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The Pythagorean Tree: A New Species

H. Lee Price

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1 Background

A *Pythagorean triple* is a triple $[a, b, c]$ of positive integers that are the sides of a right triangle; in other words, satisfy the Pythagorean equation $a^2 + b^2 = c^2$. If d is the greatest common divisor of a, b, c then $[\frac{a}{d}, \frac{b}{d}, \frac{c}{d}] = [a', b', c']$ is a Pythagorean triple called *primitive*, and when $d > 1$ the non-primitive triple $[a, b, c]$ equals a multiple $d[a', b', c']$ of a primitive triple. Thus the study of triples is most often focused on primitive Pythagorean triples (PPTs).

This study has very ancient origins. An extensive list of PPTs is found on a cuneiform tablet circa 1800 B.C.E. (see Conway and Guy [3] or Robson [10]). Van der Waerden [13] speculates that much earlier knowledge is possible. Though ancient, the field is wide open, judged by the number and depth of recent papers. We mention two central results: a) a simple two-parameter generation process, and b) an infinite ternary tree whose set of nodes is just the set of PPTs. The generation by parameters is found in Euclid and Diophantus, but the ternary tree was overlooked until the second

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half of the twentieth century. It was first discovered in 1963 by F.J.M. Barn-
ing [1], and was then independently discovered by A. Hall seven years later.
A number of rediscoveries have occurred more recently.

In this article, we announce the discovery of an entirely different ternary
tree. Indirectly the new tree grew out of joint work which I and Frank
Bernhart have been doing for several years (see: [2] and [8]).

2 On Parameters, Angles, and Fibonacci Boxes



Figure 1: Half-angle tangents

An easy geometric argument, using figure 1 shows that the acute angle
marked 2α in the right triangle with legs (a, b) is twice the size of the
angle marked α in the right triangle with legs $(b, a + c)$, the unshaded triangle
being isosceles. Hence the tangent of the half-angle α is just $b/(c + a) =$
 $(c - a)/b$. By symmetry, there is another half-angle tangent (or HAT),
namely $a/(c + b) = (c - b)/a$. It follows that when the sides a, b, c are ratio-
nal (or integral) then the HATs are rational, and can be reduced to lowest
terms.

These two reduced fractions $\frac{q}{p}, \frac{q'}{p'}$ are here displayed as the two columns of
a 2×2 matrix, with the numerators in the first row, and denominators in the
second row. The reduced fraction with the largest denominator is put in the
second column. These numbers will be revealed below to be the parameters
that show up in the two variations of the ancient parametric solution. Some
examples are now given.

$$\begin{bmatrix} q & q' \\ p & p' \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$$

(a,b,c) (3,4,5) (5,12,13) (15,8,17) (7,24,25) (21,20,29)

In the third example, $\frac{(17-8)}{15} = \frac{9}{15} = \frac{3}{5}$ and $\frac{(17-15)}{8} = \frac{2}{8} = \frac{1}{4}$.

Several remarkable properties are visible in this display. There is a single even number, and it is always in the first column. The column products are (almost) the legs of the right triangle. More precisely, the second column product equals the odd leg, whereas the first column product is *half* the even leg. The hypotenuse is the *permanent* (the sum of the diagonal products). It is also the difference of the two row products:

$$17 = (1 \times 5) + (3 \times 4) = (4 \times 5) - (1 \times 3).$$

Less obviously, the two diagonal products and the two row products are the *radii* of four circles: the *in-circle* and the three *ex-circles* of the right triangle. The largest radius (second row product) is the sum of the other three. The product of all four numbers is the *area* of the triangle. In another article [2] we show that the four radii satisfy the Descartes Circle Equation!

For triple $[5, 12, 13]$ above, the radii are found to be 2, 15 (row products) and 3, 10 (diagonal products) and the area is $1 \cdot 2 \cdot 3 \cdot 5 = \frac{1}{2}(5 \cdot 12)$. Obviously, $2 \cdot 15 = 3 \cdot 10 = 1 \cdot 2 \cdot 3 \cdot 5$. Some of these claims are easy corollaries of others. However, the most basic fact about the boxes is this: the sum and difference of the first column numbers give the second column, that is: $q' = p - q$ and $p' = p + q$. These are the *key* equations. Suppose that we convert the 2×2 into a 4-tuple, $K = [q', q, p, p']$, by appending the second row to the reversal or flip of the first row. The key equations now say that this 4-tuple, K , obeys the Fibonacci rule. The sequence K sits in the 2×2 box like the letter “C”, counter-clockwise from the upper right (see fig.2).

