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N-Dimensional Variations on Themes of Pythagoras, Euclid, and Archimedes

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the literary, musical, and artistic creations of all times to be reproduced and distributed at low cost, so that everyone—at least in the prosperous parts of the world—has available the whole of the human cultural heritage. There is surely another reason for our archeophilia, namely, the weakening of common cultural values under the profound historic changes we are undergoing planet-wide. Feeling great uncertainty about the future, one naturally turns to the past to find inspiration and direction, or just comfort.

The Renaissance is the archetype of such a backward look yielding progress. Without a past we lose the future. Hence the importance of regular revisiting of masterpieces: Euripides and Shakespeare, Cervantes and Hugo, Monteverdi and Schubert, Giotto and Delacroix will always help us to live, love—and die. Provided, that is, that these great works are properly re-created (reinterpreted), not in the vain attempt of finding their original meaning, but rather seeking to get new meanings from them. We must listen, read, and look at what comes from the past with the present's ears, eyes, and minds; Bach cannot be the same after Stravinsky, or Titian after Picasso.

Modern advances can give us unexpected insight into old results, casting light on recent developments.

These ideas may be evident as regards art, but they are not so clear when applied to science, looked at in its relation (or lack thereof) with culture. Indeed science, at least since the beginning of the twenty-first century, often vaunts its absolute modernity and demands a radical contemporaneity, even an essential amnesia, relegating all interest in the past to the status of an optional embellishment. Scientists today show a lack of historical culture unmatched in any other intellectual profession. To be sure, the

most creative minds may feel an active interaction with their forerunners, and many of the great advances of the last century show the mark of an explicit dialogue with the past. Einstein was perfectly aware of confronting Galileo and Newton; and more particularly Abraham Robinson, in developing non-standard analysis, explicitly related it to Leibniz. But at the ordinary level of teaching, popular writing, and even research, such ties to the past are unusual. We may teach and publish on history of science, but ordinarily without connection to actual scientific practice. This is too bad. Modern advances can give us unexpected insight into old results, casting light on recent developments.

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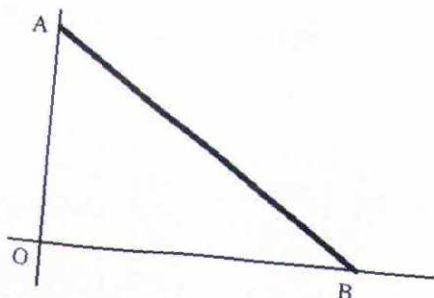


Figure 1. Pythagoras in the plane.

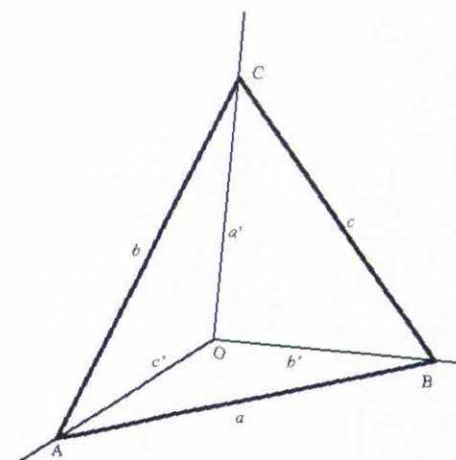


Figure 2. Pythagoras in space.

just as a modern production of *Antigone* or *King Lear* can reveal new meanings and have current impact.

This may be too pompous an introduction for a few examples I want to offer. They are drawn from one of the oldest fields in the world, geometry—understood in the physicist's sense, as "measure of space." Some classical (even ancient) results have interesting generalizations to N dimensions. I think they show the permanence of these traditions for higher space dimensions. The collective amnesia I have been complaining about makes it hard to know how much originality to claim for the present results. That most mathematician and physicist colleagues had no references to give me indicates mostly that they shared the amnesia. I found by good luck that one of the results anyway (the first), was published in a long-ago article by the great geometer H. S. M. Coxeter. I will be grateful for any further information on the questions taken up here.

