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Abstract: This paper argues that there was a shift in China in the nature, and use, of geometrical figures between around the beginning of the Common Era and the third century. Moreover, I suggest that the emphasis mathematicians in ancient China placed on generality as a guiding theoretical value may account for this shift. To make this point, I first give a new interpretation of a text often discussed, which is part of the opening section of The Gnomon of the Zhou (first century B.C.E. or C.E.). This interpretation suggests that the argument presented in this text for establishing the so-called "Pythagorean theorem" is based upon a certain kind of drawing. Secondly, I contrast this passage with Chinese texts from the third century on the same topic, but relying on a completely different type of drawing. What commands the difference in the kinds of drawing is that the latter drawings are "more general" than the former, in a sense to be made explicit. This paper hence aims at making a contribution to the study of geometrical figures in ancient China.

Commenting on one of the latter figures, one of the authors of the third century, Liu Hui, describes how various algorithms emerge out of the same transformation of one particular figure. His remarks provide grounds for commenting on the link between the general and the particular, in relation to figures and how algorithms rely on them, as the issue was perceived by the practitioners themselves. The particular figure in question and its transformation are exactly what we find in the Meno, though in relation to a different mathematical issue. The link of that very figure to the one that is perceived as its "generalization" for several algorithms, including the so-called "Pythagorean theorem," is made not only in Liu Hui, but also by Thabit ibn Qurra (ninth century C.E.), in a letter where he explicitly addresses the purpose of generalizing the reasoning of the Meno. This parallel offers an appropriate basis to highlight differences in terms of conception and use of figures.

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# Geometrical Figures and Generality in Ancient China and Beyond: Liu Hui and Zhao Shuang, Plato and Thabit ibn Qurra

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## Argument

This paper argues that there was a shift in China in the nature, and use, of geometrical figures between around the beginning of the Common Era and the third century. Moreover, I suggest that the emphasis mathematicians in ancient China placed on generality as a guiding theoretical value may account for this shift. To make this point, I first give a new interpretation of a text often discussed, which is part of the opening section of *The Gnomon of the Zhou* (first century B.C.E. or C.E.). This interpretation suggests that the argument presented in this text for establishing the so-called “Pythagorean theorem” is based upon a certain kind of drawing. Secondly, I contrast this passage with Chinese texts from the third century on the same topic, but relying on a completely different type of drawing. What commands the difference in the kinds of drawing is that the latter drawings are “more general” than the former, in a sense to be made explicit. This paper hence aims at making a contribution to the study of geometrical figures in ancient China.

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The particular figure in question and its transformation are exactly what we find in the *Meno*, though in relation to a different mathematical issue. The link of that very figure to the one that is perceived as its “generalization” for several algorithms, including the so-called “Pythagorean theorem,” is made not only in Liu Hui, but also by Thabit ibn Qurra (ninth century C.E.), in a letter where he explicitly addresses the purpose of generalizing the reasoning of the *Meno*. This parallel offers an appropriate basis to highlight differences in terms of conception and use of figures.

The earliest extant mathematical texts from ancient China, ranging from the second century B.C.E. to the first century C.E., contain almost no information about the kind of visual aids practitioners of mathematics were using at the time these writings were composed. They came down to us through two kinds of channels. The *Book on*

*mathematical procedures*,<sup>1</sup> recently excavated thanks to archeological research, presents some geometrical procedures to compute areas or volumes. However, it does not mention any visual aid in relation to their treatment. Two other writings were handed down through the written tradition, which may be correlated to the fact that they were granted the status of “Canons.” Like the *Book on mathematical procedures*, *The Nine Chapters on Mathematical Procedures* (*Jiu zhang suanshu*, hereafter *The Nine Chapters*), a book compiled in the first century B.C.E. or C.E.,<sup>2</sup> does not contain any reference to either a kind of visual aid or a specific geometrical representation. The case is, however, slightly different regarding *The Gnomon of the Zhou*, a book dealing with mathematics and astronomy and compiled probably earlier than *The Nine Chapters*.<sup>3</sup> Its opening chapter contains a development in which one is tempted to identify the first extant reference to a graphical support for a reasoning in the mathematical corpus written in Chinese.

Given the fact that the last two books held a place of pride in the Chinese mathematical literature, commentaries were regularly composed on them, until as late as the thirteenth century. Among the commentaries that are still extant, some were to be selected by the written tradition to be handed down with the Canons themselves. The link between these Canons and the commentaries selected was so tight that there exists no ancient edition of the Canons in which these commentaries would not be inserted between the sentences of the Canon.

As regards geometrical figures, the commentaries drastically differ from the Canons, since they do refer to visual aids, and even contain generic names for them.<sup>4</sup>

<sup>1</sup> This is the translation I prefer for the title *Suanshushu*, which others translate as *Book of arithmetic*. For the first annotated edition of this text, see Peng 2001.

<sup>2</sup> Modern authors still maintain divergent views regarding the time when the compilation of the book was completed. Guo Shuchun presents the theses sustained by various historians and argues that *The Nine Chapters* was completed in the first century B.C.E. (see his chapter “Histoire du livre” in Chemla and Guo 2004, 43–56). In my view, the completion of the book occurred in the first century C.E. See my arguments in Chemla and Guo 2004, 201–5, 475–8. Chemla and Guo 2004 offers a critical edition and a translation into French of *The Nine Chapters* and the traditional commentaries on it.

<sup>3</sup> Modern scholars also disagree on the mode of composition and the time of completion of *The Gnomon of the Zhou*. Qian Baocong 1963, 4, argues that the book was composed around 100 B.C.E. In contrast, Cullen 1996, 139–56, lists arguments in favor of the thesis that it consists of a collection of texts written at different time periods and later gathered together. He evaluates possible dates for each of them, considering the opening section to be the most recent of them all and having been composed for the occasion of the conference convened by Wang Mang at the Court in the year 5 C.E. I do not find the evidence gathered in Li Jimin 1993a compelling enough to accept his conclusion that this part of *The Gnomon of the Zhou* may date from the eleventh or the twelfth century B.C.E. However, as will become clear below, I find that his paper offers important new insights for the interpretation of the text. Needham with Wang Ling 1959, 19–20, mentions several other views on this question. Cullen 1996 provides a translation into English of the Canon. We follow him in using the original title, *The Gnomon of the Zhou* (*Zhoubi*), rather than the modified title under which it was included in the Tang dynasty collection, the *Ten Classics of mathematics: The Mathematical Classic of the Gnomon of the Zhou* (*Zhoubi suanjing*).

<sup>4</sup> On the question of figures in ancient Chinese mathematical texts, one can find a first description and a bibliography in Chemla 2001a. This paper emphasizes continuities and discontinuities in this respect between the third century and the thirteenth century.

We shall concentrate here on the two earliest extant commentaries, both composed in the third century: Liu Hui's commentary on *The Nine Chapters* and Zhao Shuang's commentary on *The Gnomon of the Zhou*.

Liu Hui refers to visual aids: figures (*tu*) for plane geometry and blocks (*qi*) for solid geometry, whereas Zhao Shuang only refers to figures (*tu*). Liu Hui's visual aids were not handed down through the written tradition. In contrast to this, the earliest extant edition of *The Gnomon of the Zhou*, carried out in 1213, contains reproductions of figures that can be attributed to Zhao Shuang.

The aim of this paper is to establish that the nature of the figures experienced a significant change between the time of the composition of *The Gnomon of the Zhou* and the time of the writing of the commentaries. To capture this turn, we shall concentrate on a geometrical topic on which both Canons, and hence both commentaries, overlap: the right-angled triangle, and the related so-called "Pythagorean theorem."<sup>5</sup>

Let me stress right at the outset that, although they deal with the same topic, the way of presenting mathematical knowledge in the two books differs. *The Nine Chapters* is mainly composed of problems and their solutions, followed by algorithms solving them. In contrast to this, *The Gnomon of the Zhou* consists of a running text, in which, however, tables are also inserted. As we shall see below, in correlation with this, the nature of the commentaries also differs.

## 1. The opening passage of *The Gnomon of the Zhou*

If, as already mentioned, the various scholars who studied *The Gnomon of the Zhou* obtained different conclusions regarding the date of completion of the Canon, the divergence reaches its peak with respect to the opening section of the book. I shall not attempt to settle the issue here. My aim is to bring to light a change in the use of geometrical figures in China between the time of composition of this text and the third century C.E., when the earliest commentary that came down to us was composed. To this end, it is sufficient to know that historians agree on the fact that *The Gnomon of the Zhou* was not written later than the first decades of the first century C.E. In addition, even though we know close to nothing about the author of its earliest commentary, Zhao Shuang, on which we focus in this paper, no one seems to have questioned the fact that he composed his commentary in the third century C.E.

The opening passage of the book has been the subject of a heated historiographical debate: some historians contended that it contained the reference to a proof of the correctness of the algorithm that, in ancient China, corresponded to the "Pythagorean

<sup>5</sup> The *Book on mathematical procedures* does not seem to attest to either an interest in, or some knowledge about, the right-angled triangle and the relations between the lengths of its sides. This may relate to the fact that it was composed in relation to the *milioux* of the administration of finance rather than those dealing with astronomy or the calendar. However, this assertion should be qualified (see Chemla and Guo 2004, 5–8, especially 7, n. 1).

theorem,”<sup>6</sup> whereas other scholars rejected this assertion.<sup>7</sup> The scholarly discussion tends to become sometimes unreasonably passionate when it comes to such a topic as mathematical proof. It would be quite interesting if a historiographical inquiry could cast light on what comes into play in what could appear to be as innocuous a field of inquiry as any other.

In the first part of this paper, I provide a new interpretation of the relevant passage of *The Gnomon of the Zhou*. On this basis, we shall be in a position to assess which kind of statement and proof is to be found in it.<sup>8</sup> If, as a side conclusion, I shall suggest that we do have an argument establishing the “Pythagorean theorem” and that the commentator Zhao Shuang also read the piece in that way, this does not constitute the main reason for me to engage in this new interpretation. My aim in doing so is to focus on the kind of visual aids on which the argument relies and on the use the argument makes of them. I shall thus first translate the text. I then give a graphical interpretation of the geometrical process described in *The Gnomon of the Zhou*, based on this interpretation of the text. And, at the end of this section, I shall derive some remarks on the type of visual aids on which the proof was based. Let me stress right at the outset that most of the conclusions obtained in terms of the nature and use of geometrical figures contemporary of *The Gnomon of the Zhou* do not depend on whether one agrees with my interpretation or not. However, as we shall see, the conclusion on the basis of which I shall point to an evolution, in the type of figure

<sup>6</sup> As will become clear below, what in Euclid’s *Elements* was the assertion of a theorem, in ancient China took the form of a statement of a procedure. Since they are both interpreted today as relating to the so-called “Pythagorean theorem,” I use this expression to refer to both. The reader should keep in mind that this common designation hides a difference in kind between the statements in the original sources.

<sup>7</sup> Cullen 1996, 87–88, alludes to several publications that, in his view, wrongly read a proof in this text. Moreover, after having provided his interpretation of it and Zhao Shuang’s commentary on it, he concludes that “there is nothing in the main text that could be considered an attempt at a proof.” He makes his point with stronger emphasis in Cullen 2002, 786, where he offers the same interpretation of the text as earlier. As will become clear below, my conclusion is different. In any event, the topic seems to me to be worth further historiographic analysis.

<sup>8</sup> For the edition of the passage of *The Gnomon of the Zhou* and its commentary translated here, I relied heavily on the critical edition given by Li Jimin 1993a, which makes several key suggestions regarding both the edition and the interpretation. I give the edition of the text (Canon and commentary) on which my translation is based, with editorial annotations, in Appendix A. Otherwise, for *The Gnomon of the Zhou*, I take as my basis the critical edition of the text provided in Qian 1963. This passage can be found in Qian 1963, 14. Several papers have translated or interpreted the passage dealt with here. To mention a few: Needham 1959, 22–3, 95–6; Chen Cheng-Yih 1987, 35–44, translated this part of the Canon; Ch’en Liang-ts’o (Chen Liangzuo 1993a), Li Jimin 1993a, Cullen 1996, 82–87, and Qu Anjing 1997 dealt with both the Canon and Zhao Shuang’s commentary. See also my introductory chapter to chapter 9, “Base and height,” in Chemla and Guo 2004, 661ff. Among them, Ch’en Liang-ts’o argues that Zhao Shuang’s reasoning differs from the one alluded to in *The Gnomon of the Zhou*. I do not agree with this conclusion. Chen Cheng-Yih 1987, 35–44, and Cullen 1996, 82–87, translate the text as edited by Qian Baocong. Needham 1959, 22–3, adopts “the interpretation of Mr. Arnold Koslow,” without making clear on which edition of the main text it is based. Ch’en Liang-ts’o (Chen Liangzuo 1993a) relies on his own edition of the text, as does Li Jimin 1993a; Qu 1997 follows the latter, even though, in some cases, he diverges from it, either on editorial points or for the interpretation. Comments on all differences of interpretation between the papers previously published and what is presented here would greatly exceed the scope of a paper. I shall simply indicate the points that seem to me to require elucidation or mention.

used, between these visual aids and those produced in the third century depends more significantly on the interpretation I provide.

Let me hence first offer my interpretation of the key passage of the opening section of *The Gnomon of the Zhou* that, in my view, relates to a statement and a proof of the “Pythagorean theorem.” The context in which it is stated consists of a dialogue reported to have taken place between the Duke of Zhou and Shang Gao, an interlocutor presented right at the outset as “excelling in numbers” (*shu*), a term that could also be interpreted as “mathematics.”<sup>9</sup> The Duke of Zhou asks his interlocutor to explain how, in the past – in fact, at the beginning of civilization – Baoxi could determine quantities regarding the Heaven, whereas one cannot reach the Heaven, or actually measure the Earth. In replying to the request of elucidating the origin of these numbers (*shu*, or “procedures”), Shang Gao offers a development with a cosmological import that I cannot analyze here. In his account, he states the procedure that *The Gnomon of the Zhou* demonstrates to be at the basis of the astronomical knowledge presented in it: the “Procedure of the base and the height,” which, in our terms, corresponds to the so-called “Pythagorean theorem.” In the translation that follows, I give the text of the Canon in capital letters and insert, between its sentences, Zhao Shuang’s comments, in lower case letters, in the same way as they appear in all the editions that came down to us:<sup>10</sup>

SHANG GAO SAID: “THE METHOD FOR THE NUMBERS/PROCEDURES (*SHU*) EMERGES FROM THE CIRCLE AND THE SQUARE. (...)”<sup>11</sup> THE CIRCLE EMERGES FROM THE SQUARE;<sup>12</sup> THE SQUARE EMERGES FROM THE RECTANGLE (*JU*).<sup>13</sup> The numbers of the circumference and the circle

<sup>9</sup> In Chemla and Guo 2004, I append a glossary of mathematical terms used in Classical Chinese. I shall rely on this glossary to interpret some terms here. For example, this character *shu*, which can be glossed as “number,” can also refer to a “step in a computation” as well as to a “procedure.” In relation to this, it is used to designate the nine “fundamental parts of mathematics” or even mathematics as such (Chemla and Guo 2004, 984–986). It is interesting to note that Zhao Shuang, commenting on this passage, rewrote *shu* into *suan* “counting rod, step of a computation, computing, quantity formed with counting rods, mathematics” (see my glossary in Chemla and Guo 2004, 988–989).

<sup>10</sup> For the sake of clarity, here as well as in most translations below, I insert paragraphs, although the original text most probably ran continuously. Moreover, in order not to make the reader lose the sense of the unity of the text, here as well as below, I do not insert my own comments in the main text of the paper, but in footnotes.

<sup>11</sup> Here Zhao Shuang develops a cosmological interpretation which we cannot analyze in the context of the paper. Suffice it to mention the fact that, on the basis of, respectively, the circumference and the perimeter, he associates 3 with the circle, 4 with the square. Extending these lines to become the base and height of a right-angled triangle, he introduces its hypotenuse, 5. This interpretation accounts for the mention of these very values by the Canon below.

<sup>12</sup> The sentence cannot be interpreted with certainty. It may refer to the possibility of computing the circumference and the area of the circle, when the side and area of the square are known. This seems to be the lines along which Zhao Shuang interprets this statement. It is, however, interesting to note here that, when he discusses the correctness of the algorithm given by *The Nine Chapters* for the area of the circle, the third century commentator Liu Hui makes use of the “Pythagorean theorem,” i.e. of squares. This may be another sense in which the quantities attached to the circle are determined on the basis of the square.

<sup>13</sup> For *ju* “rectangle, gnomon (geometrical figure),” see the relevant entry in the glossary (Chemla and Guo 2004, 943). One of the main points on which I do not agree with (Li Jimin 1993a) is his interpretation of *ju* in this passage, and more generally in ancient Chinese mathematical texts. In addition to the meaning of

are settled with the square. The circumference of the square is the perimeter. Things that are exactly square are made to emerge with the rectangle (*ju*). The rectangle is width and length.<sup>14</sup>

THE RECTANGLE EMERGES FROM THE MULTIPLICATION TABLE.<sup>15</sup> To deduce the *li*'s of the circle and the square,<sup>16</sup> to understand the values of the width and length,<sup>17</sup> necessarily

“carpenter’s square,” attested to elsewhere in the book, Li Jimin 1993a, 31–35, considers that, in this passage, *ju* takes on two different meanings. One would be the figure of “two lines forming a right angle,” the other one, the “geometrical figure of the gnomon.” In contrast to this, Li Jimin rejects the idea that *ju* may have meant “rectangle” in the ancient terminology. Cullen 2002, 786, n. 8, makes the same claim explicitly. However, this seems to me difficult to establish. In my view, in this sentence as well as below, *ju* does mean “rectangle,” a meaning which, in my glossary, I prove to be attested in ancient Chinese mathematical texts. Chen Cheng-Yih 1993, 478, and Ch’en Liang-ts’o 1986, 257, express the same view. Qu 1997, 206, also suggests that some of the occurrences of *ju* in this passage must be interpreted as “rectangle” or, along the same lines, “square.” Ch’en Liang-ts’o 1993b, 115–118, discusses the meanings of *ju*. Besides the meanings of “carpenter’s square, gnomon, rectangle (including square),” on which we agree, he suggests that, outside mathematical texts, *ju* may have referred to the “right angle.” A rectangle is introduced here that plays two parts in the text analyzed. On the one hand, it generates the right-angled triangle, when, as is explained immediately after, it is cut into two halves along its diagonal. On the other hand, it is that with which, and with the dissection of which, the figure obtained by bringing together the squares made on the base and the height is analyzed below. This meaning of *ju* is attested quite early, and this is also the way in which Zhao Shuang interprets the term in his commentary. Again, it is difficult to interpret what the “emergence of the rectangle from the square” refers to with certainty. It may refer to the dependence of our knowledge of the square on our knowledge of the rectangle. This is true as regards the determination of the area. This appears to be true also when the figure of the rectangle must be used to determine the square of the hypotenuse, when one has the square of, respectively, the base and the height of a right triangle.