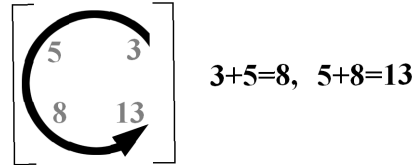


Figure 2: A “Fibonacci box”

This example, or “Fibonacci box”, is made up of four consecutive terms of the Fibonacci sequence, and it generates the primitive triple $[39, 80, 89]$.

The Fibonacci box (**FB**) is uniquely determined by any two numbers in it, hence the same is true of the triple. For example pick a column (one of the HAT fractions q/p .) The PPT is then $T = [p^2 - q^2, 2pq, p^2 + q^2]$ if we pick the first column, or $\frac{1}{2}T$ if we pick the second column. These are the two

standard solutions, known in essence for millennia. We also have a *mixed* solution(s) employing the column, row, and diagonal products that we gave earlier.

$$[q'p', 2pq, qp' + q'p] = [q'p', 2qp, pp' - qq'].$$

We think it is more elegant (and informative!), even though it uses all four parameters, when two are sufficient.

Egyptian Fractions and Pythagorean Triples

One perhaps surprising application using all four parameters is to the writing of Egyptian Fractions (EFs). Consider the *perimeter* to *area* ratio given by $\frac{2}{qq'} = \frac{1}{qq'} + \frac{1}{qp'} + \frac{1}{q'p} + \frac{1}{pp'}$. The sum of the denominators (radii) on the right-hand side *is* the perimeter of the corresponding PPT. Subtract $\frac{1}{qq'}$ from both sides to obtain a 3-term EF variant. Now, subtract $\frac{1}{pp'}$. The resultant 2-term EF, $\frac{1}{q'p} + \frac{1}{qp'}$, gives the ratio of *hypotenuse* to *area*. From the 4-term EF above, we also have $\frac{2}{q'p'} = \frac{1}{q'p} + \frac{1}{pp'}$ which, together with its multiples, produces all of the 2-term solutions found in the Rhind Mathematical Papyrus 2/n Table. Thus each **FB** produces both PPTs and EFs.

In the process of developing our work, Bernhart and I used the key equations and the “HAT” box for some time before we noticed the Fibonacci property. This property is found also in Horodam [8], who cites Dickson [4] for references to special cases noticed in the 19th century. Horodam does not use the same expression(s) for the hypotenuse as we do, and he has no reference to the HATs.

3 Connections

The reader may wish to fill in the details needed for all the claims made in the last section. We give some hints before turning to the matter of the trees. The two HATs, $x = \frac{a}{p}$ and $y = \frac{q'}{p'}$, satisfy $xy + x + y = 1$. This may be proved by substituting the values $x = \frac{(c-a)}{b}$ and $y = \frac{(c-b)}{a}$.

Since the sum of the half angles equals half of a right angle, we could also apply the tangent sum formula: $\tan 45^\circ = 1 = (x + y)/(1 - xy)$ and arrive at the same result. From there it is a small step to the following.

$$\frac{q}{p} = \frac{p' - q'}{p' + q'}, \quad \frac{q'}{p'} = \frac{p - q}{p + q}$$

The right-hand sides are both reduced, up to a factor of two for one case only.

A different perspective uses the complex number $z = (p + qi)$, where the argument of z is one of the half-angles, and q/p is a HAT fraction. Then $z \rightarrow z^2 = (p^2 - q^2 + 2pqi)$ is a transformation that doubles the angle and squares the modulus, obtaining $p^2 + q^2$ for the hypotenuse. Numerous references to this approach exist.

A similar technique is to find point $(\frac{a}{c}, \frac{b}{c})$ on the unit circle $x^2 + y^2 = 1$ in the coordinate plane using a line through $(-1, 0)$ or $(0, -1)$. See Silverman [12], for example. In terms of projective/homogeneous coordinates, “points” $(-1, 0, 1)$ and $(0, q, p)$ determine a “line” of points $\lambda(-1, 0, 1) + (0, q, p)$. Choose finite λ in order to satisfy the Pythagorean equation (this intersects the unit circle at a new point). Rescale to obtain $(p^2 - q^2, 2pq, p^2 + q^2)$, which must be proportional to (a, b, c) .

