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HIDEYA YAMAKAWA

Ο ΖΗΝΩΝ ΠΥΘΑΓΟΡΙΖΕΙ

I. Preface

In his excellent paper «Zeno of Elea's attacks on plurality» Hermann Fränkel once stated his view on Zeno's fragment B3 as follows:¹

«One glance at the text is sufficient to make us realize that for 'limited' and 'unlimited' (scil. in number) we could almost substitute 'exhausted' and 'inexhaustible'».

According to Fränkel, the subjects of Zeno's paradoxes, among all those of *Dichotomy* and *Achilles* reported in Aristotle's *Physica* Z, chap. 9, were no other than the problems of 'exhaustibility' and 'inexhaustibility' of the magnitude. Basically, I agree with his opinion, and here in my paper, developing his opinion, I would like to discuss the paradoxes *Dichotomy* and *Achilles* in connection with the problem of incommensurability.

In my view, Zeno in his *Achilles* calls into question the incommensurability (or irrationality) of the unequal distances covered by two runners, and moreover, it seems to be very noticeable, he argues there after the model of some mathematical thought patterns which originated in the earlier Pythagorean way of thinking. In the following, firstly, I would like to support the thesis through my reconstruction of *Achilles* and, secondly, I would like to verify this in the examination of the paradox *Dichotomy*.

Making liberal use of the Pythagorean way of mathematical thinking, Zeno, in my view, formulates his paradoxes on motion. Zeno's way of argumentation in paradoxes makes a necessary presupposition of some Pythagorean thought patterns. I assert that «Ζήνων πυθαγορίζει».

II. The orientation for the paradox Achilles

Suppose that the swift runner Achilles could overtake a tardy tortoise at a spot *n*, and now on his back triumphantly says as follows:

Achilles: «I conquered you. Now, you should admit your loss».

Tortoise: «I cannot believe my state of affairs. Make clear your way of running, so I may concede my loss».

Achilles: «Well, listen to my saying and satisfy yourself of your situation. My way of running was such a skillful one as follows; In the first step, I covered a distance greater than half of the distance which is showed as a length between the starting point and your final spot here; and in the second step I covered a distance greater than the half of that which was left; thus repeating the process continually, now, I am on your back comfortably. Did you understand?».

Tortoise: «Oh, I see. Your way of running was really based on the Axiom of Archimedes».

Achilles: «Not 'Archimedes', but *Elements* X. prop. 1.² You are not enlightened concerning the history of mathematics».

Tortoise: «Don't mind such a trifling matter. Now, I recommend you to listen to Zeno's argument. He says:

«τὸ βραδύτατον οὐδέποτε καταληφθήσεται θεόν ὑπὸ τοῦ ταχίστου ἔμπροσθεν γὰρ ἀναγκαῖον ἐλθεῖν τὸ διώκον ὅθεν ὤρμησε τὸ φεῦγον, ὥστ' αἰεὶ τι προέχειν ἀναγκαῖον τὸ βραδύτερον. (Even the slowest runner, when he is going, can never be overtaken by even the fastest, for, inevitably, the pursuer must first reach the point from which the pursued started, which means that the slower runner will always keep ahead.) [Arist. *Phys.* Z 9, 239b 14-18] If your way of running were Zeno's, I could never be overtaken by you, since he says that the two distances covered by you and I were incommensurable».

III. Zeno's Subsoil, Surface soil and Enemies

Before attempting to interpret *Achilles* in line with Fränkel's thesis, I would like to identify Zeno's standpoint, his weapons and his enemies.

The spanish philosopher José Ortega y Gasset made an apt remark concerning the matter. I epitomize the essence of his argument³.

A thinker's thought has always its *subsoil*, *surface soil* and *enemies*. The *subsoil* as an aged strata of the collective thinking and as a birthplace of the thinker's thought is ordinarily ignored by him. On the contrary, the *surface soil* is a recent product and an established fact of which the thinker is aware and gives principally his approval. Taking his position on the *surface soil*, the thinker gives birth to his original thought. But, he does not speak about the *surface soil*, and in opposition to which one can formulate his own original doctrine.

Now, let us identify Zeno's *subsoil*, *surface soil* and *enemies*. I present here the following as a working hypothesis:

I. Subsoil:

The tradition from Ionian philosophy to Pythagorean philosophy⁴.