<sup>14</sup> A figure is determined by its “fundamental dimensions,” those by multiplication of which the area is determined. Zhao Shuang has mentioned the perimeter of the square. The next sentence of the Classic relates to computing the area of the rectangle.

<sup>15</sup> Literally “nine (times) nine eighty-one,” which designates the table by means of its first sentence. The translation into English seems to indicate a closer link of the table to multiplication, whereas, in ancient China, the table was perceived as having a symmetrical link to multiplication and division, as is made clear below. It is interesting that, in his preface to his commentary on *The Nine Chapters*, Liu Hui also puts the multiplication table in a fundamental position with respect to mathematics (Chemla and Guo 2004, 127, 749–750, n. 5).

<sup>16</sup> One could also interpret: “To compute the *li*’s . . .” In this case, *li* refers to the integers that express (approximately) the ratio between the circumferences of the circle and the circumscribed square, or between the areas of these figures (3 and 4). They are *li* in that they are determined only up to a multiplicative factor. The term *li* is introduced by the other Han mathematical Classic, *The Nine Chapters*, within the framework of the rule of three. In the commentaries on the latter book, *li* also designates the procedures that, on the basis of these numbers, provide the values of the magnitudes mentioned. See *li* in my glossary (Chemla and Guo 2004, 956–959). Below, *li* is used to refer to the values of the sides of a right-angled triangle. In his *Pronunciation and meaning of The Gnomon of the Zhou*, the Song author Li Ji then interprets the word as meaning: “values corresponding to each other.” This implies that, once the base and the height are given, the hypotenuse must be linked to them and is, hence, determined. As a result, the shape of the triangle is known. Using the term of *li* to refer to the sides of a right-angled triangle may relate to the various ways in which, in *The Nine Chapters*, the rule of three is put into play for dealing with these triangles. The basic example for the right-angled triangle is 3, 4, 5. The previous passage of the commentary, left untranslated, explicitly links the base with the circle, the height with the square. More on this point below.

<sup>17</sup> Probably, width and length refer here to the dimensions of the rectangle the area of which equals the area sought for, i.e., its “fundamental dimensions.” For each of the areas, for the evaluation of which *The Nine Chapters* provides an algorithm, such dimensions are brought forward. It is not impossible that the juxtaposition

one must multiply and divide to compute them. The multiplication table<sup>18</sup> is the origin of multiplication and division.

THIS IS WHY ONE BREAKS THE RECTANGLE, “This is why (*gu*)” is a term to indicate that one highlights a situation. One has the intention to produce the *liu*’s of the base (*gou*) and height (*gu*),<sup>19</sup> this is why one says:<sup>20</sup> “one breaks the rectangle.” TO TAKE,<sup>21</sup> AS BASE

of the two dimensions refers to the figure itself, in which case Zhao Shuang would recapitulate here the sequence of generation described by *The Gnomon of the Zhou* in other terms.

<sup>18</sup> Literally “Nine (times) nine.”

<sup>19</sup> See footnote 16. The expression “the *liu*’s of the base (*gou*) and height (*gu*)” designates the base and height, i.e., respectively, the shorter (*gou*) and longer (*gu*) sides of the right angle, in the triangle, by stressing their quality of being *liu*’s. My introduction to chapter 9 “Base and height” (in Chemla and Guo 2004, 665ff.) discusses in which respect the dimensions of right-angled triangles are produced in ancient China as *liu*’s. Zhao Shuang interprets the text of the Canon as going on spelling out the generation of mathematical objects one from the other, the next step being the introduction of the right-angled triangle on the basis of the rectangle. I shall always translate the shorter side of a right triangle “base (*gou*),” its longer side “height (*gu*),” omitting the pinyin in case there is no ambiguity.

<sup>20</sup> This expression “this is why one says . . .” is typical of Zhao Shuang’s commentary, and it concludes several of the developments he makes to account for the meaning or the intention of sentences of the Canon. Clearly, he understands the intention of the operation of *zheju*, which I translate as “breaking the rectangle,” to be the introduction of the base and the height. And, indeed, the relation of these sides of the right-angled triangle to the sides of the rectangle constitutes the topic of the next sentences in *The Gnomon of the Zhou*. The key expression *zheju* has previously regularly been interpreted along the same lines (Needham 1959, 22; Chen Cheng-Yih 1987, 36; Ch’en Liang-ts’o 1993a, 5; Ch’en Liang-ts’o 1993b, 118–120). This amounts to interpreting *zhe* as having one of its common meanings here (“break”), and *ju* as meaning “rectangle” (see above). However, more recently, some historians interpreted it in various other ways. Understanding *ju* as referring to the pair of two lines, base and height, Li Jimin suggests that the operation refers to the breaking of the right angle they constitute into two lines, to be thereafter arranged, side by side, horizontally. However, I do not consider this meaning of *ju* as sufficiently established. Cullen 1996, 84, translates: “fold a trysquare.” It seems to me that taking *ju* as meaning “trysquare” here makes the passage difficult to interpret, and this may be one reason why the author came to the conclusions recalled above. Qu Anjing understands that *ju* means here a particular kind of rectangle, i.e., “square,” and he translates the sentence as follows: “So, convert (*zhe*) [the numbers 3 and 4] into *ju* (squares) in order to [arrive at the result] . . .” The text would state the “Pythagorean theorem” in a particular case, before setting out to establish it. In my view, *ju* should be interpreted in these two succeeding occurrences in the Canon with the same meaning, which seems to me possible as I wish to make clear here.

<sup>21</sup> The base (*gou*) and height (*gu*) of the right-angled triangle are introduced on the basis of the figure of the rectangle. The right-angled triangle first occurs here in relation to a surface that is half that of the rectangle (see fig. 1). However, most probably, in ancient China, the right-angled triangle was a geometrical shape that was essentially different from the other geometrical bodies. The latter – including the general triangle (*guitian*) – were essentially extensions having areas or volumes, and lines were introduced in relation to them, as their sides or their fundamental dimensions (for instance, the “height” for the triangle; see footnote 14). In contrast, the right-angled triangle was a configuration of lines. This, in my view, may explain why *The Gnomon of the Zhou* contains only base and height, and not the hypotenuse (see the following footnote). It is possible that, in such texts as *The Nine Chapters* as well as in the commentaries, this configuration was designated by the expression “base and height” (*gougu*), (see my glossary in Chemla and Guo 2004, 926), even though, so far, the evidence I could find does not allow one to conclude on this point with certainty. At least, this would seem to be consistent with the idea that the right-angled triangle was a figure of the special kind we alluded to. Concerning such triangles in *The Gnomon of the Zhou*, Cullen 1996, 77–80, rightly notes that “plane figures bounded by three straight lines just do not figure as a unit of discourse,” a point which Raphals 2002 analyzes in great detail. Qualifying this stand would exceed the scope of this paper. However, to put it in a nutshell, my

(*gou*), THE WIDTH, 3, this corresponds to the circumference of the circle. That which is longitudinal is called width. The base is also the width. The width is shorter. AND TAKE, AS HEIGHT (*gu*), THE LENGTH, 4. This corresponds to the perimeter of the square. That which is transversal is called length. The height is also the length. The length is longer. THAT WHICH GOES THROUGH THE CORNERS<sup>22</sup> IS 5. These are the *liu*'s that spontaneously correspond to each other.<sup>23</sup> "That which goes through" goes straight. The "corners" are the angles. It is also called "hypotenuse."

argumentation would hinge on the previous two remarks. What is at stake in the sentence of *The Gnomon of the Zhou* commented upon is articulating the rectangle and the right-angled triangle: the base appears as width, the height as length; the way in which the hypotenuse is introduced highlights the connection between the triangle and the rectangle. Thereby, the figure of the right-angled triangle is linked with an ordinary geometric figure, and hence with areas. This allows proving procedures involving the base, the height and the hypotenuse by resorting to transformations of areas (*shu*), as is done when proving the correctness of procedures for computing the areas of given figures. Chen Cheng-Yih 1993, 478, expresses a point of view close to this one. However, the assertion that this is what the "accumulation of areas" (*juji*, see below) refers to would require a stronger argument. The relationship between the rectangle and the triangle will appear to be a key element in the reasoning that follows.

<sup>22</sup> As Li Jimin 1993a noted, *The Gnomon of the Zhou* does not mention the hypotenuse. In contrast to this, the commentator Zhao Shuang glosses the passage by introducing the technical term *xian* "hypotenuse," which is found in all the other writings dealing with the right-angled triangle, from *The Nine Chapters* onwards. In translating the expression *jingyu* by "that which goes through the corners," I hence take this expression as that by which *The Gnomon of the Zhou* refers here to the hypotenuse: the "(line) going through the corners." I interpret *jing* as verbal ("go through") and *yu* as referring to the opposite "corners" of the rectangle within which the triangle is read. As a consequence, the hypotenuse is understood to be associated to the base (*gou*) and the height (*gu*), on the basis of the figure of the rectangle. This fits with the other evidence we have (Chemla and Guo 2004, 878, n. 5). The expression *jingyu* has been translated in many other ways. Chen Cheng-Yih 1987, 36, translates *jing* as "hypotenuse," leaving *yu* aside. I cannot see the evidence that can account for this interpretation. Regularly, historians have interpreted *jing* as "diameter" or "line going straight through," hence "diagonal." Needham 1959, 22, has "the diagonal between the (two) corners." Li Jimin 1993a, 37, glosses "the straight-line distance between the corners." Although *jing* does take on the meanings of "diameter" or "line going straight through" (see my glossary, Chemla and Guo 2004, 941–942), this interpretation seems to me difficult from a syntactical point of view. Overcoming this difficulty, Cullen 1996, 84, translates: "the diameter is five aslant." However, I cannot see how *yu* "corner" can be interpreted as "aslant." Ch'en Liang-ts'o 1993a, 7, n. 5, provides evidence showing that *jing* has been glossed in Han times as "oblique road." If, as I do, he understands *jingyu* as referring to the oblique line joining the corners of the rectangle, our syntactic and semantic analyses differ. Lastly, Qu Anjing 1997, 197, suggests interpreting: "[corresponding to] the corner is an oblique line 5 in length." Here, too, I am not convinced by the syntactic analysis. If we now go back to the overall statement, it can be taken as an assertion deriving from using a procedure equivalent to the "Pythagorean theorem." From here onwards, a development starts, which is concluded by the assertion that joining the squares of the base and the height yields 25, a method for which a name, "accumulating/piling up the rectangles," is provided. In between, the development should thus allow drawing this conclusion, which bears on the sum of the squares of, respectively, the base and the height and relates it to obtaining the value 5. If, as several scholars believe, the development describes the transformation of the surface obtained by joining together the two squares into the square of the diagonal of the rectangle, it would not be shocking. The main point is to interpret the text in not too far-fetched a way. Note that, as is usual in most mathematical texts of ancient China, the procedure as well as the development on it are expressed within the framework of a paradigm (Chemla 2003).

<sup>23</sup> The commentary stresses that a value is yielded, which corresponds to the others. This value is, hence, the result of a computation, and is not freely chosen.

ONCE THESE (BASE AND HEIGHT) HAVE BEEN MADE SQUARE,<sup>24</sup> ONE TAKES OUT FROM THE OUTSIDE THE HALF OF RECTANGLES OF THE OTHER TYPE.<sup>25</sup>

<sup>24</sup> This refers to the shaping of two squares, the sides of which are, respectively, the base and the height. These squares are kinds of “rectangles (*ju*)” as is made explicit later in the text. This can be shown on the basis of the final assertion of this passage. After having asserted that the line joining the corners had the length 5, *The Gnomon of the Zhou* starts here a development ending with the sentence: “THE TWO RECTANGLES HAVING JOINED THEIR LENGTH MAKE 25, HENCE THIS IS CALLED “PILING UP RECTANGLES” (my emphasis). The two rectangles that make 25 are the squares based on, respectively, the base and the height. It is clear that they are here designated by the term *ju*. Moreover, this conclusion provides a hint as to how these squares were placed with respect to each other: set by one another in such a way that two of their respective sides coincide with each other, they were made to “join lengths,” to “bring their length (that is, the transversal side) together.” This is the geometrical configuration that is taken as a starting point on fig. 2.a. (See the set of figures 2, for a graphical interpretation of the process. The figures I present here are inspired by the interpretation suggested by Li Jimin 1993a, 37. I strongly advise the reader to take a look at these figures while reading the following part of the original text and my explanations.) Making “squares (*fang*)” on the basis of the base and the height is precisely the operation opening Liu Hui’s commentary on the “Procedure of the base and the height,” the statement of *The Nine Chapters* equivalent to the “Pythagorean theorem.” On this point, see below. Li Jimin 1993a, 37, suggests interpreting: “All being made squares, . . .”

<sup>25</sup> I understand the “rectangles of the other type,” an interpretation of the expression inspired by Li Jimin 1993a, as being like the rectangle first discussed, in contrast to the squares just considered. The latter would be “rectangles of one type,” the former “rectangles of the other type.” The figure that joins the two squares is analyzed thanks to the form of the former rectangle, as shown on fig. 2 (b). These rectangles are cut in exactly the same way as was described previously. On this last point, I agree with Ch’en Liang-ts’o 1993a, 5. Moreover, Ch’en Liang-ts’o 1993a, 7, n. 7, understands that this operation is what Zhao Shuang’s commentary on this sentence describes (see below). Once these rectangles are exhibited, the text prescribes that one take out the “outer” half. The opposition between inner parts and outer parts of figures is common in the commentaries on *The Nine Chapters*. Qu Anjing 1997 restores this interpretation of the term *wai*, which Li Jimin 1993a, 37, had discarded. However, he bases his interpretation on the fact that the square mentioned has a side equal to the sum of the base and the height. If such is the case, it becomes difficult to interpret the closing sentence of the passage discussed in the previous note. Li Jimin suggests interpreting *wai* as “take out.” What the latter thinks is taken out is half of the gnomon that appears in the area produced by placing the squares on the base and the height side by side. Although this designates the same pieces as those I think are taken out, we do not understand the way in which the text refers to them in the same way: for him, they are referred to as taken out of the shape of a gnomon, whereas, for me, they are seen as halves of two copies of the rectangle considered before. It seems to me more natural to interpret *ju* as having the same meaning as earlier. Ch’en Liang-ts’o 1993a, 5, interprets the Canon as referring to a process different from what Zhao Shuang describes. Regarding *The Gnomon of the Zhou*, he suggests that the square mentioned is the one built on the hypotenuse. He then understands that, outside, half rectangles are placed so as to constitute a square with a side equal to base + height. On this point, he concurs with Chen Cheng-Yih 1987, 36, and Chen Cheng-Yih 1993, 479. However, if we were to follow Ch’en Liang-ts’o’s interpretation, we would have to admit that *The Gnomon of the Zhou* would have left completely implicit the further reasoning based on this configuration, which seems difficult to admit. Ch’en Liang-ts’o 1993b, 133, concedes this point and interprets it as a mode of writing adopted by the Canon that only mentions the key facts and leaves it to the reader to complete the reasoning. Cullen 1996, 84–85, discusses possible interpretations of the text as given by the various ancient editions. However, I think that his interpretation of *ju* again as “trysquare” makes here the task of understanding the text more difficult. The translation reproduced in Needham 1959, 22–23, on one part, does not respect the syntax of the original text and, on the other part, adds several elements that are not in the text. The operation to which the pieces cut from the whole are submitted is the object of the next sentence of the Canon.

The method of the base and the height<sup>26</sup> (allows), when first knowing two values, to then deduce the other one. When there appear base and height,<sup>27</sup> and one, then, seeks for the hypotenuse, one first multiplies each by itself to generate the corresponding area (*shi*).<sup>28</sup> The areas are reshaped by relying on the configuration,<sup>29</sup> hence only is the transformation carried out. This is why one says: “Once these (base and height) have been made square . . . .” As for “from the outside . . . ,” if one sums/assembles the squares of the base and the height<sup>30</sup> to seek for that of the hypotenuse, in the squares, one thus seeks for the (efficient) cutting<sup>31</sup> of the sum/assemblage of that of the base and the height.

<sup>26</sup> What corresponds to what we call “Pythagorean theorem” appears in *The Nine Chapters* as a procedure, called the “Procedure (*shu*) of the base and the height” (see section 3 of this paper). In later writings, “procedure (*shu*)” is often replaced by “method (*fa*).” Zhao Shuang is hence referring to this procedure here. What he says fits quite well with the actual structure of the procedure in *The Nine Chapters*, since, under this name, the Canon groups three procedures, allowing one to determine each of the sides of the right-angled triangle, when knowing the two others. Zhao Shuang’s wording, when he describes the procedure below, also agrees with the wording of *The Nine Chapters*. Zhao Shuang explicitly refers to a book called *The Nine Chapters (jiuzhang)* (Qian 1963, 22). Since the algorithms he quotes for computing the area of the circle are exactly those contained in *The Nine Chapters on Mathematical Procedures* too (ibid., 23), we may assume that he refers to the latter book.