Pythagoras and the Orthosimplex in N Dimensions

Let us begin by revisiting the very ancient theorem of Pythagoras, and generalizing it to arbitrary dimension. Of course, I am not talking about the metric version, which is now taken as an axiom in defining euclidean spaces, but of a result which may be pleasant if not profound.¹

The classical result will first be presented in the form which is to be generalized. Let two lines in the plane meet orthogonally at O . For any segment AB joining these lines (Fig. 1), the square of its length is the sum of the squares of the lengths of the two segments OA and OB , which it cuts off on the lines—its two orthogonal projections:

$$AB^2 = OA^2 + OB^2. \quad (1)$$

Now in Euclidean 3-space consider three lines meeting orthogonally at O . Let ABC be a triangle with a vertex on each of these lines, whose projections on the planes determined by pairs of these lines are then OAB , OBC , and OCA (Fig. 2). Then

Theorem of Pythagoras 3D. The square of the area of triangle ABC equals the sum of the squares of the areas of its three projections:

$$\begin{aligned} (\text{Area } ABC)^2 &= (\text{Area } OAB)^2 + (\text{Area } OBC)^2 + (\text{Area } OCA)^2 \end{aligned} \quad (2)$$

A fairly easy proof could be given starting from Heron's formula for the area of an arbitrary triangle in terms of the lengths a, b, c of its sides:

$$\begin{aligned} A^2 &= \frac{1}{16} (a + b + c)(b + c - a)(c + a - b)(a + b - c) \\ &= \frac{1}{16} (2b^2c^2 - a^4 + \text{circular perm.}). \end{aligned} \quad (3)$$

Getting the sides of the triangle ABC in terms of those of its projections by the (usual) Pythagorean theorem,

$$a^2 = b'^2 + c'^2, \quad b^2 = c'^2 + a'^2, \quad c^2 = a'^2 + b'^2 \quad (4)$$

with the notations of Figure 2. Putting this back into eq. (3),

$$A^2 = \frac{1}{4} (a'^2 b'^2 + b'^2 c'^2 + c'^2 a'^2)$$

which is just the result announced in eq. (2).

Actually, as often happens, giving a more appropriate proof enables us to see the generality of the result. We can express the area of triangle ABC as a vector orthogonal to its plane whose length is the vector product of any two of its sides:

$$\mathbf{A} = \frac{1}{2} \mathbf{a} \wedge \mathbf{b} \quad (5)$$

Now we just notice that

$$\mathbf{a} = \mathbf{c}' - \mathbf{b}', \quad \mathbf{b} = \mathbf{a}' - \mathbf{c}', \quad \mathbf{c} = \mathbf{b}' - \mathbf{a}', \quad (6)$$

and we see that

$$\begin{aligned} \mathbf{A} &= \frac{1}{2} (\mathbf{c}' - \mathbf{b}') \wedge (\mathbf{a}' - \mathbf{c}') \\ &= \frac{1}{2} \mathbf{b}' \wedge \mathbf{c}' + \frac{1}{2} \mathbf{c}' \wedge \mathbf{a}' + \frac{1}{2} \mathbf{a}' \wedge \mathbf{b}' \end{aligned} \quad (7)$$

¹The result was, I thought, new. I then happened to find it in the article by H. S. M. Coxeter and P. S. Donchian, "An n -dimensional extension of Pythagoras's theorem," *Math. Gazette* 19 (1935), 206. Their result is just the same, but their proof is more straightforward.

The three terms are the vector areas of triangles OAB, OBC, OCA, respectively. These vectors being mutually perpendicular, the usual Pythagorean theorem immediately gives the result in eq. (2).

This demonstration fits into a more general interpretation. Let us define the "vectorial area" of a surface Σ as the vector

$$\mathbf{A}_\Sigma := \int_\Sigma \mathbf{n} \, d^2s \quad (8)$$

where $\mathbf{n}$ is the normal to the surface. Then for every closed surface, the vectorial area is zero. The physical interpretation is simple:² for any vector $\mathbf{u}$, the quantity $\mathbf{u} \cdot \mathbf{A}_\Sigma$ is the flux of the constant field $\mathbf{u}$ across the closed surface Σ , which is zero (if it is necessary to convince oneself of this, one may transform the flux to the volume integral of the field's divergence, which is zero). Applied to any tetrahedron, this says that the sum of the vectorial areas of the four faces is zero. If three of these faces are mutually orthogonal, as they are for the tetrahedron OABC under consideration here, this means that the vectorial area of the fourth face is the vector sum of those of the three mutually orthogonal faces; now eq. (2) follows by applying the usual 3-dimensional Pythagorean theorem.