II. Surface soil:

- (1) Parmenidean philosophy as a legacy to be succeeded⁵.
- (2) Weapons utilized to attack his enemies but of which he was not always an inventor.

- 1) *Reductio ad absurdum* as a way of argumentation⁶.

- 2) The method of successive subtraction $\alpha\nu\theta\nu\phi\alpha\iota\rho\epsilon\sigma\iota\varsigma$ or $\alpha\nu\tau\alpha\upsilon\alpha\iota\rho\epsilon\sigma\iota\varsigma$ which originated in the earliest stage of the history of the Pythagoreans' activity⁷.

- 3) The archaic number theory as seen in *Elements* VII and IX, among all

(1) The theory of *even* and *odd*⁸, and

(2) The way of construction of figured number using *gnomon*⁹.

- 4) *The theorem of Pythagoras*¹⁰

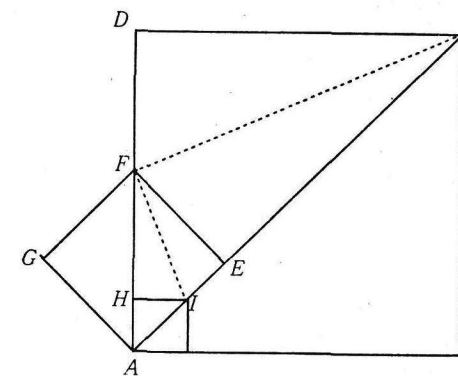
- 5) Rather hot news about the proof which utilize the theory of *even* and *odd* in order to show that the side and diagonal of a square are linearly incommensurable¹¹.

III. Enemies

It is very difficult to identify Zeno's real enemies. But according to Plato's remark in his *Parmenides*, Zeno's enemies were none other than the pluralists who ridiculed Parmenides' argument and who defended their position on the $\pi\alpha\lambda\acute{\iota}\nu\tau\rho\omicron\pi\omicron\varsigma\ \kappa\acute{\epsilon}\lambda\epsilon\nu\theta\omicron\varsigma$. They were the empiricists who were confident of the reality of the sensible world¹².

IV. Reconstruction of the Achilles paradox

I turn towards the *Achilles*. I remember here a brilliant achievement of Rademacher and Töplitz¹³. They used the Proposition X.2 in order to reconstruct an archaic method



[fig. 1]

which does vividly show that the side and the diagonal of the unit square are linearly incommensurable. They utilized the above diagram:

In the above diagram, the figure $CDFE$ is repeated in the similar figure $FEIH$. This figure is a kind of *gnomon* which repeats itself ad infinitum¹⁴.

Now, allow me here to use a slightly different construction than Rademacher & Töplitz. Suppose that when Achilles traverses the distance AB , the tortoise too traverses the distance BC and takes his position at C at the same time. Now, let the ratio of AB to ZC be equal to the ratio of the length of a side to the length of the diagonal of a square. Then, we can reconstruct *Achilles* after the model of *Elements* X.2.

Given a square $ABEF$ let AB be a side $[= \alpha]$ and AE be a diagonal $[d]$. Mark off the length $AG [= AB = \alpha]$ along AE and draw the perpendicular GI from G to the side BE ; then $EG = GI = BI = d - \alpha$ [let it be α_1]. Hence $EI = BE - BI = \alpha - \alpha_2$ [let it be d_1].

Now BI and EI respectively form the side $BC [= \alpha_1]$ and diagonal $BH (= d_1)$ of a new square $BCHI$. The procedure can be repeated for the smaller square $CDJK$ too, thus a smaller side α_2 and diagonal d_2 will appear for the square $CDJK$ again. Continuing

Suppose that the lengths of AC and BC are prime one another. Then, from the fact $AC^2 = 2BC^2$, we can conclude that $AC^2 (=d^2)$ is even, thus d is even and that $BC (=a)$ is odd.