<sup>27</sup> What “appears” (*xian*) is what is known; on the term *xian*, see my glossary in Chemla and Guo 2004, 1009–1010.

<sup>28</sup> Zhao Shuang regularly uses the word *shi* “area,” where the commentators of *The Nine Chapters* would use *mi*. On these two terms, see my glossary in Chemla and Guo 2004, 959–961, 977–978.

<sup>29</sup> On this term *shi*, translated here as “configuration,” see my glossary in Chemla and Guo 2004, 979–982, where it is transliterated as *shi*.

<sup>30</sup> The square of the base (resp. height) is here expressed as *gou shi* (resp. *gu shi*). This expression conforms exactly to the way in which Liu Hui uses the term *mi*: the term *fang* “square” refers to the geometrical surface of a square, whereas the term *shi* refers to the numerical value and its extension as a square.

<sup>31</sup> This proposition refers to the way in which parts in the squares assembled are brought to light, some of which will be moved so as to allow reshaping and hence metamorphosis into the square of the hypotenuse. The half rectangles introduced above by the Canon, in the sentence on which Zhao Shuang comments here, define a cutting of both squares, which I believe is the one discussed here. In this passage of Zhao Shuang’s, Ch’en Liang-ts’o 1993a, 6, reads an argument establishing the “Procedure of the base and the height” very similar to the one I read in both *The Gnomon of the Zhou* and the commentary. He notices that, in contrast to the extremely concise formulation of the argument in *The Gnomon of the Zhou* and in Liu Hui’s commentary examined below, Zhao Shuang’s description is more extensive and clearer. However, Ch’en Liang-ts’o takes the Canon to allude to an argument different from that of the commentator. Since Zhao Shuang uses later on, for his own development, a figure much richer than the one alluded to here, I do believe that, in this passage, he is interpreting the Canon, rather than displaying an argument of his own. Let us stress that here, like below, the terms base, height, hypotenuse refer to the squares built on these magnitudes. Historians who translated, or commented on, Zhao Shuang’s commentary offer here a variety of interpretations. Li Jimin 1993a, 37, also understands that which is considered to be the squares built on the base and the height assembled. However, he punctuates the final part of the sentence differently. He considers *fen* “cutting” and *bing* “sum” to be parallel terms, respectively referring to the opposed transformations of “taking out (*chu*)” and “bringing in (*bing*).” I cannot recall evidence in mathematical texts for supporting this interpretation. Qu Anjing 1997, 199, punctuates in the same way and translates “. . . the area [of the square on the hypotenuse] is made by piecing together (*fen*) and piling up (*bing*) the two squares.” I am not sure I understand the syntactical analysis behind this interpretation. On the basis of Qian Baocong’s edition of the text, Cullen 1996, 85, translates: “in the midst of the hypotenuse area one may seek the separation and addition of the base and altitude [areas].” He thus punctuates as Li Jimin did. However, his interpretation of the first area mentioned as the “hypotenuse area” differs from Li Jimin’s.

If the squares are not exactly equal,<sup>32</sup> they exchange with one another that which is taken out and given, and together they have something that is added.<sup>33</sup>

<sup>32</sup> This clause is quite important. Cutting half the rectangles has two effects. First a piece from the square of the height is taken out to be joined to a corresponding piece of the square of the base to make a half rectangle (one could also consider the situation conversely): what goes out on one side equals what goes in on the other side. The shape of the half-rectangle dictates what one square gives to the other. *This does not occur in case the two squares are equal* – This may be one of the points where the square comes from the rectangle. This first aspect of the transformation might be the part of the process described by Liu Hui, in his commentary on the “Procedure of the base and the height” with the expression: “what goes out and what comes in compensate for each other” (see below). Secondly, as we see below, the two half-rectangles are taken out from the conjunction of the squares to be added elsewhere to the remaining pieces. Whether the sides of the right angle are equal or not, this operation is the same in the two cases. Liu Hui will consider exactly the same opposition between two cases, see below. The latter part of the process may be what, in his commentary on *The Nine Chapters*, Liu Hui regularly refers to as: “with what is in excess, one fills up the void.” Some authors considered that the different wordings of the two principles referred in fact to the same principle. If such were the case, in the proof of the correctness of the “Procedure of the base and the height,” one would thus have twice: “what goes out on one side equals what goes in on the other side.” On the former principle, see Wu Wenjun 1982. On the latter: “with what is in excess, one fills up the void,” see Volkov 1992 and Chemla 1992, where a bibliography is given. I consider this conditional clause that starts the sentence to refer to the two squares (*shi*) built on the base and the height. In my view, the same term *shi* hence refers in its two consecutive occurrences to the same figures. Not all historians agree on this point. This clause inspired different interpretations. Li Jimin 1993a, 37, understands it as follows: “Since the area (obtained by assembling the squares on the base and the height) is not a square, . . .” This would explain why, to transform it into a square, one has to carry out transformations. Although, from a syntactical point of view, this interpretation is perfectly possible, if so, the text would appear to me as awkward. Here is how Cullen 1996, 85, interprets this clause in its context: “These areas are not exactly the same, so one may proceed to take from and give to them, and [so that] each has something to receive.” Again, syntactically, this is possible. I suppose that this reflects his understanding of the whole passage as referring only to the right-angled triangle with dimensions 3, 4, 5. If that were not the case, one can certainly think of such triangles for which the base and the height are equal. For them, the statement would be wrong. They are precisely the topic of Liu Hui’s commentary discussed below. On the basis of my overall interpretation of the text as describing general operations on the basis of a paradigm (see my argument in favor of this reading in Chemla 2003), I prefer the interpretation that takes the first part of the sentence as a condition for the remaining part. It presents in my view the advantage of highlighting a real condition for the transformation described in the following clause to be meaningful. This would bring to light one more similarity between Zhao Shuang’s and Liu Hui’s commentaries on the right triangle. Qu Anjing 1997, 199, translates: “Because their forms (of the square on the hypotenuse and of the two squares) are not congruent . . .” This implies to interpret *shi* as “form” and *zhengdeng* as “congruent,” two suggestions that would require evidence. Let us notice that, if the base and height of the triangle discussed were the same, the “rectangles” (*ju*) that the squares built on them constitute and the “other type of rectangle” (*qi yi ju*) mentioned in *The Gnomon of the Zhou* would be the same. It is hence interesting to note that Zhao Shuang introduces this clause, exactly at the point where he is commenting on the latter expression of the Canon.

<sup>33</sup> “Something that is added” is literally expressed as “something that is obtained,” an expression that evokes, as its counterpart, “that which is lost (*shi*).” The way of looking at the situation of the two squares assembled as losing something is in fact the perspective stressed by *The Gnomon of the Zhou*, when it prescribes: “one takes out the half of rectangles of the other type.” The commentary hence seems to be making explicit the other perspective on the same situation: the configuration of the two squares is, on the other hand, gaining something. The two half rectangles are lost on one side and gained on the other side, hence the area remains the same, even if its shape has experienced a metamorphosis. I interpret here *hu as bing* “together.” The *Hanyu dacidian* gives this meaning as attested to only from the Jin dynasty onwards, in a text by Lu Ji (261–303) and thus probably slightly later than the moment when Zhao Shuang composed his commentary. However, in my view, this provides a

This is why one says: “one takes out the half of rectangles of the other type.”<sup>34</sup>

As for the corresponding procedure,<sup>35</sup> base and height each multiplied by itself – 3 times 3 makes 9, 4 times 4 makes 16 –, the sum makes the area of the hypotenuse multiplied by itself, 25. Subtracting (the area of) the base from that of the hypotenuse makes the area of the height, 16. Subtracting (the area of) the height from that of the hypotenuse makes the area of the base, 9.

ROTATING<sup>36</sup> AND BRINGING TOGETHER THE (PIECES) THAT REVOLVED,<sup>37</sup> ONE OBTAINS GENERATING 3, 4, 5. *Pan* reads like the *pan* of *panhuan* (revolving, describing circles). This

meaningful interpretation here. Li Jimin 1993a, 37, interprets “what goes out and what comes in compensate for each other, what is taken and what is given counterbalance each other.” Qu Anjing 1997, 199, translates: “cut them and move them so as to match each other.” It seems to me that this fails to render the nuances of, not only “taking out,” but also “giving.”

<sup>34</sup> At this point, Zhao Shuang has commented upon the general idea of the transformation that yields the square of the hypotenuse, and he then turns to asserting the procedure of the base and the height in its various cases. In the following sentence, *The Gnomon of the Zhou* makes clear how the geometrical transformation should actually be carried out, what Zhao Shuang glosses by discussing the terms employed (see below). The fact that Zhao Shuang concludes at a point that differs from where *The Gnomon of the Zhou* concludes might relate to the difference in the drawing they are considering, while discussing. More on this below.

<sup>35</sup> Note, as a confirmation, that Zhao Shuang reads this passage as relating to the “Pythagorean theorem,” or “Procedure of the base and the height.” Moreover, the phrasing is the same as that of the “procedure of the base and the height” in *The Nine Chapters* (see below). In addition, Zhao Shuang opposes what precedes to the statement of the procedure prescribing computations, which indicates that the preceding passage is of another nature than the mere statement of the procedure. This is a hint that he himself reads the passage of *The Gnomon of the Zhou* as stating the result, at the same time as establishing the correctness, of the procedure. We shall find other hints of the same fact below. Qu Anjing 1997, 199–200, draws the same conclusion. However, Cullen 2002, 785, seems to understand the text in another way, when he writes: “... the opening dialogue includes a passage which (though rather obscurely phrased) amounts to no more than the statement of the *gonggu* relation for the case [... 3, 4, 5 ...]. No premodern Chinese commentator has ever claimed to see anything more substantial here ...”

<sup>36</sup> This is how I translate *huan*. Later, in *The Gnomon of the Zhou* (Qian 1963, 22), one reads: “Rotating (*huan*) the rectangle (*ju*) to make the circle.” I hence interpret the text as prescribing to rotate the half rectangles that, according to the previous sentence of the Canon, were to be detached from the two squares assembled. This is the main idea introduced by Li Jimin 1993a for interpreting this passage of the Canon. However, it seems that Ch’*en* Liang-t’*o* (1993a, 6–7) arrived at the same idea in his interpretation of Zhao Shuang’s commentary.

<sup>37</sup> Here and below, I understand *gong* as “assemble, bring together.” This verbal use of *gong* is attested to in later mathematical writings. For example, it is quite common in Li Ye’s *Sea-mirror of the circle measurements* (*Ceyuan haijing*, 1247). However, to my knowledge, it is not attested to in other mathematical sources of the time of *The Gnomon of the Zhou*. Zhao Shuang suggests reading *pan* as *panhuan* “revolving, circling round, describing circles, whirling.” This is a meaning for which the *Hanyu dacidian* provides evidence. Li Jimin 1993a, 37, interprets the text as referring to the same transformation as I adopt, but on the basis on a different term-by-term interpretation. He glosses *huan* as “revolve,” and *gong* as “yielding by assemblage.” However, he interprets *pan* as referring to a square. He thus does not follow Zhao Shuang’s suggestion, which I do find quite useful. I interpret the occurrence of *pan* in this passage of *The Gnomon of the Zhou* as designating the pieces that were rotated and hence circled round the configuration (see fig. 2.b, and 2.c, for the result of the operation). Cullen 1996, 85–87, translates “Placing them round together in a ring,” which he finds difficult to interpret mathematically in a meaningful way. Qu Anjing 1997, 197, suggests: “Move it (the half rectangle) around the square.” He thus adopts Li Jimin’s suggestion to interpret *pan* as “square.” However, I am not sure that I understand how he renders *gong*. Chen Cheng-Yih 1987, 35, suggests for the passage, opting for the same text as Qian Baocong: “Now, after drawing a square on the outside [of the hypotenuse], circumscribe it by half rectangles so as to form a square plate.” I do not understand how *ban zhi yi ju* is interpreted from a syntactic point of view. Nor do I see either how *gong* is translated.

means that one refers to taking out the corresponding areas that are subtracted together,<sup>38</sup> rotating, contracting and bringing together the (pieces) that revolved. Dividing this by extraction of the square root<sup>39</sup> yields one of its sides. This is why one says:<sup>40</sup> “one obtains generating 3, 4, 5.”

THE TWO RECTANGLES HAVING BROUGHT TOGETHER/JOINED THEIR LENGTH MAKE 25, HENCE THIS IS CALLED “PILING UP RECTANGLES.”<sup>41</sup>

The “two rectangles” are the areas of the base and the height each multiplied by itself. That which (arises from) “bringing their length together” has the value of the sum of the areas. One has the intention of extending this to all situations and hence first presents the corresponding *li*’s.<sup>42</sup> THEREFORE THAT WITH WHICH YU ORDERED THE WORLD IS THAT FROM WHICH THESE NUMBERS/PROCEDURES AROSE.<sup>43</sup>

Here Shang Gao’s reply to the question of the Duke of Zhou, i.e., the passage of *The Gnomon of the Zhou* in which I am interested, comes to an end. See figs. 1 and 2, which summarize the graphical process alluded to – let me stress that they are all restored, since the text neither contains figures, nor refers to any.

Let us now concentrate on the hints this passage of *The Gnomon of the Zhou* contains regarding the nature and use of the graphical aids alluded to.

<sup>38</sup> The word used for “area” *ji* refers to the area as a number. In contrast to what Cullen 1996, 85, suggests, I do not think that *bing jian* can be understood as “sums and differences,” since, when referring to a sum, *bing* is not an antonym of *jian* “subtracting” – *jian* is rather opposed to such verbs as *jia* (adding). On these terms, see my glossary in Chemla and Guo 2004. I agree with Qu Anjing 1997, 199, on this point. This implies that, according to Zhao Shuang’s understanding, there are more than one piece that are subtracted. This probably refers to the two half-rectangles mentioned by *The Gnomon of the Zhou*, a fact that is confirmed by Zhao Shuang’s following gloss of the sentence. As regards the sentence commented here, Ch’en Liang-ts’o 1993a, 7, n. 7, considers the original text as in conformity to what the Southern Song edition contains (see Appendix A). He punctuates in a way different from mine.

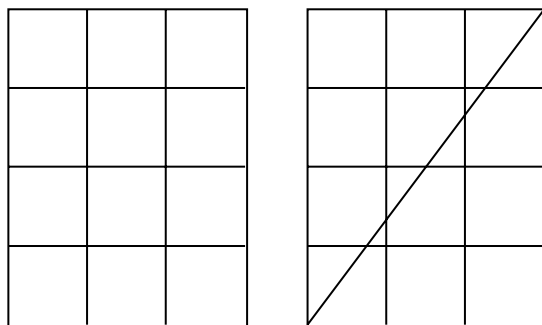
<sup>39</sup> This corresponds to the prescription of a square root extraction. The expression stresses the fact that the operation is a kind of division. The logic of the passage implies that Zhao Shuang understands that “rotating, contracting and assembling the pieces” yields a square the side of which is the hypotenuse. This is in agreement with the interpretation provided. We may have here a description of the physical operations corresponding to implementing in this particular case the general principle: “with what is in excess, one fills up the void.”

<sup>40</sup> In his account for the correctness of what the Canon asserted, Zhao Shuang makes explicit the reasoning establishing the conclusion. Exegesis and mathematical proof are closely connected. On this, see Chapter A, in Chemla and Guo 2004.

<sup>41</sup> The latter expression can be understood in two ways. Either assembling the two rectangles (see footnote 24) represent “accumulating (*ji*) rectangles,” or, once the rectangles that portray the squares of the base and the height, respectively, are assembled, they are reshaped as two other rectangles and a square, which would be conceived of as piling up two cuttings of the same area in terms of rectangles. In relation to the interpretation I give to Liu Hui’s commentary below, I would favor the second interpretation. In fact, it would fit quite well with the opposition described by *The Gnomon of the Zhou* between “rectangles (*ju*)” and “rectangles of the other type (*qi yi ju*).” Ch’en Liang-ts’o 1993b, 118, favors the first one.

<sup>42</sup> This sentence corresponds to Liu Hui’s commentary on the beginning of the chapter “Base and height” of *The Nine Chapters*, where he makes explicit the reason why, in his view, the authors of the Canon placed the “Procedure of the base and the height” at the beginning of the chapter (Chemla and Guo 2004, 704–5, 878–9, n. 6). This remark of Zhao Shuang’s again increases the plausibility that he is reading a general statement in this passage of the Canon. About *li*, see footnotes 16 and 19 above.

<sup>43</sup> On Yu and his deeds, see Cullen 1996, 87. The “procedure of base and height” is thus given to be the basis for major topographical and, hence, cosmographical, enterprises.