Consider any **FB** that contains four positive integers and satisfies the key equations. Define r_1, r_4 to be the products in the first and second rows, define r_2, r_3 to be the rising and descending diagonal products, with sum c , and let b', a be the two column products. Put $b = 2b'$. The product of all four numbers is grouped in three ways to get $r_1 r_4 = r_2 r_3 = ab' = ab/2$.

Quite easily $r_1 + r_2 + r_3 = r_4$, and

$$a = r_1 + r_2 = r_4 - r_3,$$

$$b = r_1 + r_3 = r_4 - r_2,$$

$$c = r_2 + r_3 = r_4 - r_1,$$

and $a^2 + b^2 = c^2$ is equivalent to $r_1 r_4 = r_2 r_3$, which we have just seen. So every **FB** gives a Pythagorean triplet, not necessarily primitive.

If some $k > 1$ divides all four parameters of the box, divide each by k . Clearly this removes a factor of k^2 from the triple.

Now suppose the second column contains an even parameter. Combine key equations, obtain $q' + 2q = p'$, and infer both numbers in the second column are even. Divide the second column by two, and exchange the two columns. This will result in another **FB**, and remove a factor of two from the triple.

When the second column is odd with no common factor $k > 1$, we call the box *primitive*. In this case the triple is primitive.

When the first row ends in one, the box is clearly primitive. All triples of this sort form a family attributed to Pythagoras, also described as the triples $[a, b, c]$ with $b + 1 = c$.

If the first row begins with one, and ends with any odd number, again the box is clearly primitive. This time the family of primitive triples is attributed to Plato, also described as the triples $[a, b, c]$ with $a + 2 = c$.

We can also get these families in a simple manner, the first by finding consecutive square numbers whose difference is an odd square, and the second by finding consecutive *odd* squares whose difference is an even square. Both families are central to understanding the Barning-Hall tree (**BH**) and the new tree.

Consider the operation which takes the second row of a primitive box as the first row of a new box; i.e. $K \rightarrow K'$ where $K = [\cdots, \cdots, x, y]$ and $K' = [y, x, \cdots, \cdots]$. Then K' is also primitive. For instance

$$\begin{aligned} [1, 1, 2, 3] &\rightarrow [3, 2, 5, 7] \rightarrow \\ [7, 5, 12, 17] &\rightarrow [17, 12, 29, 41] \rightarrow \dots [Q_n, P_n, P_{n+1}, Q_{n+1}] \end{aligned}$$

and we encounter the standard Pell sequences $(P_n), (Q_n)$. Ratios Q_n/P_n approach $\sqrt{2}$ as a limit. The resulting triples satisfy $|a - b| = 1$ and were studied by Fermat (see also Eckert [5] or Hatch [7]). This family is simply related to the **BH**-tree. It was also studied by Hatch. Recall the two row products (r_1, r_4) and the two diagonal products (r_2, r_3) formed from an **FB**. Let i, j, k rotate the values 1, 2, 3. Then the compact expression $r_i + r_j = r_4 - r_k$ finds all three sides of the triangle for us (these three equations were given separately above).

From this it is easy to set up four tangent circles (K_i) for $i = 1, 2, 3, 4$, where K_i has radius r_i and center C_i . If we arrange the centers so that C_2, C_3 and C_1, C_4 are the diagonals of a rectangle of dimensions $a \times b$, the equations show that all six pairs of circles are tangent, with K_4 enclosing the other three.

We established in [2] that four of the six points of tangency lie on a line. Reflection in that line exchanges the other two points of tangency. It follows that there is a congruent set of circles using the same six points of tangency. Comparison of these circles with the four copies of the right triangle (a, b, c) contained in the rectangle shows that these circles are also the in-circle and ex-circles! So the radii r_i are as claimed.

We have now laid the groundwork for viewing the old and the new tree.