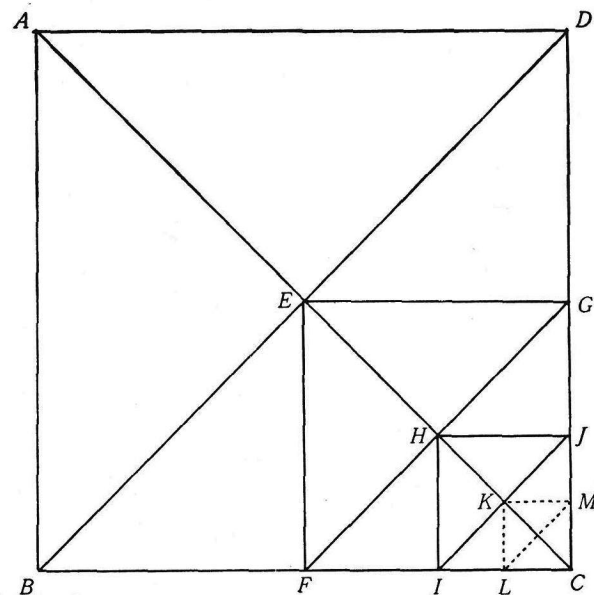
Now, from the figure, the length of $AC (=d)$ is $2 \cdot \alpha_1 + d_1$. On the one hand, in the square $CHIG$, $CI^2 = 2CG^2$, thus $d_1^2 = 2\alpha_1^2$. Therefore d_1^2 is even, thus d_1 is even. On the other hand, from our hypothesis, the length of $BC (=d_1 + \alpha_1)$ must be odd, thus $d_1 + \alpha_1$ is odd, while d_1 is even; therefore α_1 must be odd.

Let GH be joined; then the diagonal GH bisects at K another diagonal CI . Thus the lengths of IK and CK are respectively $1/2 d_1$. But, since $CG^2 (=a_1^2) = 2CK^2$, it must be concluded that a_1^2 is even, so that a_1 is even. The same a_1 is odd and even at the same time. It is impossible.

Similar recurrent construction can be continued infinitely in the diminishing squares, and at each step we can verify that any side or diagonal is expressible as even and odd at the same time.

VI. Dichotomy

The fact that Zeno's paradoxes are $\mu\mu\eta\mu\alpha\tau\alpha$ of the Pythagorean way of mathematizing is verified more clearly by the examination of Zeno's first paradox on motion i.e. the *Dichotomy*.



[fig. 5]

Given the square $ABCD$, let the length of a side of the square be α and let $AC (=d)$ be the diagonal. Let BD be another diagonal of the square. Suppose that the side and the diagonal are prime one another. The line BD bisects the line AC at E . Draw the perpendiculars to the BC and CD respectively and mark off F and G respectively. F and G bisect the sides BC and CD respectively. $EFCG$ is a new and smaller square.

Let EC be a diagonal of the square $EFCG$. Let FG be another diagonal of the square. The line FG bisects the line EC at H . Draw the perpendiculars to the CF and CG respectively and mark off I and J respectively. I and J bisect the sides CF and CG respectively. $HICJ$ is a new and smaller square.

The process can be repeated successively as though the *gnomon* for the diminishing squares were continually applied, so that the points

$E, H, K, \dots$

are marked on the diagonal AC , and in parallel with this process the sequence of the diminishing distances

$1/2d, 1/4d, 1/8d, \dots, 1/2^n d$

is formed.

Now, it is very easy to see the following:

- (1) $AC^2 = 2BC^2$ so, AC^2 is even, thus, AC is even, therefore, BC is odd.
- (2) $BC^2 = 2CE^2$ so, BC^2 is even, thus, BC is even, therefore, CE is odd.
- (3) $CE^2 = 2CF^2$ so, CE^2 is even, thus, CE is even, therefore, CF is odd.
- (4) $CF^2 = 2CH^2$ so, CF^2 is even thus, CF is even, therefore, CH is odd.

.....
This process can be repeated *ad infinitum*. In each step we discern an impossible fact that the same length can be expressible as even and odd at the same time.

In the above reconstruction of Zeno's paradox *Achilles* and *Dichotomy*, the archaic Pythagorean method was used. So, it is not too much to say that Zeno here in the paradoxes of *Achilles* and *Dichotomy* attacks motion after the fashion of the earlier Pythagorean method. Thus, we may safely conclude that $Z\eta\nu\omega\nu \pi\nu\theta\alpha\gamma\omicron\rho\iota\zeta\epsilon\iota$.