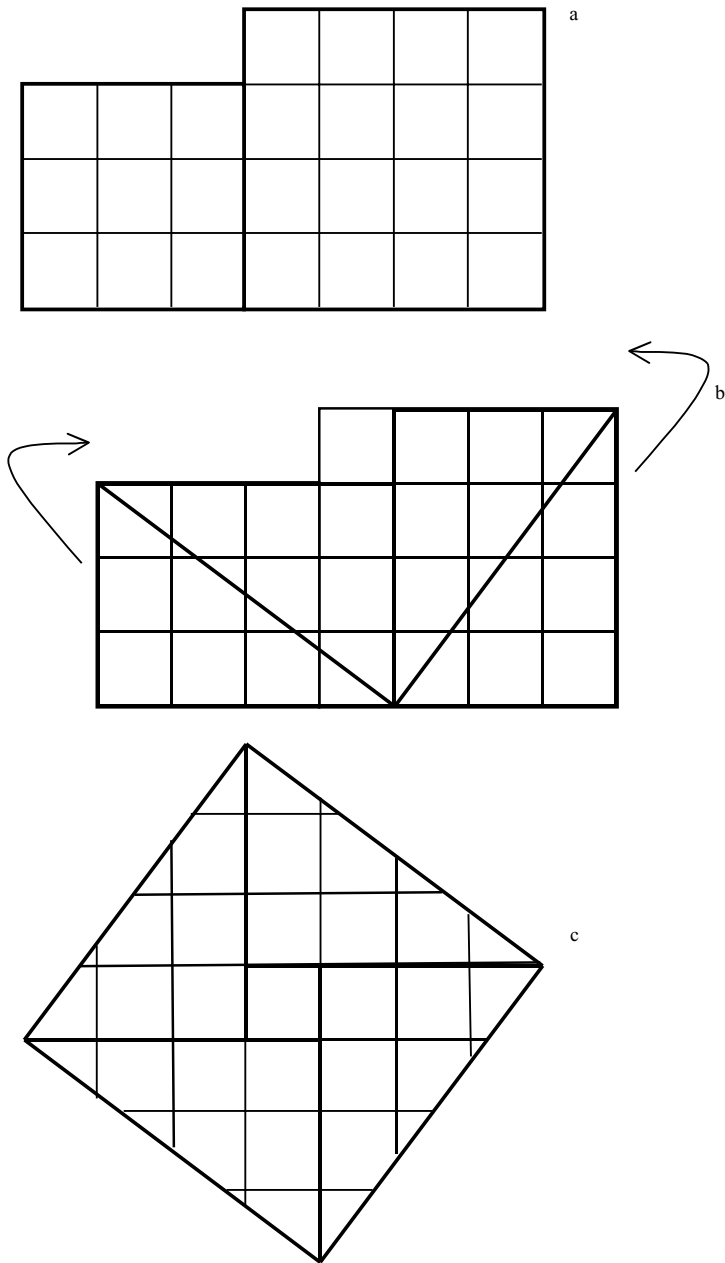
**Fig. 1.** Breaking the rectangle: (left) rectangle; (right) “One breaks the rectangle, to take, as base (*gou*), the width, 3, and take, as height (*gu*), the length, 4. That which goes through the corners is 5.”

Whatever the interpretation of the text may be, some terms are quite telling: “break the rectangle,” “take out the half of rectangles,” “rotate.” These terms seem to indicate that a “drawing” was physically designed in such a way that it could be submitted to concrete transformations of this sort. Shapes were assembled, cut; parts of them rotated, and so on. This description fits with one of the graphical processes described some centuries later by the third century commentator Liu Hui, in a passage of his commentary on *The Nine Chapters* where he refers to drawing figures on paper, cutting them, coloring them, rearranging the pieces cut.<sup>44</sup> Whatever the interpretation of this passage of *The Gnomon of the Zhou* may be, there is no doubt that reference is made to a visual process that is to be carried out while reading the text. In this sense, we do have here the earliest known reference to visual aids in ancient Chinese mathematical texts, even though no word refers to the type of representation used *per se*.

Depending on the interpretation of the text, however, is the type of figures restored.<sup>45</sup> In the reconstitution to which I adhere, the visual aids are composed

<sup>44</sup> See the commentary on problem 9.15 in Chemla and Guo 2004, 728–729. See also Chemla 2001a. Most probably, however, drawings were not made on paper as early as the date of composition of *The Gnomon of the Zhou*.

<sup>45</sup> Beyond differences in the interpretation of the text, Li Jimin 1993a and Ch'en Liang-ts'o 1993a restore the sequence of figures in the same way. However, Ch'en attributes this sequence of transformation only to Zhao Shuang and considers that the reasoning alluded to by *The Gnomon of the Zhou* is different. Chen Cheng-Yih 1993 considers that this sequence of figures captures Liu Hui's reasoning for establishing the “Procedure of the base and the height.” To my knowledge, the first mathematician to have suggested this sequence of transformations is Xiang Mingda, in his *Six procedures for the right-angled triangle (Gougu liushu)*, 1825, p. 5. Note, however, that Qu Anjing 1997, 198, restores another kind of graphical process under the opening passage of *The Gnomon of the Zhou*. If we now leave aside the content of the figures restored to concentrate on their physical appearance, it appears that only Li Jimin restores the drawings with shapes cut in a paper marked by a grid with unit squares. For several reasons, I follow him on this point (Chemla 2001a). If this feature rightly captures an aspect of the ancient figures, this would establish a parallel between the figures used and the problems found in the mathematical sources: all would be paradigms (Chemla 1997). However, I wonder whether this was not a feature that was adopted only later, as we shall see, when shapes were drawn and cut on paper, probably around the third century (Chemla 2001a).



**Fig. 2.** Restoration according to Li Jimin 1993a. (a). “Once these (base and height) have been made square (i.e. rectangles),” (b). “one takes out from the outside the half of rectangles of the other type.” (c). “Rotating and bringing together the (pieces) that revolved, one obtains generating 3, 4, 5. The two rectangles having brought together their length make 25, hence this is called ‘piling up rectangles’.”

and physically transformed while the argument establishing the correctness of the procedure explained develops. This means that, in contrast to what Euclidean geometry accustomed us to, there did not exist a unique figure on which to follow the text. The shapes designed to support the argument evolved through the text, the squares being first brought together, then cut, some of their pieces being moved and so on. In addition to this first remark, one should also stress that there is no other procedure that refers to the same sequence of drawings. These points constitute the main features, with respect to which a new type of figures that emerged in third century commentaries contrasts with earlier visual aids.<sup>46</sup>

## 2. The emergence of a new type of figure attested to in the third century

Right after the passage of *The Gnomon of the Zhou* just examined, Zhao Shuang inserts a long development on the right-angled triangle, entitled “Figures of the base and the height, the square and the circle.”<sup>47</sup> He opens this development with three figures, followed by a running text. Today, the earliest pictorial evidence of the nature of these figures is to be found in the 1213 edition, by Bao Huanzhi, of the Canon and its early commentaries. The last two figures were clearly damaged in the process of transmission. However, I do not need to consider them for my argument here, since, in my view, the first figure suffices to highlight all the new features that are shared by them all and that are meaningful for us here.<sup>48</sup> I shall hence focus on only the first of these figures, entitled “Figure of the hypotenuse” (fig. 3). In fact, it appears to represent a

<sup>46</sup> In this sense, Cullen 1996, 87–8, rightly criticizes Needham 1959, 22–3, according to whom the opening section of *The Gnomon of the Zhou* refers to the diagram contained in Zhao Shuang’s third-century commentary. This stand consists in projecting what is found in the commentary back onto the Canon, a historiographic error that is quite common. However, I do not agree with Christopher Cullen on another point of crucial importance here. Cullen rightly stresses that, in his preface, Zhao Shuang claims to have “drawn diagrams (*tu*) on the basis of the Canon.” But this statement does not seem to me to imply, as it seems to me Cullen states, that there could not be any reference to visual aids in *The Gnomon of the Zhou*. We hence disagree on whether the opening section of the Canon refers to visual aids or not.

<sup>47</sup> One can also interpret this title as “Figures of the right-angled triangle, the square and the circle” (see footnote 21).

<sup>48</sup> The reader interested in the whole set of figures and an edition of the last two ones is referred to my introduction to chapter 9, in Chemla and Guo 2004, 673–684, 695–701. In my view, Qian Baocong 1963, 15–16, restores the set of figures in an incorrect way. Not only are the figures themselves redundant (which contradicts the very idea of this new type of figure), but he also deletes what made the characteristic feature of these figures, i.e. that they be represented as drawn or cut in a paper with a unit square grid. One point needs further clarification. Cullen 2002, 786–787, n. 13, casts doubts on the authenticity of the “Figure of the hypotenuse” in the form found in the 1213 edition. He writes: “. . . it is clear from Zhao’s commentary that the diagram he used was not in the form seen in most versions of the *Zhou bi* nowadays, in which a 7 by 7 square has four 3–4–5 triangles inscribed in its corners, so as to enclose an inclined 5 by 5 square in which four further 3–4–5 triangles are inscribed so as to enclose a unit square. Such a diagram might be used to give a graphical dissection proof of the *gougu* relation, although Zhao does not do this . . . . In fact, it is clear from his description that his *xian tu* “hypotenuse diagram” consisted only of the 5 by 5 square with its inscribed triangles and central unit



Fig. 3. The Figure of the hypotenuse, according to the 1213 edition of *The Gnomon of the Zhou*.

square. The outer part of the usual diagram is never referred to by Zhao and probably originates in the draftsman's construction lines used to construct the inner square and its triangle." However, in his development entitled "Figures of the base and the height, the square and the circle," as is clear from the quotation below (passage (\*\*)), Zhao Shuang, in fact like Liu Hui, does refer to an outer square, whose side equal the sum of the base and the height and that is thus identical to the one found in the 1213 version of the "Figure of the hypotenuse." I hence do believe that, in this respect, it is faithful to Zhao Shuang's diagram. On this question, see also footnote 78.

reshaping of the figures on the basis of which the argument examined previously was developed.

Translating the text on the diagram in fig. 3 from top to bottom, right to left, the two characters at the top: *xian tu*, indicate that this is the “Figure of the hypotenuse.” Now proceeding from right to left, we read:

The square (*shi*)<sup>49</sup> of the hypotenuse, 25, is vermillion and yellow.

The square of the hypotenuse

The base is 3.

Central yellow area (*shi*).

(in horizontal characters) The height is 4.

Vermillion area (*shi*)

(slantwise) The hypotenuse is 5.

The vermillion areas<sup>50</sup> are 6. The yellow area is 1. (Qian 1963, 17, reproduction of the 1213 edition)

If we reproduce it with colors, as indicated on the 1213 drawing (fig. 3), this yields fig. 4.<sup>51</sup>

In fact, as mentioned above, the ancient editions of Liu Hui’s commentary on *The Nine Chapters* handed down to us do not contain any figure. However, in some of Liu Hui’s developments, reference is explicitly made to figures (*tu*). In the case of the drawing above, the reference is so clear that it enables us to recognize essentially the “Figure of the hypotenuse,” with, even, the same colors laid on the same pieces. How are we to account for such stability?

It is first important to recall that our two commentators, Liu Hui and Zhao Shuang, lived in different parts of China, most probably at roughly the same time. None of them refers to the other, but Zhao Shuang mentions a book called *The Nine Chapters*.<sup>52</sup> Zhao Shuang’s development that we partly examine here presents, as a whole, strong correlations with one of the two parts of Liu Hui’s commentary on the chapter “Base and height” of *The Nine chapters*.<sup>53</sup> Unless it can be proved, on the basis of new evidence, that one of the commentators depended on the other, the correlation between their texts and their figures seems to indicate that they both drew on earlier sources, in which case the figure under study would have been shaped before the composition of both commentaries. Whatever the case, our question can hence be rephrased as

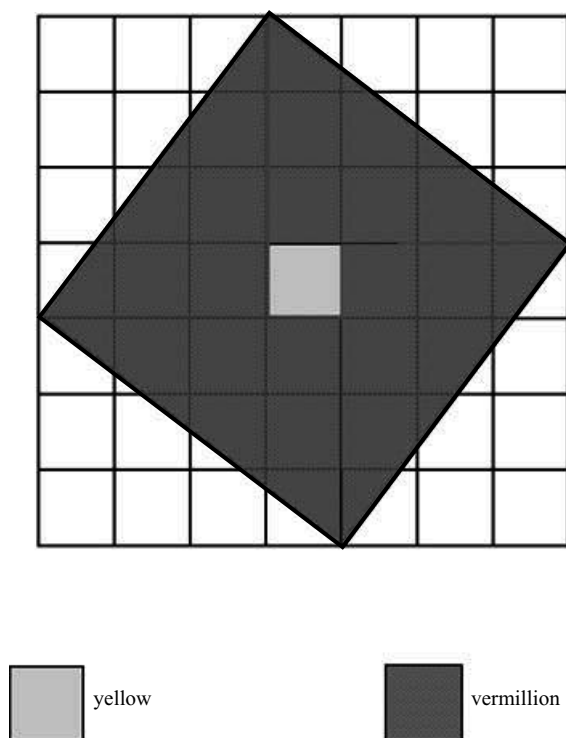
<sup>49</sup> On the translation of *shi* sometimes as “square of . . .,” sometimes as “area,” see footnotes 21, 28, 30 above.

<sup>50</sup> I am indebted to the anonymous referee for suggesting that this sentence could also be interpreted as referring to a unique vermillion triangle. Correspondingly, we would have a drawing with a yellow square and a unique vermillion triangle. This is possible. However, taking into account the introductory sentence “The square (*shi*) of the hypotenuse, 25, is vermillion and yellow,” I do believe that understanding a plural here is a better choice.

<sup>51</sup> In the 1213 figure (fig. 3), the lines of a grid drawn slantwise mark the central square with a side equal to the hypotenuse.

<sup>52</sup> See footnotes 26 and 35 above.

<sup>53</sup> Shen Kangshen 1982 develops an exhaustive parallel between Liu Hui’s and Zhao Shuang’s texts in this respect. Also see my introduction to chapter 9 in Chemla and Guo 2004.



**Fig. 4.** A modern reproduction of the “Figure of the hypotenuse.”

follows: why is it that, once it had taken shape, the figure apparently experienced no change when being incorporated into new texts? My suggestion to account for this fact is that it may have been a consequence of the new practice with figures that I shall now describe.

To establish this point, let us first concentrate on the use made, in Zhao Shuang’s commentary, of the figure reproduced above and on how this bears witness to a new kind of, and practice with, figures. On this basis, we shall then turn to the evidence found in Liu Hui’s commentary and compare both authors in this respect.

As already mentioned, the “Figure of the hypotenuse” constitutes, together with the two other twin figures, the opening component of Zhao Shuang’s own development on the right-angled triangle. The key fact is that these figures form the basis for the whole development in the following sense: all the algorithms placed after the figures derive from them, in that the reasons for their correctness are drawn from these figures and only from them. To highlight this point, let us translate here those passages of the

development relating to the figure under inspection – the same conclusions can be drawn with respect to the other figures.<sup>54</sup> They read as follows:

Figures of the base and the height, the square and the circle.

(1) Base and height being each multiplied by itself, summing up these (results) makes the square of the hypotenuse.<sup>55</sup> Dividing this by extraction of the square root hence (gives) the hypotenuse.

(2) (\*)<sup>56</sup> Relying on the “Figure of the hypotenuse,” one can *further* consider the multiplication of the base and the height by one another as 2 samples of the vermilion area (*shi*); doubling this (result) makes four samples of the vermilion area. One takes the multiplication by one another of the difference between the base and the height and itself as the central yellow area (*shi*). Adding one sample of the square (*shi*) of the difference (to the four obtained previously) *also* generates the square of the hypotenuse.

(3) Subtracting the square of the difference from the square of the hypotenuse, halving the corresponding result, taking the difference as “joined divisor,”<sup>57</sup> dividing

<sup>54</sup> The reader can find a full translation of the whole passage in Gillon 1977, in Cullen 1996, 208–17, and in my introduction to chapter 9 in Chemla and Guo 2004, 695–701. As above, in order not to break the continuity of the text, I insert my comments in footnotes. However, I introduce in the main text numbers between brackets: they are attached to the successive algorithms described by Zhao Shuang, and I shall use them to refer to the algorithms more conveniently.

<sup>55</sup> Two details are worth noting here. First, in contrast to the following algorithm (2), there is no argument given to establish the algorithm (1) here. Secondly, again in contrast, there is here no reference to the figure. These facts constitute an additional argument indicating that Zhao Shuang considers the text previously examined as containing a proof of the correctness of this assertion. Several other points lead to this conclusion. The first one is precisely what we stress in this section: all the following algorithms Zhao Shuang states are based on the figures placed before the whole development. This one must hence also be. In fact, the sequence of figures restored behind the argument of *The Gnomon of the Zhou* can easily be embedded into figure 3, and thus the argument too can be made on the basis of the latter. This might have been one of the facts Zhao Shuang had in mind when, in his preface, he claimed to have “drawn diagrams (*tu*) on the basis of the Canon.” Moreover, the fact that, in his description of algorithm (2), Zhao Shuang uses both the words “further” and “also” indicates that the sentence examined here also refers to the “figure of the hypotenuse,” as a transformation of the earlier graphical devices. Geometrically, with respect to the argument, both the graphical device and figure (3) amount to the same. However, graphically and in practice, as we shall see, they constitute two different types of figure.

<sup>56</sup> I introduce the mark (\*) to be able to refer to this passage of the text below. It does not belong to the original text.

<sup>57</sup> This technical term refers to the coefficient in  $x$  of a quadratic equation. The coefficient in  $x^2$  of such an equation was, at that time, always implicitly taken to be equal to 1 and its constant term was called “dividend” (*shi*, same term as area, see my glossary in Chemla and Guo 2004, 977–978). In this case, the introduction of the term “joined divisor” leads retrospectively one to understand that, in the algorithm, (3) the area computed previously is the “area/dividend/constant term” of the equation, which states:

$$\frac{1}{2}(c^2 - (b - a)^2) = (b - a)x + x^2.$$

The equation allows, when knowing the hypotenuse and the difference between the base (*gou*,  $a$ ) and the height (*gu*,  $b$ ), to determine the dimensions of a right-angled triangle. The base  $a$  is its solution, what Zhao Shuang describes as a «restoring» (*fu*). By reference to fig. 4, the area/constant term can be interpreted as a rectangle composed of two vermilion areas (we shall see below that such reasoning is carried out by Liu Hui as well as by Zhao Shuang). The square of the unknown,  $a^2$ , leaves in it a rectangle, the dimensions of which are respectively  $x$  and  $(b - a)$ . This provides a geometrical figure of the quadratic equation that can be read on fig. 4. In ancient China, on the basis of this geometrical figure, quadratic equation was linked to square root extraction and solved

this by extraction of the square root yield, as a restoring, the base. Adding up the base to the difference hence (gives) the height. (...)