4 The First Tree

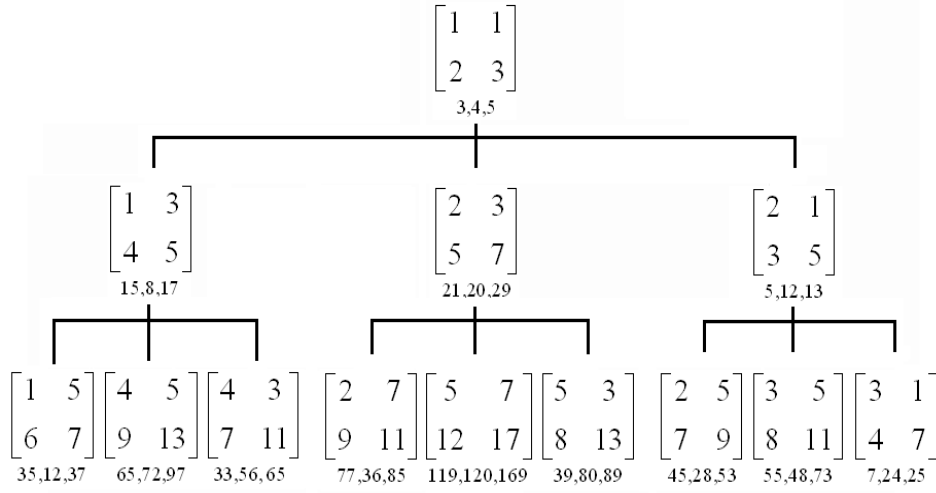


Figure 3: The Barning-Hall Tree (first 3 generations)

Using our convenient **FB** representation, we can quite easily describe both trees. Each one has a first level ($L = 0$) containing only the triple $[3, 4, 5]$, or equivalently, the box containing the sequence $K = [1, 1, 2, 3]$.

Each node on level n generates three successors (children) on the next level. These successors may be called the *left child*, *middle child*, and *right child* (the order is a bit arbitrary). Hence level $L = n$ has 3^n occupants.

On the Barning-Hall Tree, flipping a column of a box is exchanging the two numbers. If M is a primitive Fibonacci box, let M_1, M_2, M_3 be obtained from M by flipping the second column, both columns, and the first column. The result is not an **FB**, but there is a unique M_i which has the *same first row*. In other words, discard the second row, and recompute. This defines the three children.

It immediately follows from our definition that the first row product of any child is the same as one of the other products for the parent. For the (left, middle, right) child the first row product is exactly the product in the parent of the (main diagonal, second row, back diagonal). Briefly, the value r_1 has been replaced by a different radius (r_3, r_4, r_2) . Our name for this is *circle promotion*. In effect, one of the three ex-circles is selected, then we say “you are now the in-circle”.

Suppose we have any primitive triple, with its primitive **FB** (box). It is straightforward to reverse the process of column flipping, and show that this triple/box has a unique place on the tree. Find the positive difference for each row, place the results in the first column (the larger value must go on top), then ‘complete the box’. Barning and Hall both used a different method, the method of matrix transforms. If each PPT is written as a column vector, the children are obtained by left multiplication by these three matrices. They differ in sign only.

$$\begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix}$$

We omit the details that confirm that our construction and theirs give the same tree. Starting at the top, and following *exclusively* the (leftmost, middle, rightmost) path downward, one is running through the family of triples of (Plato, Fermat, Pythagoras).

For Hall, the tree is the same, except it is rotated ninety degrees, and grows from left to right. For Barning the tree is not rotated, but the left and middle children are exchanged. Thus the families of Fermat and Plato are exchanged.

5 The Second Tree

In this section an entirely new tree is revealed. Again we start with the box M , which uncoiled, has the sequence $K = [q', q, p, p']$. This time we move the last element back one, three, or two steps. That is, $K = [x, \dots, y, \dots]$ becomes $K_1 = [x, \dots, y, \dots]$ or $K_2 = [y, x, \dots, \dots]$ or $K_3 = [x, y, \dots, \dots]$. Note the slight change of order.

Using these templates, we now construct valid boxes K_i such that the second column elements x, y of M are repositioned as indicated. The first column elements a, b do get altered. A visual display looks like this.

$$M = \begin{bmatrix} a & x \\ b & y \end{bmatrix} \quad K_1 = \begin{bmatrix} 2a & x \\ y & \bullet \end{bmatrix} \quad K_2 = \begin{bmatrix} x & y \\ 2b & \bullet \end{bmatrix} \quad K_3 = \begin{bmatrix} y & x \\ 2b & \bullet \end{bmatrix}$$

Notice here that one element of the first column is bumped, and the other is *doubled*.