VII. Epilogue

The Hungarian scholar, Arpad Szabó, has earnestly tried to explain «why a predominantly empirical mathematical tradition suddenly and for no apparent reason became anti-empirical and anti-visual» in the history of Greek mathematics, and gave his answer as follows:¹⁵

«I believe that the influence of Eleatic philosophy was responsible for the rejection of empiricism and visual evidence in Greek mathematics, as well as for the introduction of indirect proof. The construction of Greek mathematics as a deductive system was a result of this same influence and that, had it not been for the philosophy of Parmenides and Zeno, it would not have been possible to build up so ingenious a system as Euclid's *Elements* »¹⁶.

This is a very attractive statement; and this is also very similar to Burkert's and Philip's judgements on the development of Greek mathematics. But, I think that his doctrine has some major defects in his following theses;¹⁷

(1) The method of indirect proof is a monopoly of the Eleatics.

(2) The Eleatics could hardly have adopted the method of indirect proof from mathematics to philosophy.

Concerning Szabó's thesis (1), the philological studies of today can cite some significant refutals from the Pre-parmenideans. Concerning Szabó's thesis (2), we can neither affirm nor deny it immediately because of the almost entire deficiency of the early fifth century's mathematical materials. But if we may detect some underlying mathematical structures or mathematical way of thinking in Eleatic philosophy, we can indirectly reject Szabó's thesis (2) too.

My task in this paper was precisely this.

NOTES

1. Hermann Fränkel in his «Zeno of Elea's attacks on plurality» in Allen and Furley, *Studies in Presocratic Philosophy*, pp. 104-105 interprets B3 as follows: «In order to explain the second part of the fragment, there seems to be left only the alternative of operating with things, or parts, of indefinite magnitude. If we assume plurality, i.e. divisibility, of any unit, some part of it is here and some other part is

there. But even Here and There are very close together, nothing can prevent us from making the Something here and the Something there small enough to allow for third thing to be squeezed in between them. The operation can be repeated indefinitely without reaching a limit. The premise of plurality and divisibility does not admit the assumption of an ultimate indivisible unit, and strict logic does not allow a gradual transition from the very small to the extended. For it is the nerve of many of Zeno's arguments that lack of magnitude must be radically distinguished from any magnitude, however small... The idea of decreasing quantities, ... recurs in Zeno's dichotomy, Achilles, and frag. B1. The closest parallel, however, is to be found in Plato's *Parmenides* in a dialogue, which follows Zeno's book closely in theme and in treatment».

2. *Elements* X. prop. 1 says: «Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out.» See T. L. Heath, *The Thirteen Books of Euclid's Elements*, Vol. III pp. 14-16.

3. See José Ortega y Gasset, *Origen y epílogo de la filosofía in Obras completas de José Ortega y Gasset*, Revista de Occidente 2 ed., 1965.

4. See my book *The Origin of Philosophy and Science*, Sekai-Shisō-sha, 1987. pp. 16-106; especially see, chapter 6 «Parmenides' challenge», pp. 76-106.

5. Fragment 8, 1-6 says:

«μόνος δ' ἔτι μῦθος ὁδοῖο λείπεται, ὥς ἔστιν ταύτη δ' ἐπὶ σήματ' ἔασι πολλὰ μάλ', ὥς ἀγέννητον ἔδν καὶ ἀνώλεθρόν ἐστιν, οὐλον μονογενές τε καὶ ἀτρεμές ἢ δ' ἀτέλεστον, οὐδέ ποτ' ἦν οὐδ' ἔσται, ἐπεὶ νῦν ἐστιν ὁμοῦ πᾶν, ἔν, συνεχές· τίνα γὰρ γένναν διζήσεται αὐτοῦ;».

It is a noticeable fact that among many signs there are signs «οὐλον μονογενές» and «ἀτρεμές». Being is entire-unique and unmoved, since it is now all together, one, indivisible. «ἔν», it seems, gives a firm basis for the entirety of Being (οὐλον μονογενές) and «συνεχές» for the unmovingness of Being, since the not-one being could mean the plurality divided into many parts (≠ entire-unique) and since in turn the divisible thing implies that it cannot be now all together and so implies also its generation (or origin). Thus we may identify the problem of refutability of plurality and indivisibility of Being as a legitimate legacy for Zeno which was handed down from Parmenides.