(4) (\*\*) The reason why, when doubling the square of the hypotenuse and subtracting<sup>58</sup> from it the square of the difference between the base and the height, there appears the square of the sum is that, if one examines it with the figure, doubling the square of the hypotenuse fills up the big outer square and there is a yellow area in excess. This yellow area in excess is the square of the difference between the base and the height. Subtracting from this (the former result) the square of the difference and extracting the root of the corresponding remainder hence yields the side of the big outer square. The side of the big square is the sum of the base and the height.

(5) Carrying out the multiplication of the sum by itself and then subtracting it from the double of the square of the hypotenuse, extracting the root of the corresponding remainder yields the side of the central yellow square. The side of the central yellow square is the difference between the base and the height. Subtracting the difference from the sum and halving this (result) makes the base. Adding up the difference to the sum and halving this (result) makes the height.

If the double of the hypotenuse is taken as the assembling of the width and the length (...).<sup>59</sup> (Qian 1963, 18; my emphasis)

Several remarks can be made here on the relationship between the text and the figure.

The figure is used to provide a geometrical interpretation of operations prescribed by various algorithms. In this process, two pieces of area appear to play a key role: on the one hand, the central square, that is colored in yellow and the side of which is the difference ( $b-a$ ) of the height ( $b$ ) and the base ( $a$ ), and, on the other hand, the triangle of dimensions ( $a, b, c$ ), colored in vermillion.<sup>60</sup> These pieces are those that are

as if extracting a square root, which explains that prescribing to solve the equation amounts to prescribing the extraction of a square root. I can by no means dwell on these points. For a more accurate treatment, see my introduction to chapter 9, in Chemla and Guo 2004. Up to here, the passage can hence be interpreted with respect to fig. 4. There follows a passage relating to the other figures, which we skipped, after what Zhao Shuang comes back again to fig. 4.

<sup>58</sup> I adopt here an emendation suggested by Guo Shuchun and Liu Dun in their new critical edition of *The Gnomon of the Zhou* (Guo and Liu 1998, 3, 35 n. 23). All ancient sources have *lie* instead of the graphically similar character *jian* “subtract.” They hence suppose that a copyist mistakenly copied one for the other. Moreover, I introduce the mark (\*\*) to make easier the reference to this passage of the text below.

<sup>59</sup> Here starts a new reading of the same figure, where the sum of base and height is interpreted as the double of the hypotenuse of another triangle and where the whole figure is read in a different way. I refer the reader to my introduction to chapter 9 in Chemla and Guo 2004, 700–701, for a more extensive discussion of this passage.

<sup>60</sup> On the figure, the triangle is a surface, that of the half-rectangle, in the sense that, in terms of relations, the area of the rectangle is given to be twice that of the triangle. The triangle is physically marked as a surface, since its area is colored. As mentioned above, the right-angled triangle appears in ancient China as a configuration of lines, base, height and, later, hypotenuse. The distinction between the two concepts can be grasped thanks to a remark: never, except in the passage of *The Gnomon of the Zhou* quoted above and on the figure included in Zhao Shuang’s commentary, where the two are articulated, are the sides of the vermillion triangle designated by the names of “base,” “height” and “hypotenuse.” Two kinds of geometrical representations are hence articulated here.

marked by colors in Zhao Shuang's figure. The same pieces are marked by the same colors when Liu Hui refers to the figure. These pieces are the elements with which, respectively, the squares based on either the difference or the sum of the base and the height, as well as the square of the hypotenuse, can be decomposed. The figure hence appears to be a way of displaying the relationships between these three areas, in terms of the two basic elements constituted by the yellow and the vermillion pieces.

If we look at the algorithm (2) described by Zhao Shuang, the computation of  $ab$  is interpreted as corresponding to two samples of the vermillion areas; its double,  $2ab$ , corresponds to 4 such samples. Added to  $(b-a)^2$ , interpreted as the yellow piece, the area obtained is interpreted as the square of the hypotenuse. On the one hand, we have the statement of an algorithm computing the side of the hypotenuse, when one knows the base and the height. On the other hand, the step-by-step interpretation, based on the figure, brings to light the reasons underlying the correctness of the algorithm. The yellow and vermillion pieces are the units of a geometric computation allowing establishing algorithms.

It is in this way that, as stated above, the figures placed at the beginning of Zhao Shuang's development are the basis for accounting for the correctness of all the algorithms described after them. They are fundamental in this respect. This constitutes, in my view, the earliest extant example of a phenomenon for which we find many other examples in subsequent Chinese writings: some figures, considered fundamental, are placed at the beginning of sections of books, or at the beginning of books, that are based on them. In this regard, one can think of the so-called "Pascal triangle,"<sup>61</sup> placed by Yang Hui at the beginning of chapter 4 of his commentary of *The Nine Chapters* completed in 1261, *Detailed explanations of The Nine Chapters on mathematical methods* (*Xiangjie jiuzhang suanfa*). The same phenomenon can be identified when, in the same book, fundamental figures are placed at the beginning of the chapter devoted to the right-angled triangle. One can further think of the drawings placed at the beginning of *Sea-mirror of circle measurements*, by Li Ye (1248), or of *Jade mirror of the four elements* by Zhu Shijie (1303). If we now go back to Zhao Shuang's figure discussed in this section, its position in the commentary may express the fact that it was considered fundamental with respect to the development following it. This feature would in turn account for its stability in space and time, i.e. for the fact that we find it unchanged in several writings by different authors.

To sum up, two key differences appear to characterize this figure in contrast to the graphical process described in *The Gnomon of the Zhou*.

First, the figure is no longer an object that is created in relation to an algorithm and reshaped while proving its correctness. It is completed before the statement of any algorithm. It is then used to interpret the results of the successive steps of procedures and

<sup>61</sup> See Chemla 1994. The reader finds there reasons to believe that there was direct mathematical exchange between China and the Arabic world. This is the background against which one can read the final section of this paper.

show how they combine to yield the result sought for. The figure remains unchanged along the sequence of interpretations that establishes an algorithm. In correlation with this, the figure is comprehensive enough to encompass all the links between magnitudes as well as the geometrical reshaping needed. As mentioned earlier (footnote 55), the set of figures that are succeeding each other in the graphical procedure sketched by *The Gnomon of the Zhou* can all be embedded in the “Figure of the hypotenuse,” and the reasoning outlined in its opening passage discussed above can be carried out on the basis of this one of Zhao Shuang’s figures. The first statement of the commentator’s development is probably to be understood with the previous text of *The Gnomon of the Zhou* and Zhao Shuang’s commentary on it in mind.

Secondly, and probably most importantly, the figure is the basis for establishing in a uniform way the correctness of *several* algorithms. This fact is illustrated by the passage of Zhao Shuang’s development translated above: it contains all the algorithms, the correctness of which can be brought to light by examining the first fundamental figure.<sup>62</sup> This key feature characterizes figures like the “Figure of the hypotenuse,” in contrast to the earlier visual aids: they are *general* figures, in the sense that each such figure offers a basis for showing the correctness of various distinct algorithms. Zhao Shuang stresses this property right at the beginning of his development, when he emphasizes that he uses *again* the “Figure of the hypotenuse” to establish that another algorithm *also* yields the hypotenuse.<sup>63</sup>

This manifestation of an interest in generality is not surprising: it seems to constitute, for figures, the reflection of the emphasis more generally placed on generality in the mathematics of ancient China.<sup>64</sup> However, it does not yet seem to be perceptible as such with respect to figures in *The Gnomon of the Zhou*. In contrast, as we shall soon see, Liu Hui’s commentary on *The Nine Chapters* attests to the same use as Zhao Shuang of figures such as the “Figure of the hypotenuse.” Moreover, we shall show that this can also be interpreted as linked to an interest in generality, taken in the same sense. This convergence thus highlights that we are dealing here not with one of Zhao Shuang’s peculiarities, but with a more general phenomenon. Such constraints bearing on figures understandably led to elaborate specific figures that were to experience a certain stability.

These remarks lead me to put forward the hypothesis that figures like the “Figure of the hypotenuse” bear witness to the emergence of a new type of diagrams, different from what earlier writings like *The Gnomon of the Zhou* alluded to, and that their

<sup>62</sup> My introduction to chapter 9, in Chemla and Guo 2004, shows how the remaining part of Zhao Shuang’s development can be based on the other two figures, as restored by Li Jimin 1993a, 34.

<sup>63</sup> Chen Cheng-Yih 1993, 482, and Qu Anjing 1997, 200–201, also emphasize these words, but, like Ch’en Liang-ts’o 1986, 278, they interpret this passage of the text as describing another proof of the “Procedure of the base and the height,” put forward by Zhao Shuang, or another diagram for it. I suggest rather that Zhao Shuang is describing *another algorithm* – algorithm (2) – to derive the value of the hypotenuse, and, at the same time, accounting for its correctness on the basis of the same figure.

<sup>64</sup> See, for example, Chemla 2001b.

emergence was driven by the search for general figures, commanding the greatest number of algorithms possible. These figures are dissected into elementary constituents that are distinguished by colors. They thereby allow interpreting algorithms on the basis of these constituents, and, in fact, they allow interpreting several algorithms in the same way. Such new figures and uses are attested to, at the latest, in the third century. However, we saw reasons, above, to believe that they might have appeared earlier. The main argument for this is that the “Figure of the hypotenuse” and the uses we described belong to a body of knowledge relating to the right triangle that the two third-century commentators Liu Hui and Zhao Shuang seem to share. This might come from the fact that they both drew on the same older sources.

To complete my overall argument, I shall hence now turn to a passage from Liu Hui’s commentary demonstrating that the third-century commentator of *The Nine Chapters* was using the “Figure of the hypotenuse” in the same way as Zhao Shuang did. However, this passage not only allows providing evidence for the interest in generality highlighted above; it also reveals yet another way in which Liu Hui conceived of generality in relation to figures. We shall hence develop our argument with the aim in mind to bring to light this second aspect.

### 3. Generality and geometrical figures in third-century China

Let us first examine the context within which Liu Hui inserts the development in which we are interested.

Chapter 9 of *The Nine Chapters*, entitled “Base and height,” is entirely devoted to the right-angled triangle and, as for the whole book, displays mathematical knowledge in the shape of problems and algorithms to solve them. It starts with a series of abstract problems that read as follows:<sup>65</sup>

SUPPOSE THAT THE BASE (GOU) IS 3 CHI AND THE HEIGHT (GU) 4 CHI.

ONE ASKS HOW MUCH THE HYPOTENUSE MAKES.

ANSWER: 5 CHI

SUPPOSE THAT THE HYPOTENUSE (GOU) IS 5 CHI AND THE BASE (GU) 3 CHI.

ONE ASKS HOW MUCH THE HEIGHT MAKES.

ANSWER: 4 CHI

SUPPOSE THAT THE HEIGHT (GU) IS 4 CHI AND THE HYPOTENUSE (GOU) IS 5 CHI.

ONE ASKS HOW MUCH THE BASE MAKES.

ANSWER: 3 CHI

PROCEDURE OF THE BASE AND THE HEIGHT: (...)

<sup>65</sup> Again, I translate the text of the Canon in capital letters and Liu Hui’s commentary in small letters. Regarding *The Nine Chapters* and its commentaries, unless otherwise stated, I follow the critical edition provided in Chemla and Guo 2004.

BASE AND HEIGHT BEING EACH MULTIPLIED BY ITSELF, ONE ADDS (THE RESULTS) AND DIVIDES THIS BY SQUARE ROOT EXTRACTION, HENCE THE HYPOTENUSE. (...) <sup>66</sup>

FURTHER, THE HEIGHT MULTIPLIED BY ITSELF IS SUBTRACTED FROM THE HYPOTENUSE MULTIPLIED BY ITSELF. ONE DIVIDES THE CORRESPONDING REMAINDER BY EXTRACTION OF THE SQUARE ROOT, HENCE THE BASE. (...)

FURTHER, THE BASE MULTIPLIED BY ITSELF IS SUBTRACTED FROM THE HYPOTENUSE MULTIPLIED BY ITSELF. ONE DIVIDES THE CORRESPONDING REMAINDER BY EXTRACTION OF THE SQUARE ROOT, HENCE THE HEIGHT. (...)

It is in his commentary on the eleventh problem of the chapter (problem 9.11) that Liu Hui makes an explicit reference to a figure clearly identical to the “Figure of the hypotenuse” discussed above. Let us translate the problem and algorithm, together with the relevant passage of Liu Hui’s commentary, before commenting on it. The problem is cast in terms of a particular situation: <sup>67</sup>

SUPPOSE ONE HAS A ONE-LEAF DOOR, <sup>68</sup> THE HEIGHT OF WHICH IS GREATER THAN THE WIDTH BY 6 *CHI* 8 *CUN* AND THE (OPPOSITE) CORNERS OF WHICH ARE AT A DISTANCE OF EXACTLY 1 *ZHANG* FROM ONE ANOTHER. <sup>69</sup> ONE ASKS HOW MUCH ARE RESPECTIVELY THE HEIGHT AND THE WIDTH OF THE DOOR.

ANSWER:

THE WIDTH IS 2 *CHI* 8 *CUN*;

THE HEIGHT IS 9 *CHI* 6 *CUN*.

PROCEDURE: ONE CARRIES OUT THE MULTIPLICATION OF 1 *ZHANG* BY ITSELF, WHICH MAKES THE DIVIDEND (*SHI*). <sup>70</sup> ONE TAKES HALF OF THAT BY WHICH ONE IS GREATER THAN

<sup>66</sup> All the passages left untranslated are commentaries. Liu Hui’s comment on this procedure that is inserted here is translated below.

<sup>67</sup> Compare with the English translation in Shen et al. 1999, 476 sq. The piece of Liu Hui’s commentary translated here seems to have been badly damaged through transmission. It is hence one of those whose critical edition raises the most difficult problems. The various authors who tackled this question adopted very different views as to the restoration of the original text. In his afterword to Shen et al. 1999, 560–561, Prof. Shen Kangshen explains that the translation in modern Chinese on which the English translation of *The Nine Chapters* is based relies mainly on Qian Baocong’s 1963 edition of the text, while incorporating suggestions coming from Bai Shangshu, (Guo Shuchun 1990) and (Li Jimin 1993b) (not 1994, compare Shen 1997, III). This Chinese text, published in Shen 1997, does not however make clear the modalities of choice between these various sources. For this passage, Shen Kangshen seems to follow Qian Baocong’s edition, which is quite far from what the ancient sources contain. As mentioned above, I follow here the critical edition provided in Chemla and Guo 2004, 716–718, except for one sentence (see below). For the reader’s convenience, in appendix B, I repeat the Chinese text from Chemla and Guo 2004, though with some modifications in the punctuation. However, I do not repeat the critical apparatus, referring in this respect the reader to the book. Only for the sentence for which I suggest an additional emendation, do I add here a footnote.

<sup>68</sup> *Hu* “one-leaf door” is to be distinguished from the door with two leaves (*men*), considered by the problem placed immediately before in the Canon. Names for the dimensions of the door indicate that its shape is rectangular.

<sup>69</sup> 1 *zhang* = 10 *chi*. 1 *chi* = 10 *cun*.

<sup>70</sup> Remember that the word *shi* means both “dividend” and “area.” When it refers to the “dividend,” whether it is for a division or a root extraction, it designates the position of the dividend on the counting surface. If some computations carried out change the value of the number stored in the position, the name of the position

THE OTHER AND ONE CARRIES OUT ITS MULTIPLICATION BY ITSELF. ONE DOUBLES THIS (RESULT) AND SUBTRACTS FROM THE DIVIDEND. ONE TAKES HALF OF THE CORRESPONDING REMAINDER<sup>71</sup> AND DIVIDES THIS BY EXTRACTION OF THE SQUARE ROOT. FROM WHAT ONE OBTAINS, ONE SUBTRACTS HALF OF THAT BY WHICH ONE IS GREATER THAN THE OTHER, HENCE THE WIDTH OF THE DOOR; TO IT, ONE ADDS HALF OF THAT BY WHICH ONE IS GREATER THAN THE OTHER, HENCE THE HEIGHT OF THE DOOR.

Let the width of the door make the base (*gou*), the height (*gao*) make the height (*gu*, of the right-angled triangle),<sup>72</sup> the distance from one corner to the (opposite) one, 1 *zhang*, make the hypotenuse, that by which the height is greater than the width, 6 *chi* 8 *cun* make the difference between the base and the height. One determines their position on the basis of the figure.<sup>73</sup> The square (*mi*) of the hypotenuse fills exactly 10000 *cun*. If one doubles it, subtracts the square (*mi*) of the difference between the base (*gou*) and the height (*gu*), and divides this (result) by extraction of the square root, that which is obtained gives the value of the sum of the height and the width.<sup>74</sup> Subtracting the difference from the sum and halving this (result) gives the width of the door. Adding the value of that by which one is greater than the other gives the height of the door.

Now, this procedure first looks for their halves.<sup>75</sup> “1 *zhang* multiplied by itself”<sup>76</sup> makes 4 samples of vermillion areas and 1 sample of yellow area. “Half” the difference

will then refer to the new value. This accounts for the use of the “dividend” in what follows: the “dividend” on which the square root extraction algorithm is carried out is the last value placed in the position bearing this name. However, in addition to this meaning, Liu Hui will introduce a figure on which to interpret these successive values as areas (*shi*).