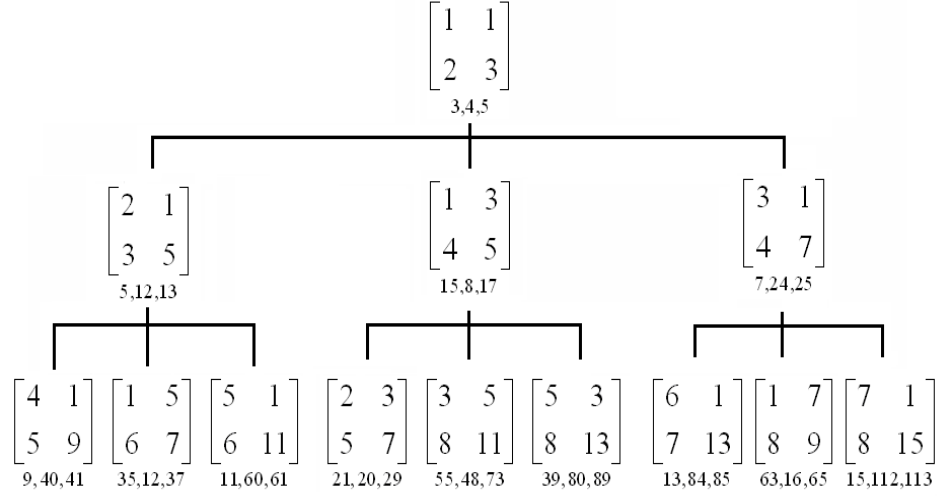


Figure 4: The New Pythagorean Tree (first 3 generations)

The reason that this works is that there is a simple inverse to the operation. It goes as follows. Move the odd element of the first column into the bottom right corner, flip the second column if necessary (the smaller number must be on top), divide the even element of the first column by two, and recompute the missing element. Here is an example that reduces completely in six steps. The details of the first step are spelled out, then all six steps are summarized (see figure 4).

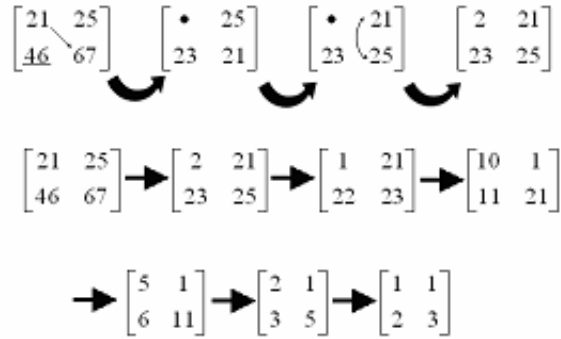


Figure 5: Reversal process in six steps

The process must end because the final element is always decreasing. In our example the final box contains $K_0 = [1, 1, 2, 3]$ and represents the simplest triple $T_0 = [3, 4, 5]$. But the reverse process in general stops only because the first row is constant, which is $K = [k, k, \dots, \dots] = [k, k, 2k, 3k] = k[1, 1, 2, 3] = kK_0$.

The first column of the box contains the HAT fraction q/p . We recall that this is the traditional generator of the triple: $a = p^2 - q^2, b = 2pq, c = p^2 + q^2$. There is a simple description of the change in this fraction when we go one step up the tree.

Divide the number that is even by two, then replace the other number by the positive difference. For example

$$\frac{13}{24} \rightarrow \frac{(13-12)}{12} = \frac{1}{12} \rightarrow \frac{(6-1)}{6} = \frac{5}{6} \rightarrow \frac{(5-3)}{3} = \frac{2}{3} \rightarrow \frac{1}{(3-1)} = \frac{1}{2}.$$

Again, if a triple is written as a column vector v , there are matrices M_1, M_2, M_3 such that the three successors of v are $M_i v, i = 1, 2, 3$. The determinants are 8, -8 , -8 .

$$\begin{bmatrix} 2 & 2 & 1 \\ -2 & 2 & 2 \\ -2 & 1 & 3 \end{bmatrix} \quad \begin{bmatrix} 2 & 2 & 1 \\ 2 & -2 & 2 \\ 2 & -1 & 3 \end{bmatrix} \quad \begin{bmatrix} 2 & -2 & 1 \\ 2 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix} \quad \begin{array}{|c|c|} \hline \text{shaded} & \text{white} \\ \hline \text{white} & \text{shaded} \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \text{white} & \text{shaded} \\ \hline \text{shaded} & \text{white} \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \text{white} & \text{white} \\ \hline \text{shaded} & \text{shaded} \\ \hline \end{array}$$

Figure 6: Matrices M_1, M_2 , and M_3 .