6. R. E. Allen in his book *Plato's Parmenides*. Translation and Analysis, Basil

Blackwell 1983, has put it: «As a pattern of argument, *reductio ad absurdum* did not originate with Zeno. It is used by Parmenides, and it is a characteristic and pervasive method of Greek mathematics, one which may well antedate not only Zeno but Parmenides» (p. 70); and further commenting on this topic he says: «Heath suggests that the substance of Euclid, *Elements* VII-IX, which relies on indirect proof in many places goes back to the Pythagoreans and that there is clear indication that number theory had been reduced to elements by the time of Archytas (c. 430-365 B.C.). *TBEE* ii, 294-95» (p. 303). According to K. R. Popper's «Back to the Presocratics» we can discern a typical type of argumentation by *reductio ad absurdum* in «Anaximander's theory of the suspension». Popper asserts as follows: «How did Anaximander arrive at this remarkable theory? Certainly not by observation but by reasoning. His theory is an attempt to solve one of the problems to which his teacher and kinsmen Thales, the founder of the Milesian or Ionian School, had offered a solution before him. I therefore conjecture that Anaximander arrived at his theory by criticizing Thales' theory. This conjecture can be supported, I believe, by a consideration of the structure of Anaximander's theory. Anaximander is likely to have argued against Thales' theory (according to which the earth was floating on water) on the following lines. Thales' theory is a specimen of a type of theory which if consistently developed would lead to an infinite regress. If we explain the stable position of the earth by the assumption that it is supported by water—that it is floating on the ocean (*Okeanos*)—should we not have explained the stable position of the ocean by an analogous hypothesis. But this would mean looking for a support for the ocean, and then for a support for this support. This method of explanation is unsatisfactory: first, because we solve our problem by creating an exactly analogous one; and also for the less formal and more intuitive reason that in any such system of supports or props failure to secure any one of the lower props must lead to the collapse of the whole edifice» (*Conjectures and Refutations*, The Growth of Scientific Knowledge, p.139). Anaximander's way of refutation of Thales' theory was a type of *reductio ad absurdum* which discloses an absurdity as an *infinite regress*. We should pay attention to a significant fact that the tradition of Ionian philosophy originally had its anti-empirical and anti-visual aspect as a notable feature. The most secure method in order to refute a hypothesis is to locate on the contradictory consequences from the hypothesis. Anaximander realized well this matter.

The argumentation by *reductio ad absurdum* is never the monopoly of Eleatic philosophers. For example, we are justified in detecting a type of *reductio ad absurdum* in Xenophanes' attack on the anthropomorphic conception of the gods. Especially, in fragments B14 or B15 we may point out a thought pattern as *reductio ad absurdum*. But, here I would like to reconstruct Xenophanes' indirect proof from the doxography. Arist. *Rhetorica.*, 1399b put it:

«οἷον Ξενοφάνης ἔλεγεν ὅτι ὁμοίως ἀσεβοῦσιν οἱ γενέσθαι φάσκοντες

τοὺς θεοὺς τοῖς ἀποθανεῖν λέγουσιν. ἀμφοτέρως γὰρ συμβαίνει μὴ εἶναι τοὺς θεοὺς ποτε».

(For example, Xenophanes used to say that «those who assert that the gods are born are as impious as those who say that they die; for in both cases it follows that the gods at some time fail to exist».)

The argument can be reconstructed as a type of *reductio ad absurdum* as follows:

Thesis: To assert that the gods had birth is impious.

1) Suppose that the assertion «the gods had birth» is not impious.

2) But, whoever asserts that «the gods die» is impious, because of the proposition «the gods die» implies a time when the gods are annihilated.

3) In the same way, the assertion «the gods had birth» implies a time when the gods do not exist.

4) Whoever asserts that there is a time when the gods do not exist is impious.

Therefore, to assert that the gods had birth is impious.

Xenophanes' argumentation thus reconstructed not only possesses a perfect bone-structure of the indirect proof, but also in its entirety corresponds to Parmenides' negative argumentation of the generation and the extinction of Being in the fragment 8.

7. Euclid, *Elements* X. prop. 2: «If, when the less of two unequal magnitudes is continually subtracted in turn from the greater, that which is left never measures the one before it, the magnitudes will be incommensurable (*ἀσύμμετρα*)». Concerning the fact that the theory of successive subtraction originated in the Pythagorean study of musical proportion see a fine description by A. Szabó, *The Beginnings of Greek Mathematics*, Akademia Kiad, Budapest 1978, especially pp. 199-203.