<sup>71</sup> The description of the procedure leading to computing this quantity appears to be quite indirect. We’ll come back to this detail below.

<sup>72</sup> Liu Hui’s commentary starts by reading the dimensions of the rectangle that the door makes in terms of a right-angled triangle. This reminds us of the opening section of *The Gnomon of the Zhou*, translated above. We translate as “height” two different Chinese terms: the height of the door (*gao*) and the greater side of the right-angled triangle (*gu*). The context helps to distinguish between both, and, in case there is a danger of ambiguity, I add the pinyin between brackets.

<sup>73</sup> Below, we shall find reasons to understand that the figure referred to here is the same as Zhao Shuang’s “Figure of the hypotenuse.” In relation to this, it is interesting that, even though, in his opening sentence, Liu Hui introduced the technical terms of the right-angled triangle, when he accounts for the correctness of the procedure, he rather makes use of the terms of the problem.

<sup>74</sup> One recognizes the algorithm described by Zhao Shuang in relation to the “Figure of the hypotenuse” that we indicated by the sign (\*\*).

<sup>75</sup> Liu Hui first stated a procedure that solved the problem. In parallel to the description of the procedure, he made clear the meaning of its main steps. This highlights why, in the end, the procedure yields the result. It is highly probable that the reasoning supporting the interpretations is based on the “Figure of the hypotenuse,” in the way in which, as we saw above, Zhao Shuang was accounting for it as well as the other algorithms. The first hint in favor of this stand is that Liu Hui refers to a figure. The second hint is that, below, when accounting for the correctness of another procedure derived from the former one and leading to establishing the correctness of the algorithm as stated in *The Nine Chapters*, Liu Hui clearly refers to this figure. This is why I suggest understanding that both form one and the same figure. To turn to the procedure stated by *The Nine Chapters*, Liu Hui makes explicit what he understands is its intention: computing half the sum and half the difference of the base and the height, instead of their full value. This is how he accounts for why the procedure is indirect.

<sup>76</sup> Liu Hui quotes step by step the procedure of the Canon and interprets their meaning in exactly the same way as we saw Zhao Shuang do, when establishing the meaning of the steps of the first procedure on the “Figure of the hypotenuse.” I mark the quotations in Chinese by quotation marks, even though they cannot appear as quotations in English, due to the difference of grammar between Classical Chinese and English.

“multiplied by itself,” further “double this” makes two-fourths of the yellow area.<sup>77</sup> “Subtracting from the dividend, halving the corresponding remainder,” one has 2 samples of vermilion areas and one-fourth of yellow area. With respect to the greater square, this makes one fourth.<sup>78</sup> Consequently, dividing by extraction of the square root yields half of the value of the sum of the height and the width. Subtracting half the value of the difference from half the value of the sum yields the width; adding yields the height of the door.

Furthermore, on the basis of the area (*mi*) of this figure, the square (*mi*) of the sum of the base (*gou*) and the height (*gu*) with one another<sup>79</sup> to which one adds the square (*mi*) of their difference *also* generates the square of the hypotenuse.<sup>80</sup> One makes these areas,

<sup>77</sup> Here is the key step accounting for the form of the procedure as described in *The Nine Chapters*: a unit of area was introduced – a fourth of the yellow central square – and this, together with the vermilion triangle, will be the basis on which computations will be interpreted.

<sup>78</sup> This sentence is quite important for what it reveals concerning the figure on the basis of which the reasoning is carried out. On the one hand, regarding the appearance of the figure, it implies that the greater square has its sides equal to the sum of the base and the height (see footnote 48 above). Moreover, the yellow and vermilion areas are the same as those in the “Figure of the hypotenuse.” It hence seems reasonable to identify the figure alluded to here and Zhao Shuang’s figure. Furthermore, as a side result, this incites one to believe in the authenticity of the main features of Zhao Shuang’s figure as represented in the 1213 edition. On the other hand, the sentence is highly revealing of the kind of geometrical computation that was made on the basis of the figure and its colored units. The values computed correspond to areas that are analyzed in terms of two unit pieces: the vermilion, and the fourth of yellow, pieces. With respect to these units, it appears that the area computed consists of one-fourth of the pieces composing the whole area, hence the result. Let us stress the point here: on one side, as geometrical entities, the surfaces do not coincide, but, on the other side, it is not only the values of the areas that are equal. A third mode of relationship between areas is introduced: they can be decomposed into the same unit pieces. In this respect, it appears that it is not only the figure that is fundamental, but also the pieces into which it was dissected: the vermilion and the yellow sectors are crucial for the way in which they allow interpreting a set of algorithms based on the figure and establishing their correctness. In the case of the algorithm provided by *The Nine Chapters* after problem 9.11, a new fundamental unit piece is needed: the fourth of the yellow central square, and Liu Hui may be interpreting that it was for the sake of introducing it that *The Nine Chapters* adopted this indirect description of the algorithm. In any case, his practice with the “Figure of the hypotenuse” proves, here, to be exactly the same as Zhao Shuang’s.

<sup>79</sup> The commentator now turns to using, in relation to the figure, the terms of the right-angled triangle rather than the terms of the situation of the problem. Moreover, he introduces the term *mi* “area, square,” which he uses in the same way as, above, Zhao Shuang was using *shii* (see footnotes 28 and 30).

<sup>80</sup> My emphasis. This sentence proves to be crucial. First, let me make clear that I deviate here from the text as restored in the critical edition provided by Chemla and Guo 2004. Instead, I follow the suggestion I made in footnote 46 to the translation of chapter 9 (Chemla and Guo 2004, 884–885). See appendix B, where I sum up the additional emendation and give my punctuation for this sentence. The emendation is inspired by the discussion in Li Jimin 1993b, 495–6, and suggests that the character *jian* “subtract” is a corruption of *cheng* “generate.” As a consequence, the conclusion of the sentence is the same as that of the passage marked (\*) in Zhao Shuang’s commentary translated above: “... also generates the square of the hypotenuse.” It reveals an interest in generating in different ways the square of the hypotenuse. We have seen how Zhao Shuang insisted on the second generation of this area that he provided. We see here that Liu Hui would then place emphasis on the same point, even though he describes yet another mode of generation. Now, the question is: what does his “also” refer to here? The answer is straightforward: in the preceding commentaries on *The Nine Chapters*, only one other mode of generating the square of the hypotenuse is mentioned: that given by the “Procedure of the base and the height.” We will come back to Liu Hui’s commentary on it below. Let us simply keep in

since one makes the corresponding hypotenuse appear first, and, then, one knows the corresponding base and height.

Suppose they are equal.<sup>81</sup> Multiplied by itself, each of them also makes a square (*fang*).<sup>82</sup> And, assembled, they make the square (*mi*) of the hypotenuse.

Taking the half of that by which one is greater than the other and multiplying it by itself, doubling this;<sup>83</sup> further, half the sum being multiplied by itself, doubling this;<sup>84</sup> (the areas obtained) *also*, assembled, make the square of the hypotenuse.<sup>85</sup>

mind for the moment that, as for Zhao Shuang, the “also” must hence refer to this other mode of generating the square of the hypotenuse by adding up the squares of the base and the height. However, as it stands, the sentence raises another question: why is it that, in the passage of Liu Hui’s commentary examined here, the commentator seems to be stating a wrong algorithm? The full and exact algorithm would have it that, by adding  $(b+a)^2$  to  $(b-a)^2$ , you get  $2c^2$ . Why does Liu Hui conclude that it “generates the area of the hypotenuse”? My interpretation for this is as follows: in this passage, Liu Hui is in fact *less* stating an algorithm than he is insisting on an alternative way of generating the square of the hypotenuse. In this respect, he lays emphasis on the fact that the generation should be carried out *directly* from the sum and the difference of the base and the height. In contrast, as the next sentence makes clear, he insists on *not* deriving the base and the height from their sum and difference, to then make use of the “Procedure of the base and the height.” This explains, in my view, why, here, he stresses the *modalities* of generation of the square of the hypotenuse rather than the exact algorithm as such. However, below, he will rely on the exact algorithm based on the full sum and difference, to state, like previously, the algorithm yielding directly  $c^2$  on the basis of the half sum and the half difference. He, hence, repeats, with respect to yielding  $c$  on the basis of  $(b+a)$  and  $(b-a)$ , the two-step reasoning that he previously developed for yielding  $(b+a)$  on the basis of  $c$  and  $(b-a)$ . Note that this is not the only case when the statement of an algorithm is concluded by that which it can yield rather than by what it actually yields. See *de* “yield,” in my glossary (Chemla and Guo 2004, 915).

<sup>81</sup> Before considering how to yield  $c$  on the basis of  $(b+a)/2$  and  $(b-a)/2$ , Liu Hui takes a step that will prove decisive for the issue we examine. He considers the first algorithm generating the square of the hypotenuse, i.e. the “Procedure of the base and the height,” and examines what the geometrical process yielding the square of the hypotenuse on the basis of the squares of the height and the base becomes, in the case when  $a=b$ . This is illustrated by fig. 6, as we justify below.

<sup>82</sup> This “also” refers to the geometrical reasoning developed by Liu Hui for the general case, after the third problem of the ninth of *The Nine Chapters*. We shall examine it in detail below, but the reader can take a glimpse at fig. 5 to get an idea of its main lines. For the general case, Liu Hui concluded this reasoning by a sentence quite similar to the one he uses here: “( $a^2$  and  $b^2$ ), assembled, generate the square of the hypotenuse.” It is interesting to notice that, in his comment on the opening passage of *The Gnomon of the Zhou*, Zhao Shuang had also considered the case when their squares of the base and the height are equal (see footnote 32). Let us note that the term used here by Liu Hui to designate the square, *fang* – the same as in his commentary on the “Procedure for the base and the height” (see below) – refers to the square as a geometric entity.

<sup>83</sup> After the digression concerning the particular case when  $a=b$ , the commentator takes up again the issue of generating, in an alternative way, the square of the hypotenuse. Note that, in transforming the algorithm yielding  $c$  on the basis of  $(b+a)$  and  $(b-a)$  into the algorithm yielding  $c$  on the basis of  $(b+a)/2$  and  $(b-a)/2$ , Liu Hui uses the same kind of indirect description as *The Nine Chapters* for problem 9.11. If we make use of the geometrical computation described above, this step yields two-fourths of the yellow square.

<sup>84</sup> The value yielded can be interpreted, in the same vein as above, as 2 vermilion pieces and two-fourths of the yellow square.

<sup>85</sup> As can be seen below, this “also” again refers to the concluding statement of Liu Hui’s commentary on the “Procedure of the base and the height,” which reads as follows: “( $a^2$  and  $b^2$ ), assembled, make the square of the hypotenuse.”

And, in case the value of the difference vanishes, whether each is multiplied by itself or they are multiplied by each other, the value, in each case makes the area of the door.<sup>86</sup>

With respect to the (situation) when the height is longer and the base shorter, from a same origin, differences flow. (...)

Before discussing the interpretation of this last sentence, some conclusions need to be drawn in relation to the argumentation developed in section 2 of this paper. As was stated above, it is clear from this long quotation that, in his commentary on *The Nine Chapters*, Liu Hui refers to a figure comparable in every point to the “Figure of the hypotenuse” as it is found in Zhao Shuang’s commentary on *The Gnomon of the Zhou*. Moreover, the figure is fundamental for him in the same way as what was described for Zhao Shuang. It is the basis on which various algorithms can be interpreted and proved to be correct.<sup>87</sup> This nicely complements the evidence on the basis of which we put forward the hypothesis that a new type of diagram emerged in China, at a moment that could be located between the time of the compilation of *The Gnomon of Zhou* and that of the composition of the two commentaries. Such figures, general in the first sense we introduced, may well have emerged in the context of commentaries and be typical of the kind of exegesis commentators developed on Canons.

At this point of the quotation, however, we have reached the crux for our second conclusion, relating to the practice of figures, as evidenced by the commentaries on *The Nine Chapters*, and its relation to generality. To be able to draw it, we need to interpret its last sentence and, in particular, what the “same origin” and the “differences” that emerge from it can be. Interpreting these terms will put us in a position to grasp a second type of interest in generality that developed in connection with figures in third-century China. For our analysis, it will prove useful to translate a last piece of Liu Hui’s commentary: his account for the “Procedure of the base and the height,” which we already mentioned several times above. The piece reads as follows:<sup>88</sup>

The base multiplied by itself makes a vermillion square (*fang*). The height multiplied by itself makes a blue-green square (*fang*).<sup>89</sup> One makes what goes out and what comes in

<sup>86</sup> Again, Liu Hui considers the case, when  $a = b$ , and concludes that the algorithms computing  $ab$ ,  $a^2$  and  $b^2$  yield the same value and, incidentally, the same geometric extension. In this case, the vermillion piece is the same as half the square of the base or of the height.

<sup>87</sup> This point is even clearer if one reads the final section of Liu Hui’s commentary on problem 9.11. On this question, I refer the reader to my introduction to chapter 9 (Chemla and Guo 2004).

<sup>88</sup> Compare with (Shen et al. 1999, 459), where the text is interpreted in a different way.

<sup>89</sup> The reader can follow the translation on the basis of fig. 5. Here, I rely on several hints to conclude that this piece of text needs to be understood as referring to, and by reference with, the “Figure of the hypotenuse.” First, as I stressed above, the commentary following problem 9.11, which makes an explicit reference to this figure, regularly uses particles like “also,” and I gave arguments showing that this indicated a comparison with this commentary on the “Procedure of the base and the height.” We just saw that Liu Hui uses the “Figure of the hypotenuse” as a fundamental figure, on the basis of which he interpreted distinct algorithms within the context of problem 9.11. Given the nature and use of this figure, I take the particles like “also” to betray

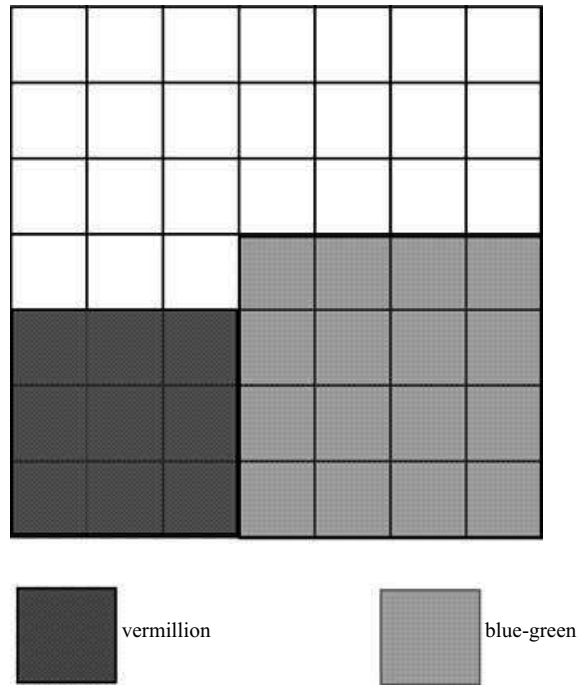
compensate for each other,<sup>90</sup> each (triangle) follow its category.<sup>91</sup> On the basis of the fact that one leaves the corresponding remaining pieces without shifting them, assembling (these) generates the area (*mi*) of the square whose side is the hypotenuse.<sup>92</sup> Dividing this by extraction of the square root hence gives the hypotenuse.

that these pieces of text are all written with respect to this figure. Sharing the same figure would certainly ease such a comparison. Our further development below lends support to this thesis. Secondly, Liu Hui appears to be extremely thrifty in the figures (*tu*) he uses. This feature may relate to the will of identifying fundamental figures from which one can derive as many algorithms as possible. One has a very interesting example of this fact in his commentary on the area of the circle (after problem 32 of chapter 1). The figure with respect to which the commentary develops contains a circle within which a hexagon is inscribed. Later on, in the same piece of commentary, Liu Hui needs to consider the relationship between the circle and the inscribed square. Yet, instead of considering a new figure, he refers to what he calls “the figure of the segment of circle.” He introduces this other figure only later, when he comments on the segment of circle, (after problem 36 of the same chapter). In effect, to analyze the (inaccurate) formula that *The Nine Chapters* provides for the segment of circle, Liu Hui starts by considering the case of the half circle, which is the topic of problem 1.35. The interpretation of the algorithm described by the Canon requires considering the square inscribed in the circle. It is to this figure that the commentator refers in his commentary on the circle. It thus appears that Liu Hui was not multiplying figures at will. To the contrary, he apparently sought to reduce the number of figures introduced. This remark hence also suggests that, since it is possible, the piece of commentary examined was to be read with respect to the same figure as the commentary on 9.11. Thirdly, as I argue in my introduction to chapter 9 in Chemla and Guo 2004, 672–701, the piece of text by Zhao Shuang examined above belongs to a development that has so much in common with Liu Hui’s commentary on the first part of the chapter, both in terms of the algorithms described and the figures mentioned, that it seems to be coherent with the previous hints to assume that both discuss the right-angled triangle with respect to the same figures. Last but not least, this hypothesis allows interpreting the key sentence that we are examining now in a plausible way. Conversely, this development seems to me to support this hypothesis. Even though we do not agree on every detail, Ch’en Liang-ts’o 1986, 261, also considers that the commentary on 9.11 provides evidence that Liu Hui’s commentary on the “Procedure of the base and the height” should be interpreted as referring to a drawing of that type. Shen et al. 1999, 459, translates this sentence as follows: “Let the square on the *gou* be red in colour, the square on the *gu* be blue.” It think it problematic that the translation deletes the reference to numerical operations and hence the way in which the Chinese original sentence articulated the numerical and the geometrical dimensions of the situation discussed. Moreover, the translation does not render the syntax of the original. In these respects, the translation by Shen Kangshen into modern Chinese is more accurate (Shen 1997, 641).