Now take N orthogonal lines meeting at O in N -dimensional Euclidean space, and an N -simplex formed by N points $P_1, P_2, \dots, P_N$, one on each line. Together with O , these make an $(N+1)$ -simplex, all of whose face angles at O are right angles, which we accordingly call an "orthosimplex." Call its $(N-1)$ -face $H := (P_1, P_2, \dots, P_N)$, its "hypotenusal" face, and call its N faces $F_k := (O, P_1, P_2, \dots, \hat{P}_k, \dots, P_N)$,³ $k = 1, 2, \dots, N$, the "right" faces; these are the projections of the hypotenusal face parallel to the axes; they too are orthosimplexes, in $(N-1)$ dimensions. Then

Theorem of Pythagoras ND. The square of the $(N-1)$ -dimensional volume of the hypotenusal face of an orthosimplex equals the sum of the squares of the volumes of its N right faces:

$$(\text{Vol}_{N-1} H)^2 = \sum_{k=1}^N (\text{Vol}_{N-1} F_k)^2. \quad (9)$$

The proof is almost trivial if one invokes the exterior calculus. Setting $\mathbf{r}_k = \text{OP}_k$ for the vector to the point P_k ($k = 1, 2, \dots, N$), one may express the $(N-1)$ -dimensional volume of the N -simplex $(P_1, P_2, \dots, P_N)$ as a vector orthogonal to the $(N-1)$ -dimensional subspace containing it, by taking the exterior product of $(N-1)$ of its edges, for example those away from the point P_1 :

$$\mathbf{A} = (\mathbf{r}_2 - \mathbf{r}_1) \wedge (\mathbf{r}_3 - \mathbf{r}_1) \wedge \dots \wedge (\mathbf{r}_N - \mathbf{r}_1). \quad (10)$$

A simple proof by recursion shows that, by antisymmetry of the exterior product, the right-hand member is equal to the sum of the terms given by the exterior product (up to sign) of $(N-1)$ of the N vectors $\mathbf{r}_k$ ($k = 1, 2, \dots, N$):

$$\begin{aligned} \mathbf{A} = & (\mathbf{r}_2 \wedge \mathbf{r}_3 \wedge \dots \wedge \mathbf{r}_N) - (\mathbf{r}_1 \wedge \mathbf{r}_3 \wedge \dots \wedge \mathbf{r}_N) \dots \\ & \pm (\mathbf{r}_1 \wedge \mathbf{r}_2 \wedge \dots \wedge \mathbf{r}_{N-1}) \end{aligned} \quad (11)$$

The first term here gives the volume of face F_1 , the N -simplex $(O, P_2, P_3, \dots, P_N)$, as vector $\mathbf{A}_1$ orthogonal to that face. The same applies to each of the other terms, one corresponding to each of the right faces. This gives (with appropriate choice of orientations)

$$\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 + \dots + \mathbf{A}_N. \quad (12)$$

Exactly as in the 3-dimensional case, this is valid for every simplex. But here, starting from an orthosimplex, we have the added feature that the projections (the right faces) are orthogonal: $\mathbf{A}_k$ is parallel to the k th axis; the usual Pythagorean theorem now gives eq. (2).

Note, however, that the generality of the result is limited: the N -simplexes, which can be defined by N points on N orthogonal lines, are, for $N > 3$, very special. Indeed, they are determined by specifying N parameters $r_1, r_2, \dots, r_N$, while the general N -hedron depends on $N(N-1)/2$ parameters (for example, the lengths of its edges).

Euclid and the N -Dimensional Pyramid

Conceptual mastery of space goes by way of geometry, regarded in the first place as science of the measure of forms—thus fundamentally a physics of space. Estimation of lengths, areas, and volumes of the simplest objects in 3-space is its core.⁴ The importance of the formulas giving the area of a triangle and the volume of a pyramid cannot be overstated; these are, after all, the first examples of integration in 2 and 3 dimensions, as was realized later. Now the geometric reasoning which establishes these two results generalizes easily to N dimensions, furnishing both a new proof of elementary results and an approach which ought to instill some N -dimensional intuition.

The case of the triangle is quickly disposed of, for every triangle has evidently area half that of the rectangle with the same base and altitude (Fig. 3). But I prefer here to start from a special case which will be easily generalizable: an isosceles right triangle, half of the square having the same side. The formula giving its area,

$$\text{Area of triangle} = \frac{1}{2} \text{Altitude} \times \text{Length of base} \quad (13)$$

then extends to an arbitrary triangle by suitable affine transformations, which make the triangle skew and dilate its dimensions, retaining validity of eq. (13) and the factor $\frac{1}{2}$ (Fig. 4).