8. O. Becker in his «Die Lehre vom Geraden und Ungeraden im neunten Buch der eukl. Elemente», *Quell. u. Studien zur Gesch. d. Math.* Abt. B., Bd. 3, 1936, pp. 533-553 has decidedly verified that the theory of *even* and *odd* originated in Pythagorean *ψηφοφορία*.

9. See W. K. C. Guthrie, *A History of Greek Philosophy* Vol. 1, pp. 239-251. See also Theo Gerard Sinnige, *Matter and Infinity in the Presocratic Schools and Plato*, pp. 73-76.

10. See B. L. Van der Waerden, *Science Awakening* Part I, Chapter 3. It is a firmly verified fact that *the theorem of Pythagoras* was known by Babylonians even more than 1000 years before Pythagoras' acme.

11. We need not suppose that a proof of the lineal incommensurability of the side and the diagonal of a unit square was given later than Zeno's or Parmenides'

time. R. E. Allen says as follows: «We know too little about the history of mathematics in the fifth century to speak with confidence, but there is no compelling reason to assign a date late in that century to the discovery of incommensurability, and quite a good reason of supposing that it was known by about 465, and perhaps earlier. For if the *Parmenides* is to be dated around 450, and if Zeno at that date was nearing forty, and if Zeno wrote his book in his twenties, then 465 would seem to be an approximate *terminus ad quem* for the discovery of incommensurability, if Zeno's work betrays familiarity with it. Now, Zeno's Fragment 3, to take but one example, argues that things that are infinite, since there are always other things between things that are, and again others between them, and this implies a relatively precise concept of the infinite divisibility of magnitudes, and of the property of density, as we should call it, which attaches to such divisibility. But infinite divisibility is an intelligible, not a perceptual, property, and it is difficult to see how Zeno could have known of it unless he had a mathematical proof of it; that proof must surely have been the proof of incommensurability, which implies of any magnitude that it has no least measure. It may well be, indeed, that a proof of incommensurability along the line of Euclid X. 117 may have given not only Zeno but Parmenides their confidence in *reductio ad absurdum*, their faith in reason, and their equally strong distrust of the senses and common opinion. If this were so, then Parmenides must have known the proof when he wrote the *Way of Truth*, putting the *terminus ad quem* of the discovery perhaps as early as 485». R. E. Allen, *Plato's Parmenides*, Translation and Analysis, Basil Blackwell, 1983, pp. 210-211.

12. I agree basically with A. H. Coxon's assertion: «Zeno and Melissus are concerned to defend Parmenides' non-physical monism against all such analyses and accordingly use the term 'many' to denote not simply any plurality but the plurality of physical substances». (*The Fragment of Parmenides*, A critical text with introduction, translation, the ancient *testimonia* and a commentary, Van Gorcum, 1986, Appendix I Zeno's argument about magnitude p. 255).

13. H. Rademacher and O. Töplitz, *Von Zahlen und Figuren*, Berlin 1930, p. 258 used *Elements* X. prop. 2 to show that the side and diagonal of a square are linearly incommensurable.

My figure is different from theirs, but the idea is basically the same.

14. Utilizing *Elements* IX prop. 23 Th. G. Sinnige has vividly shown that a recurrent figure illustrates the absurdity of the supposition. It must be conceded that my reconstruction of *Dichotomy* owes its idea to his excellent suggestion.

15. A. Szabó, *The Beginnings of Greek Mathematics*, p. 217.

16. *Op. cit.*, p. 218.

17. *Op. cit.*, p. 219. Szabó says: «The earliest application of indirect proof which I have encountered in my study of Greek language and culture occurs in the didactic poem of Parmenides. On the other hand, it is agreed nowadays that the earliest known mathematical applications of this technique are all without exception of a later date. This simple observation about chronology suggests the following conjecture. Unless the method of indirect proof was developed independently in mathematics and in Eleatic philosophy, it must have originated exclusively among the Eleatics. They could hardly have adopted it *from mathematics* since it does not appear to have been used there until a comparatively late date.» Contra Szabó's theses, see my note (4).

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