<sup>90</sup> Most probably, this refers to the same operation that is described in *The Gnomon of the Zhou*, i.e. the analysis of the joint area in terms of the rectangles of dimensions *a* and *b*, as suggested by the backdrop of the figure, and the dissection of a part of the square of the height to join a part of the square of the base, as shown of fig. 5 (a). As suggested above, this would explain the “piling up rectangles,” referred to in *The Gnomon of the Zhou*. Of the two colored squares, the blue-green one would contribute a piece to what is to be cut in the vermilion square: the composition of the overall triangle removed would hence show clearly what came out from the blue-green piece to be joined to the vermilion one. Ch’en Liang-ts’o 1986 is devoted to establishing that this principle of Liu Hui’s comes from *The Gnomon of the Zhou*. However, his interpretation of the principle encompasses more than I am sure to admit: in his view, the principle captures the process of decomposition of the area into pieces that are moved to be recomposed in another way. See my discussion in footnote 32.

<sup>91</sup> One triangle has been composed by the previous operation. The second triangle, equal to it, is taken out of the blue-green square and they are both identical to the vermilion piece of the “Figure of the hypotenuse.” One can also understand the expression as follows: “each (triangle) joins a (piece of the same) category.” This is a quotation from the *Yijing*, which indicates that Liu Hui considers that this process of transformation follows general patterns of transformation as described in the Canon. About *lei* “category,” see my glossary in (Chemla and Guo 2004, 948–949).

<sup>92</sup> One could also understand this last sentence as: “On the basis of the fact that one approaches (the previous pieces towards) the corresponding remaining pieces without shifting them, assembling (these) generates the



**Fig. 5.** Restoration of Liu Hui's proof. (left square) "vermillion"; (right square) "blue-green".

Fig. 5 illustrates the interpretation I give to this passage. Starting from the vermillion and the blue-green squares, one analyzes the joint area they form, with respect to the backdrop of the fundamental figures, into the vermillion four triangles and the yellow square (figure 5 [a]). Moving the two triangles (figure 5 [b]) to place them on the corresponding triangles, above the pieces that do not move, yields the square of the hypotenuse.<sup>93</sup> This, hence, corresponds closely to the transformation described in *The Gnomon of the Zhou*, except that it would all take place within the framework of the fundamental "Figure of the hypotenuse."

On this basis, we can now go back to our questions, relating to the last sentence quoted from the commentary on problem 9.11 of *The Nine Chapters*: how are we to interpret what the "same origin" amounts to? And, in correlation with this, how are we to understand the "differences" that arise from it?

square (*mi*) of the hypotenuse." Shen et al. 1999, 459, translates the last part of the passage as follows: "Let the deficit and excess parts be mutually substituted into corresponding positions, the other parts remain unchanged. They are combined to form the square on the hypotenuse." Some of the elements seem to me to be added to the original text ("substituted," "position"), whereas elements of the original text are lost ("compensate for," "category," "area" [*mi*]).

<sup>93</sup> In contrast to the fact that some pieces of area are said not to be moved, it seems highly plausible that the other pieces (those that "each, join its category") move.

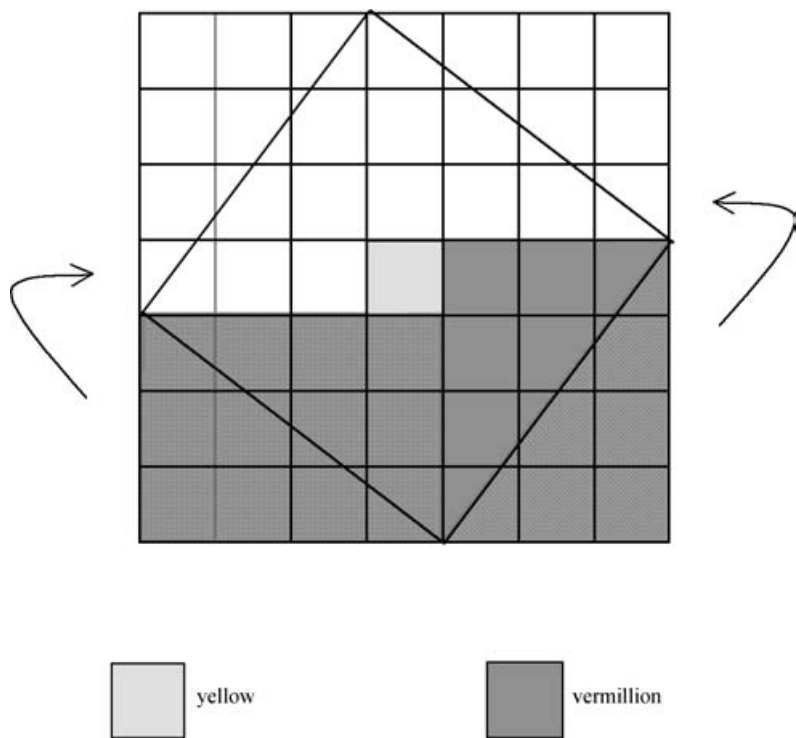


Fig. 5 (a).

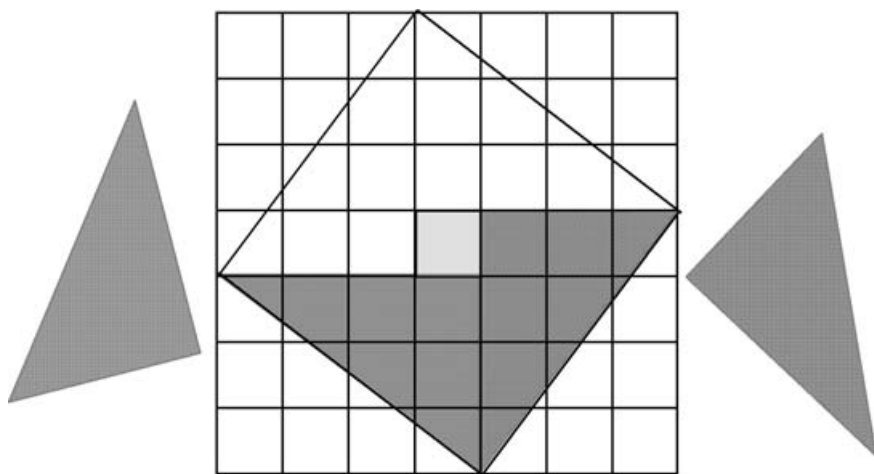
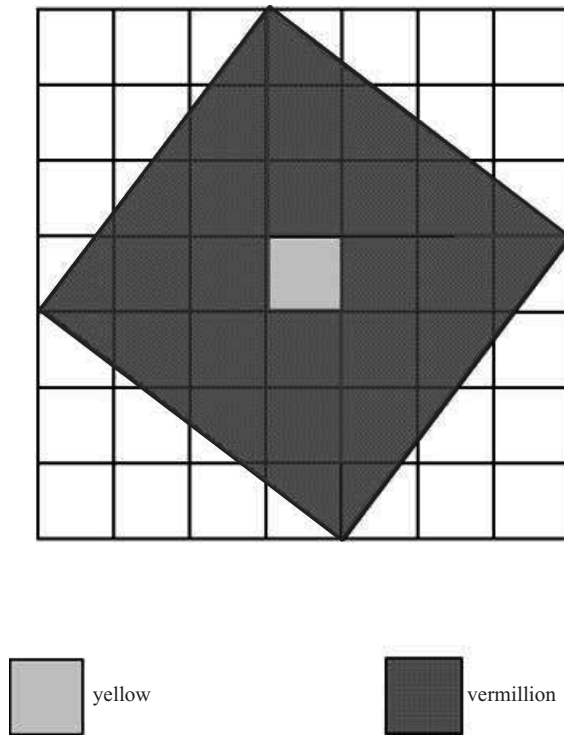


Fig. 5 (b). Moving the triangles.

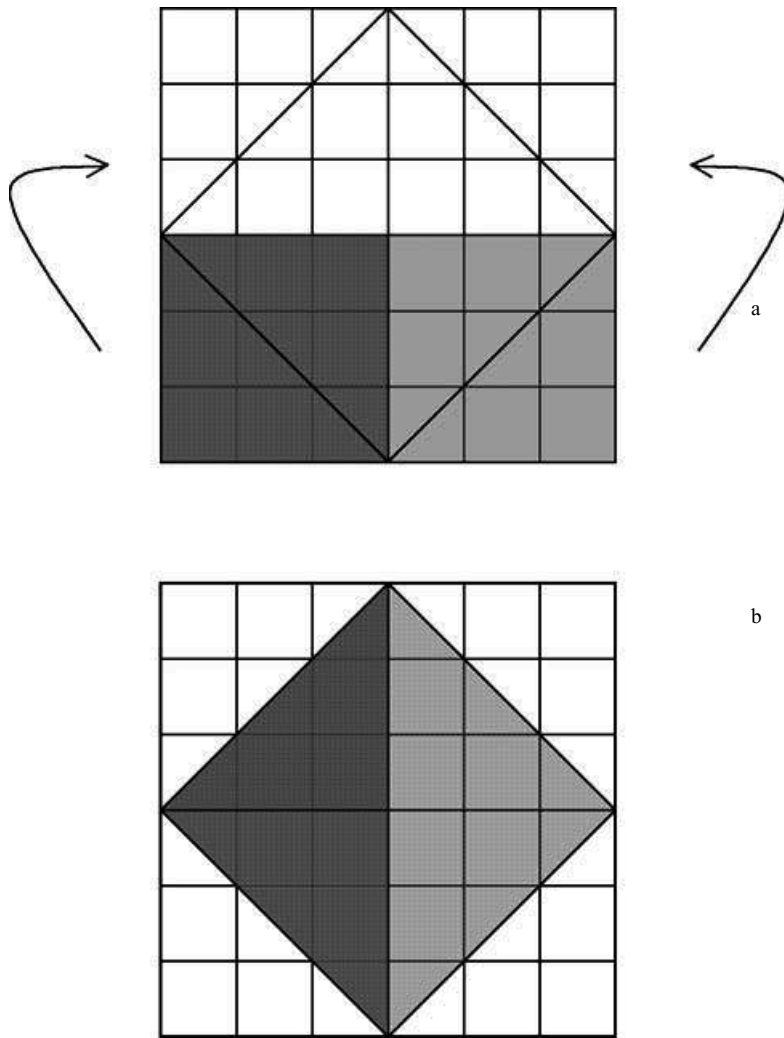


**Fig. 5 (c).** Forming the square of the hypotenuse.

Let us analyze the context of the commentary of problem 9.11, within which Liu Hui inserts the statement in question. In the sentence preceding this one, the commentator stressed, how, when the height and the base are equal, the computation of  $ab$ ,  $a^2$ , and  $b^2$  amount to the same. This remark followed considerations of two types. On the one hand, Liu Hui had previously recalled that, in the case where  $a = b$ , the process transforming  $a^2 + b^2$  into  $c^2$  remained the same as in the general case, and hence amounted to what is shown on fig. 6.

On the other hand, Liu Hui had just described a new algorithm yielding  $c^2$ , by adding  $2[(b+a)/2]^2$  and  $2[(b-a)/2]^2$ . In terms of vermillion and yellow unit pieces, following the lines of the beginning of this commentary, these new computations could be interpreted as yielding, on one side, two-fourths of the yellow square and four vermillion pieces, and, on the other side, two-fourths of the yellow square.<sup>94</sup> In case the difference vanishes, the yellow square disappears, and the surface is transformed into 2 rectangles of area  $ab$ , each.

<sup>94</sup> The same area corresponds to the algorithm (2) described by Zhao Shuang in his development translated above. In modern terms, its computations can be represented as yielding  $2ab + (b-a)^2$ .



**Fig. 6.** The transformation in the case when base and height are equal.

In this context, Liu Hui's statement, regarding the fact that, when  $a = b$ , the computation of  $ab$ ,  $a^2$ , and  $b^2$  amount to the same, implies that the two figures, i.e., on the one hand, the assembling of  $a^2$  and  $b^2$  and, on the other hand, the vermillion and yellow pieces degenerate into the same figure. In this case, the same transformation of one and the same figure, as shown of fig. 6, establishes the two algorithms.<sup>95</sup> This

<sup>95</sup> This is the reason why I reconstituted the process as I did in fig. 6. We can now go back to Zhao Shuang's mention of the case when  $a = b$  (see footnote 32). In this case, the overall geometrical process degenerates into

is, in my view, what Liu Hui designates as the “same origin.”<sup>96</sup> As for the differences arising from it, my interpretation would be that they refer to the different algorithms deduced by putting into play the same transformation on the area of fig. 5, according to whether one structures it into<sup>97</sup>  $a^2$  and  $b^2$ , or into  $2ab$  and  $(b-a)^2$ . In the former case, one proves the correctness of the “Procedure of the height and the base.” In the latter one, one establishes that the second algorithm yielding the square of the hypotenuse is correct.

This interest of Liu Hui’s in “the same out of which differences arise” constitutes what I consider to be a second kind of interest in generality, in relation to figures. Here is in my view what matters to him in the situation: a figure (6 (a)) and its transformation into another figure (6 (b)), both providing the basis for interpreting, and thereby accounting for, an algorithm, develop in two different ways into other figures, transformations, and algorithms. More precisely, the figure gets transformed in two distinct modes of structuring the same area in the “Figure of the hypotenuse.” It develops, on the one hand, into fig. 5 and, on the other hand, into fig. 5(a). In the former case, the two squares of figure 6 are interpreted as giving rise to  $a^2$  and  $b^2$ . In the latter case, they get transformed into rectangles of area  $ab$ , whereas the difference appears. On these two bases, the same transformation is efficient and, thereby, distinct algorithms can be derived. The latter algorithms are hence conceived of as all emerging from the same origin: a unique algorithm based on the former figure and a transformation.<sup>98</sup> In conclusion, Liu Hui would thus stress the fact the particular

something simpler: there is no longer an exchange between the square of the base and that of the height; only the movement taking two half rectangles above the two others is preserved. The disappearance of the first part of the process is what makes the two geometrical processes fuse into one and the same operation.

<sup>96</sup> My interpretation differs from that of Li Jimin 1993b, 4, 5 (see also Li Jimin 1998, 703–704). In his view, when speaking of algorithms, “having the same origin” refers to the fact that the procedures and their reasons both have the “same source.” In addition, the “differences” designate the differences arising in computations. According to him, in this case, the commentator stresses the fact that the computations for the case when  $a$  differs from  $b$  and for the case when  $a$  equals  $b$  differ. Their common origin would be the “figure of the hypotenuse.” However, in the two cases, the figure would not have the same structure, and from that would arise the differences. It is interesting to note that, immediately after his statement Liu Hui turns to considering, when  $a=b$ , the irrational character of values for either the sides or the hypotenuse. His interest in the common source might be guided by this question. I cannot dwell on the question of the irrationals here. For a discussion and a complete bibliography, see Chemla and Keller 2002.

<sup>97</sup> This example brings us back to the first lines we translated from *The Gnomon of the Zhou* and provides elements with which to understand better such statements as “THE SQUARE EMERGES FROM THE RECTANGLE.”

<sup>98</sup> It is interesting to recall here part of a statement by Needham with Wang Ling 1959, 24: “In the Chinese approach, geometrical figures acted as a means of transmutation . . .” I have discussed the concept of “origin” more generally on the basis of Liu Hui’s commentary at the PILM conference, organized in Nancy by G. Heinzmann (30/9/2002–4/10/2002). The paper “Fondements des mathématiques selon les commentateurs de la Chine ancienne” is in preparation and shows a fundamental connection between the stress put on this concept and the fact of prizing generality. In relation to figures, Adolf Pavlovitch Youschkevitch (Juschkevitch 1964, 62) stresses a similar point when he writes: “. . . ist es bemerkenswert, dass der erste Beweis des pythagoreischen Lehrsatzes in China nicht rein synthetisch war und einfache Konstruktionen und algebraische Umformungen in sich vereinigte.”

algorithm in the case when  $a$  equals  $b$  can be accounted for in a way that is general enough to be generalized not only to one, but to two distinct algorithms emerging from it, when  $a$  differs from  $b$ . Let us stress the fact that this interpretation strongly lends support to the hypothesis that Liu Hui's commentary on the "Procedure of the base and the height" was also read on the backdrop of the "Figure of the hypotenuse." Reading different algorithms with respect to the same fundamental figure does encourage paying attention to phenomena of the type just discussed. Such a practice of mathematics does spur comparison between the reasons accounting for the correctness of the various algorithms referred to the same figure.

What is quite surprising is that, much later, it is also in terms of the particular and the general that an Arabic author of the ninth century, Thabit ibn-Qurra, will conceive of the relationship between such figures as, respectively, figs. 6 and 5. Let us evoke his development on the topic. Our goal in doing so will be twofold. On the one hand, we shall bring to light the parallel between the two authors, in terms of an interest in generality. On the other hand, we shall get a sense of how different figures that, for us, may look alike are in fact deeply different in nature. The contrast will, I believe, help us grasp the singular character of Chinese figures, beyond the change experienced between the beginning of the common era and the third century.

#### 4. The *Meno* and Thabit ibn-Qurra

Let us look at fig. 6 (b), out of the context in which it emerged above. Doesn't it remind us of the figure that Socrates drew with the slave in the *Meno* (82–85b), to highlight how the slave was "recalling" that the square based on the diagonal is the one having a space being the double of that of the square itself?

This step was made by one of Thabit ibn Qurra's friends, who read this diagram as a "Socratic proof" of the Pythagorean theorem and became dissatisfied that it could tackle only the particular case of the isosceles triangle. He then asked Thabit ibn Qurra for a generalization of the proof.

This friend may have addressed Thabit, as someone not only well versed in mathematics, but a fine connoisseur of Greek mathematical literature. One may recall here that Thabit once revised one of the translations of Euclid's *Elements* into Arabic and that he himself translated Nicomachos's *Introduction to arithmetic*. Moreover, Thabit developed research along Euclidean lines, for example, in number theory. In addition to this, he introduced Euclidean style in domains that had first developed in the Arabic world outside Euclidean geometry, such as algebra.