The three matrices of Figure 5 are alike, except for signs. The small boxes with shading at the right show where the sign differences are between each pair of matrices (first two, first and last, last two). The New Tree is shown in Figure 6 below. The correctness of the matrices may be assured by checking that they transform as predicted the three triples $[3, 4, 5]$, $[5, 12, 13]$, $[15, 8, 17]$, since as vectors these three are independent. It is also easy to verify that when you transform any triple $[a, b, c]$, you are taking one PPT to another PPT. E.g. the following simplifies to $a^2 + b^2 = c^2$.

$$(2a - b + c)^2 + (2a + 2b + 2c)^2 = (2a + b + 3c)^2$$

Now recall the two families of Plato and Pythagoras, where:

$$\begin{aligned}
A(1) &= B(1) = [3, 4, 5] \\
A(n) &= [4n^2 - 1], 4n, [4n^2 + 1] \\
B(n) &= [2n + 1, 2n(n + 1), 2n(n + 1) + 1] = [a, b, b + 1]
\end{aligned}$$

In the new tree, the three successors of $B(n)$ are $B(2n)$, $A(n + 1)$, and $B(2n + 1)$. This is easily verified by applying the backward step to the HATs $\frac{2n}{2n+1}, \frac{2n+1}{2n+2}, \frac{1}{2n+2}$. We summarize our results and add a few easy corollaries in the following proposition.

Proposition 1

- (a) *The set of PPTs forms an infinite tree with $[3, 4, 5]$ at the root, and the successors of (column vector) v are M_1v, M_2v, M_3v , (M_i given above).*
- (b) *Let the primary HAT (first column of the **FB** of any triple v be written as reduced positive fraction $\frac{q}{p}$. For the successors this becomes $\frac{2q}{p+q}, \frac{p-q}{2p}, \frac{p+q}{2p}$. Thus the predecessor is found by dividing the even number by two and changing the other number to the positive difference.*
- (c) *From the infinite ternary tree, extract an infinite binary tree by keeping only the first and third successors. The binary tree comprises the family $B(n)$ where $n = 1, 2, 3, \dots$ reads from left to right, level by level.*
- (d) *The middle successor of $B(n)$ is $A(n)$. So to reach any triple v not in either family, one must start with $B(1) = [3, 4, 5]$ and proceed to $B(n)$ by k steps, where the base two numeral for n has $k + 1$ digits, thence to $A(n + 1)$, and further steps.*

Part (d) of the *Proposition* reads like the philosophical maxim: “Plato follows Pythagoras, and all else follows Plato.”

Fermat's Famous Triple

Any location on the tree can be reached by a path from the top, consisting of n steps to reach the n^{th} level. One step down-left (A), straight-down (B), or down-right (C). From Figure 6 we see that path code CCC takes us to triple $[15, 112, 113]$. Path code

AABAABACABACABAAABABABAAAAABAAA

takes us to a famous PPT studied by Fermat. According to Sierpinski [11], Fermat wrote a letter to Mersenne in the year 1643, in which he asserted that primitive triple

$$[4, 565, 486, 027, 761; \quad 1, 061, 652, 293, 520; \quad 4, 687, 298, 610, 289]$$

is the *smallest* in which both the hypotenuse and the leg-sum are perfect squares. In another article [9] we show that the route to Fermat's triple on the **BH**-tree is even longer (43 levels down compared to 32 on the New Tree). The route required is

BCCCBAAAAAAAACAABCCCCCCCCCCCCCCCCCBCCBAAA

Thus Fermat's triple is one of 3^{43} (on level 43, **BH**-tree) compared to one of 3^{32} (level 32 on the New Tree).

6 Concluding Remarks

An inexact general contrast between the two trees is this: the Barning-Hall tree is closely related to the geometry, and the circles, whereas the new tree is seemingly less geometric and more number theoretic. That there is more than one tree constructed by intrinsic means is somewhat surprising. Arbitrary arrangements can be concocted, but what counts here is the use of a set of three matrices, composed of constants, to generate the three successors of a fixed triple. There does not seem to be such a tree in which the Pythagorean family $B(n)$ are supplanted by the Platonic triples $A(n)$. Frank Bernhart has found a binary tree for $A(n)$ corresponding completely to my binary subtree, complete with two matrices to generate the two successors. Application of the matrices to arbitrary PPTs does not give PPTs.

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AMS Classifications

- 11-99 Number theory
- 11D68 Diophantine equations
- 11B39 Fibonacci and Lucas numbers
- 05-B07 Triple systems
- 05C50 Graph theory

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