceptible of a vivid and lucid classroom demonstration, which I describe in the remainder of this paper. I have used this demonstration with great success at several levels of instruction. A handout is useful, since the students will become absorbed in the demonstration and forget to take notes.

Objective

What one does in this demonstration is *show* the Pythagorean intervals, and their musical ratios, by 1) measuring off and marking the proper proportions on a single string of a modern stringed instrument, then 2) either plucking or bowing the string, alternately open and stopped at the marked point (octave, fifth, or fourth), so that the students *hear* these intervals, and hear that the mathematical determinations of these intervals *really work*. Violin, viola, and cello are all suitable. If the instructor is uncomfortable about actually playing the instrument with the bow, she or he might either recruit a student or colleague who is a string-player, or (in a small classroom) simply pluck the string to produce the desired open/stopped-string sounds.

Demonstration

What the Pythagoreans discovered, probably by experimenting on a monochord (single-stringed instrument), can also be demonstrated on a single string of any modern stringed instrument. One simply calculates the musical intervals by measuring the string length (the part of the string that is free to vibrate). On a violin, viola, etc., this is the string length between the nut (raised portion between scroll and fingerboard) and the bridge (thin arched slice of wood which rests on the belly of the instrument). This length (x inches) is then divided into two parts (to create the interval of the octave), or three parts (to create the fifth), or four parts (to create the fourth).⁷

For each interval, one should sound alternately the open string and the stopped string. The distance between the two sounds is the musical interval in question.

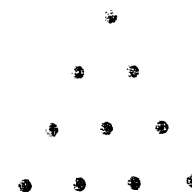
The ratio of the *octave*, the Pythagoreans saw, is 2:1. On a violin (for example) of average size, the part of the string that is free to vibrate will be about 13 inches long. Thus, the octave interval is a ratio of 13:6.5. So, if we measure 6.5 inches from the scroll of a violin, and mark this spot, we should have located the octave-point, or the note that is eight notes above the note sounded by the open string. (Measuring and marking can be done with a ruler and chalk, and should be done in full view of the students.)

The ratio of the *fifth* (five notes above the open string's note) is, according to the Pythagoreans, 3:2. Thus, staying with the example of the violin, we divide the 13 inch length into three parts, each of which will measure 4.33 inches. We then measure 4.33 inches from the scroll-end of the string and make our mark. This is the location of the interval of the fifth. The fifth is also the interval between the first and second notes of the song "Twinkle, Twinkle, Little Star"; the bravest instructor will burst into song at this point. It is in addition the interval between any two open strings of violin, viola, or cello, so that the less intrepid may simply play any two neighboring strings in succession to show that the chalk-marked, Pythagoreanly-derived fifth really is a fifth.

The ratio of the fourth (four notes above the open string) is 4:3. Thus, if we are using a violin, we will divide 13 by 4 and get 3.25. Measuring 3.25 inches from the scroll-end of the string, we will locate, mark, and can then play the interval of the fourth in succession with the open string.

These intervals (octave 2:1, fifth 3:2, fourth 4:3) were of central importance for Greek music. And they have remained fundamental in western music ever since. Modern orchestral stringed instruments are still tuned in fifths, with the exception of the double-bass, which is tuned in fourths.

In the proportions expressing the intervals, the critical numbers are 1, 2, 3, and 4. Added together, we have seen they equal 10. But they equal 10 in an important way, which the Pythagoreans represented in a figure—the tetractys.



The tetractys replicates the musical intervals perfectly, since the ratio between the top two layers corresponds to that of the octave; the ratio between the second two mirrors the ratio of the fifth; and the relation of the bottom two layers mirrors the ratio of the fourth. This "frequency" built into the figure, a kind of sonar coherence, made the tetractys a perfect candidate for the basic unit of physical reality.

Thus, according to the Pythagoreans, the world is in a very real sense composed of musical/numerical proportions. We do not hear, and therefore cannot appreciate, this cosmic symphony, precisely because it is *always there*. We tune it out, in the same way that we tune out the faint buzz of the fluorescent light fixtures in some classrooms. Some more mystical adherents of these views have maintained that, through meditation and strict mental discipline, we can develop the ability to hear the mysterious music emanating from the heart of reality. Others have maintained that its beauty is too great and would destroy us. However this may be, it is generally acknowledged that the Pythagorean musical intervals in their numerical ratios represent the first natural law to be formulated mathematically.⁸

Notes

For considerable help in working out this demonstration I am indebted to Prof. Jane M. Snyder, of Ohio State University's Classics Department. An anonymous referee of this journal provided helpful criticisms of an earlier draft.

1. Sextus, *Ad. Math.* VII. 94-5, as translated by Kirk, Raven, and Schofield in *The Pre-Socratic Philosophers*, 2nd ed., 1983, Cambridge, p. 233 n.2.

2. *Ibid.*, pp. 233-4.

3. Walter Burkert, *Love and Science in Ancient Pythagoreanism*, Harvard, 1978, p. 378.

4. *Republic* 443d-e; *Phaedo* 85e-86d (this definition, offered by Simplicius, is rejected later). Also relevant are the constitutive geometrical figures of the *Timaeus* (53c-56d), about the generation of which Plato writes, "the god began by first marking them out into shapes by means of forms and numbers" (*Timaeus* 53b). Vlastos gives a full discussion of this complex passage in his *Plato's Universe* (University of Washington Press, 1976 ch. 3). Also relevant is Aristotle's somewhat baffling discussion (at *Metaphysics* I.6-7) of the generation of the world according to Plato out of the One and the Dyad.

5. See the perfectly apt quote from Galileo which J. M. Robinson introduces into