With respect to the Pythagorean theorem, upon the request of this friend and in a letter to him, Thabit provided a "Socratic general proof." By this expression, he designated a new proof, in the spirit of the "Socratic special proof," that is, a proof

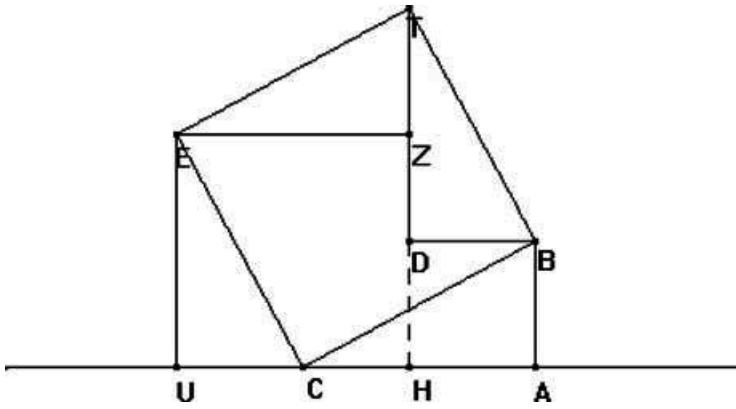


Fig. 7. Thabit ibn-Qurra's figure.

different from Euclid's (Sayili 1960).<sup>99</sup> It is easy to recognize, in the formulation of the problem and in its solution, an interest in generality comparable to what we just saw with Liu Hui. Thabit's letter is entirely devoted to discussing, and practicing, generality, and it concludes by remarks on its import for perfect knowledge. In Thabit's own words, "the equivalent [of the proof in the *Meno*] for any triangle encompasses and generalizes this *explanation* and *applies to all triangles*" (my emphasis).

In fact, Thabit yields two proofs for the Pythagorean theorem, and the reader will perhaps feel some surprise when discovering that the figure drawn by Thabit ibn-Qurra, for his first proof generalizing "Socratic special proof," is quite similar to what is to be found in Chinese sources (see fig. 7). The same interest in generality led to establishing the same figure,<sup>100</sup> the same proof as well as the same link between a particular figure with operations on it (fig. 6) and the generalizing one with the corresponding operations (fig. 7).

Furthermore, when one knows the emphasis placed by Thabit on the principles put forward by Aristotle for mathematics, especially regarding the exclusion of movement from geometry, one may also feel surprised to discover the name he gave to his method: "method of reduction and composition," or "method of reduction to triangles and rearrangement by juxtaposition." This does not catch our attention, only because it reminds us of what we saw above. It is also quite interesting to notice, through this choice of terms, a possible influence of algebra in geometry.

<sup>99</sup> Sayili 1958 provides a critical edition of Thabit's letter, in Arabic. Owing to my more than limited knowledge of Arabic, it was only thanks to the help of Maria Achek-Youssef, Sakina Önen, and Christine Proust that I had access to the contents of this letter. I thank them all wholeheartedly.

<sup>100</sup> The figures are at least the same structurally: the same triangles are made to appear in the area obtained by drawing, side by side, the squares on the base and the height of the triangle, respectively. However, in nature, the figures differ: for instance, Thabit's figure does not display the use of grid paper that Chinese sources evidence.

Let us thus read Thabit's first proof, with two aims in mind: observing the similarity of the proof with that of Chinese sources from *The Gnomon of the Zhou* onwards, on the one hand, and detecting differences in the conception of a geometrical figure, on the other. It reads as follows:<sup>101</sup>

Here is a *general construction* that I made for this: for any right-angled triangle, the sum of the squares of the sides adjacent to the right angle is equal to the square of the hypotenuse.

Let ABC a right-angled triangle, the right angle of which is BAC. The sum of the squares of the sides AB and AC is equal to the square of the side BC.

The proof is the following:

One builds on the side AB the square ABDH. One makes HCU equal to AC and one builds on it the square HZEU and one extends HZ until T, making that DT be equal to AC. One gets CE, BT, TE.<sup>102</sup>

The four triangles BAC, BDT, TZE and CUE are right-angled and their sides adjacent to the right angle are equal. For DT was built like AC and AC was built like ZE and UE, and each of them were built like HU.

HZEU is a square, hence the four sides AC, DT, ZE and EU of these triangles are equal.

The equality of the sides AB, BD, TZ and CU can be proved in an analogous way: as for AB and BD, they are equal since ABDH is a square. As for ZT, it is because DT is equal to ZH, which was built equal to AC. If one takes out the common part that DZ is, there remains ZT, equal to DH, itself equal to AB. As for CU, it is because HU is equal to AC. If one takes out the common part that HC is, there remains CU, equal to AH, itself equal to AB.

The sides adjacent to the right angles mentioned above being respectively equal, the hypotenuses are equal, and they are CB, BT, TE, CE. Hence the four triangles are equal.

The surface BTCE has equal sides and also has right angles for the angle ABC from the triangle BAC is equal to the angle DBT from the triangle DBT; as for the angle CBT, it is equal to the sum of the angles ABC and CBD; but ABD is a right angle, hence CBT is also a right angle; likewise, one proves that CET is a right angle.

One knows moreover that the surface has four equal sides; hence the two other angles are right and this surface is a square with side BC.

Therefore the sum of the two triangles ABC and CEU is equal to the sum of BDT and ZTE, for we proved that the four triangles were equal.

Let us consider the figure CBDZE.<sup>103</sup> The sum of CBDZE and the two triangles ABC and CUE is equal to the two squares ABDH and HUEZ.

Likewise, the sum of this figure CBDZE and the two triangles BDT and ZTE is the square CBTE.

<sup>101</sup> The italics are mine. The translation should be read as giving a general idea of the original text. Let us stress that the international circulation of similar proofs would be worth examining but would exceed the scope of this paper. We concentrate on Thabit here because of the link with generality.

<sup>102</sup> No such description of the making of a figure out of lines can be found in Chinese sources. Instead, one has descriptions of surfaces cut (see above and Chemla 2001).

<sup>103</sup> This corresponds to the pieces that do not move in Liu Hui's proof of the "Procedure of the basis and the height."

As for the squares  $ABDH$  and  $HZEU$ , they are equal to the squares on the sides  $AB$  and  $AC$ , for we built  $HU$  equal to  $AC$ .

As for the surface  $CBTE$ , it is a square of side  $BC$ .

As for the two squares of sides  $AB$  and  $AC$ , if one assembles them, their sum is equal to the square of side  $BC$ . q.e.d. ...if you want that I make the reasoning precise by sentences, I need to give it a name: method of reduction and composition. *The aim of what was described was to reduce the two squares*, I mean by this the squares  $ABDH$  and  $HUEZ$ , *in three pieces. One composes them so as to make a unique square, equal to the square  $BTCE$ . Or one reduces the square  $BTCE$  in three pieces, and one composes them so as to make two squares equal to  $ABDH$  and  $HZEU$ . The figures get transformed in several different ways...*

I think it is clear that the method just expounded by Thabit ibn-Qurra is quite similar to what we restored on the basis of the text of *The Gnomon of the Zhou* or Liu Hui's commentary. Moreover, the link between fig. 5 and fig. 6, in relation to the common geometrical transformation operating on them, is conceived of by Liu Hui and Thabit in an analogous way: in terms of generalization.

One difference between the two authors is, however, meaningful in this respect. Whereas, for Thabit, the figure of two equal squares placed side by side is generalized into that of the two unequal squares, for Liu Hui, it is generalized into two distinct modes of structuring the same area that figs. 5 and 5 (a) embody. Here the two types of generality identified in relation to figures in ancient China mesh with each other. The common origin to distinct algorithms is detected on the basis of a figure that is characterized precisely by the fact that it allows accounting for the correctness of several algorithms. This remark brings to light how, behind the similarities between Thabit's and Liu Hui's "generalization" of an argument, the fact that they work within different mathematical cultures brings about key differences in the nature of the figures on which they operate and the modes of generality that are attached to them.

One could certainly widen the focus and compare more generally the modes of drawing and using geometrical figures to which Thabit, on the one hand, and the Chinese sources examined attest. Clearly, beyond the evolution to which Chinese writings bear witness, the production of the figures, the discourse about them, the use to which they are put, all these elements betray sharp differences in these two mathematical cultures. This sheds light on how different human communities designed different work environments to carry out research in mathematics and how that can be correlated to the results obtained. But to go into more detail on this issue requires another paper!

## Acknowledgments

It was in the wealth of off-prints gathered by Dr. Joseph Needham and given in free access to all visitors of Needham Research Institute that I could discover Sayili 1960.

This paper may have otherwise escaped my attention. I hence take this opportunity to express my deepest gratitude to Dr. Joseph Needham for his intellectual generosity and his ability to create architectural tools for collective research, from which I greatly benefited.

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Over the years, Sir Geoffrey has been a debater whose keen eye helped me go deeper in many an issue. It is my pleasure to acknowledge my intellectual debt by a paper where the topic takes us, as he so often did, from West to East and back.

It is my pleasure to express my deepest gratitude to the anonymous referee who read the paper very carefully and made so many helpful comments. In the present-day organization of research, the exercise of refereeing papers is not rewarding. It is for me all the more moving when a colleague fulfils this duty with such care. Many thanks to Prof. Fu Daiwie too, for helping me in finding some papers that were not easily available. *Last, but not least*, my warmest thanks to Tom Archibald for his help in polishing the English of the paper.

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## Appendix A

商高曰：數之法出於圓方。[...] <sup>1</sup> 圓出於方；方出於矩；圓規之數，理之以方。方周匝也。方正之物，出之以矩。矩，廣長也。矩出於九九八十一。推圓方之率，通廣長之數，當須乘除以計之。九九者，乘除之原也。故折矩故者申事之辭也。將為句股之率，故曰"折矩"也。以為句廣三，應圓之周。橫者謂之廣，句亦廣。廣短也。股修四。應方之匝。從者謂之脩，股亦脩。脩，長也。徑隅五。自然相應之率。徑，直；隅，角也。亦謂之弦。既方之，外半其一矩。<sup>2</sup> 句股之法，

<sup>1</sup> We skip here a section of the commentary that is not essential for us in the context of this paper.

<sup>2</sup> This is the text as given by the three ancient editions on the basis of which *The Gnomon of the Zhou* can be edited: the Southern Song edition, printed in 1213 by Bao Huanzhi 鮑澣之, the edition included in the *Grand encyclopedia of the reign period Yongle* 永樂大典 (Li Jimin 李繼淵 1993a, 37, n. 1) and the edition printed by Hu Zhenheng 胡震亨 in the *Bice huihan* 秘冊彙函 collection in 1603 (Guo Shuchun 郭書春 and Liu Dun 劉鈍 2001, 68, n. 9, 10). The edition of the Collection *Wuyingdian juzhen ban* 武英殿聚珍版, edited by Dai Zhen on the basis of the Hu Zhenheng edition, which he modified with reference to the *Yongle dadian* (see the *Tiyao* added to the publication), gives the text as: "既方其外，半之一矩"，a suggestion adopted by Qian Baocong in his edition.

先知二數然後推一。見句、股然後求弦，先各自乘，成其實。實成勢化，爾乃變通。故曰“既方”。其“外”<sup>3</sup>，或并句股之實以求弦，實<sup>4</sup>之中乃求句、股之分并。實不正等，更相取與，互有所得。故曰“半其一矩”<sup>5</sup>。其術，句股各自乘，三三如九，四四一十六，并為弦自乘之實二十五。減句於弦，為股之實一十六。減股於弦，為句之實九。環而共盤，得成三、四、五。盤讀如盤桓之盤。言取其<sup>6</sup>并減之積，環、屈而共盤之謂<sup>7</sup>。開方除之，得<sup>8</sup>其一面。故曰“得成三、四、五”也。兩矩共長二十有五，是謂積矩。兩矩者，句、股各自乘之實。共長者，并實之數。將以施於萬事，而此先陳其率也。故禹之所以治天下者此數之所也。

Li Jimin 李繼明 1993a, 37, n. 1, considers the former to conform to the original. Guo Shuchun 郭書春 and Liu Dun 劉鈍 2001, 68, n. 9, also holds this view. However, these publications punctuate in different ways. I follow here Li Jimin. Guo Shuchun 郭書 and Liu Dun 劉鈍 2001, 33, punctuates: “既方之外，半其一矩”。

<sup>3</sup> This is the text as given by the Southern Song edition, the edition printed by Hu Zhenheng as well as the edition of the Collection *Wuyingdian juzhen ban*. Li Jimin 1993a, 37, n. 1, suggests that Dai Zhen modified the text of *The Gnomon of the Zhou* to make it conform to the quotation made by the commentator below. Li Jimin solves the problem of the divergence between the main text and that quoted by the commentator, by punctuating the commentary in a new way. I follow him on this point too. Guo Shuchun and Liu Dun 2001, 68, n. 11, mentions the divergence, keeps the text as it is found in all the ancient editions in both places and adopts here the following punctuation (Guo Shuchun and Liu Dun 2001, 33): “既方其外。或并。。”。

<sup>4</sup> This is the text as given by the Southern Song edition, the edition included in the *Grand encyclopedia of the reign period Yongle* (Li Jimin 1993a, 37, n. 3) and the edition printed by Hu Zhenheng (Guo Shuchun and Liu Dun 2001, 68, n. 12). The edition of the Collection *Wuyingdian juzhen ban* adds one character: “以求弦實實之中乃求。。”。 Qian Baocong 1963, 16, n. 2, adopts this suggestion. Li Jimin 1993a, 37, n. 3, considers that the ancient editions are conform to the original text. Guo Shuchun and Liu Dun 2001, 68, n. 12, also holds this view. However, these publications punctuate in different ways. I follow here Li Jimin. Guo Shuchun and Liu Dun 2001, 33, punctuates: “以求弦實，之中乃求。。”。

<sup>5</sup> This is the text as given by the Southern Song edition, the edition included in the *Grand encyclopedia of the reign period Yongle* and the edition printed by Hu Zhenheng (Guo Shuchun and Liu Dun 2001, 68, n. 10). The edition of the Collection *Wuyingdian juzhen ban* modifies it into: “半之一矩”，a point of view adopted by Qian Baocong 1963, 16, n. 3. Li Jimin 1993a, 37, n. 4 considers that the ancient editions are conform to the original text. Guo Shuchun and Liu Dun 2001, 68, n. 12, also holds this view.

<sup>6</sup> This is the text as given by Dai Zhen's edition of the Collection *Wuyingdian juzhen ban* and adopted by Qian Baocong 1963, 16, n. 4. The Southern Song edition and the edition printed by Hu Zhenheng have 之 for 其 (Guo Shuchun and Liu Dun 2001, 68, n. 13). Li Jimin 1993a, 37, n. 5; Guo Shuchun and Liu Dun 2001, 34; and Ch'en Liang-ts'o 陳良佐 1993, 7, n. 8 all adopt the latter in their editions. However, Guo Shuchun and Liu Dun 2001, 68, n. 12, gives the former as a possible option. This is the option I consider as the best one.

<sup>7</sup> This is the text as given by the Southern Song edition and the edition printed by Hu Zhenheng. The latter character *wei* is not to be found in Dai Zhen's edition of the Collection *Wuyingdian juzhen ban*. Qian Baocong 1963, 17, n. 5, adopts the latter text in his edition. Li Jimin 1993a, 38, n. 1, and Guo Shuchun and Liu Dun 2001, 34, both adopt the former option in their edition, even though they do not agree on the punctuation. Here, I follow Guo Shuchun and Liu Dun for the punctuation. Note that Guo Shuchun and Liu Dun 2001, 68, n. 14, gives the latter as a possible option.

<sup>8</sup> This character is omitted in the Southern Song edition (Li Jimin 1993a 38, n. 2) and in the edition printed by Hu Zhenheng. Dai Zhen restores it in his edition for the Collection *Wuyingdian juzhen ban*. Qian Baocong 1963, 17, n. 6, adopts this emendation, an option also followed by Li Jimin 1993a, which I find preferable. Guo Shuchun and Liu Dun 2001, 69, n. 15, considers both options possible and follow the Southern Song edition (Guo Shuchun and Liu Dun 2001, 34).

## Appendix B

今有戶高多於廣六尺八寸，兩隅相去適一丈。問戶高、廣各幾何。

荅曰：

廣二尺八寸，  
高九尺六寸。

術曰：令一丈自乘為實。半相多，令自乘，倍之，減實，半其餘，以開方除之。所得減相多之半，即戶廣；加相多之半，即戶高。令戶廣為句，高為股，兩隅相去一丈為弦，高多於廣六尺八寸為句股差。按圖為位，弦幂適滿萬寸。倍之，減句股差幂，開方除之，其所得即高廣并數。以差減并而半之，即戶廣；加相多之數，即戶高也。今此術先求其半。一丈自乘為朱幂四、黃幂一。半差自乘，又倍之，為黃幂四分之二。減實，半其餘，有朱幂二、黃幂四分之二。其於人方者四分之二。故開方除之，得高廣并數半。半并數減差半，得廣；加，得戶高。又，按此圖幂，句股相并幂而加其差幂亦成<sup>1</sup> 弦幂。為積，蓋先見其弦，然後知其句與股。今適等，自乘，亦各為方，合為弦幂。令半相多而自乘，倍之，又半并自乘，倍之，亦合為弦幂。而差數無者，此各自乘之，而與相乘數，各為門實。及股長句短，同原而分流焉。

<sup>1</sup> Following an idea put forward by Li Jimin 1993b, 495-6, without however adopting his overall restoration of the sentence, I suggest understanding here that the character "減" that the ancient editions contain here has been wrongly copied in place of the similar character "成".