The extension to 3 dimensions is still easy. Consider a cube, and make use of the 3-fold symmetry about an axis joining two opposite vertices. A clever dissection into three equal pyramids joined along that main diagonal (Fig. 5) shows that the volume of a right pyramid with square base

²Thanks to Jean-Paul Marmorat for this remark.

³The notation means that the face F_k contains all the points $OP_1, P_2, \dots, P_N$ with the exception of P_k .

⁴Euclid, *Elements*, Book XII.

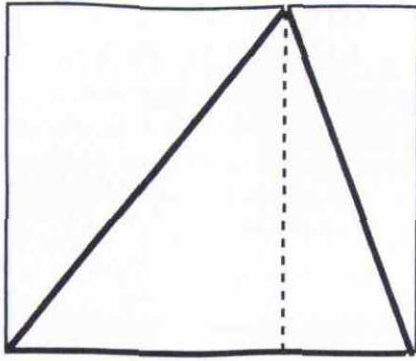


Figure 3. The area of a triangle.

and with altitude equal to the side of the square is indeed a third of the volume of the cube:

$$\text{Volume of pyramid} = \frac{1}{3} \text{ Altitude} \times \text{Area of base} \quad (14)$$

Again, suitable affine transformations carry the pyramid into any other pyramid with parallelogram base. Finally, a Cavalieri-style comparison of such a pyramid with another having the same altitude, and any base provided it has the same area as the parallelogram (Fig. 6), completes the proof that eq. (14) holds for any pyramid.⁵

Seeing the factor $\frac{1}{2}$ in 2 dimensions and the factor $\frac{1}{3}$ in 3 dimensions, one is tempted to generalize. In 4 dimensions, the picture can still be visualized. The 4-dimensional hypercube (often called a "tesseract" when one wants to sound mysterious) is easily projected down to 2 dimensions (Fig. 7). One can see that it can, in fact, be dissected into four equal hyperpyramids with (3-dimensional) cubic base, joined along a main diagonal of the hypercube. Thus the volume of each is equal to a quarter of the volume of the hypercube, and this relation generalizes to other hyperpyramids as above.

Now there is a fairly simple formal proof in arbitrary dimension N . In the euclidean space $\mathbb{E}^N$ with an orthonormal basis at the origin O , consider the hypercube $\Gamma^{(N)}$ of unit edge whose 2^N vertices are given by their coordinates $a_1, a_2, \dots, a_N$, with each a_k being 0 or 1 ($k = 1, 2, \dots, N$). Denote by $\Gamma_k^{(N-1)}$ the hypercubic $(N-1)$ -face of $\Gamma^{(N)}$ defined by setting $a_k = 1$; these N hypercubes are the N faces of $\Gamma^{(N)}$ which meet at the vertex $O' = (1, 1, \dots, 1)$ opposite the origin $O = (0, 0, \dots, 0)$. Further, consider the N hyperpyramids $\Pi_k^{(N)}$ ($k = 1, 2, \dots, N$) having O as vertex and as bases the N hypercubes $\Gamma_k^{(N-1)}$ respectively. These equal hyperpyramids, with common edge OO' , partition $\Gamma^{(N)}$ into N

equal pieces. Indeed, any point $(x_1, x_2, \dots, x_N) \in \Gamma^{(N)}$ (that is, one such that $0 \leq x_k \leq 1 \forall k = 1, 2, \dots, N$) belongs to one and only one of the hyperpyramids, namely, that $\Pi_l^{(N)}$ such that $x_l = \sup(x_1, x_2, \dots, x_N)$. An evident consequence is that each of the hyperpyramids has volume equal to $1/N$. The result generalizes to an arbitrary hyperpyramid in the same way as above, showing therefore that

Volume of N -hyperpyramid

$$= \frac{1}{N} \text{ Altitude} \times \text{Volume of } (N-1)\text{-hyperbase.} \quad (15)$$

In algebraic terms, this is evidently equivalent to integrating a monomial of exponent $(N-1)$. Indeed, consider the section of the hyperpyramid $\Pi_N^{(N)}$ (for example) by the plane $x_N = t$ ($0 < t < 1$): it is an $(N-1)$ -hypercube whose vertices are the points $(ta_1, ta_2, \dots, ta_{N-1}, t)$ with each $a_k = 0$ or 1 ($k = 1, 2, \dots, N-1$); its volume is t^{N-1} . Then the total volume of the pyramid is obtained by integrating with respect to the height t of the plane of

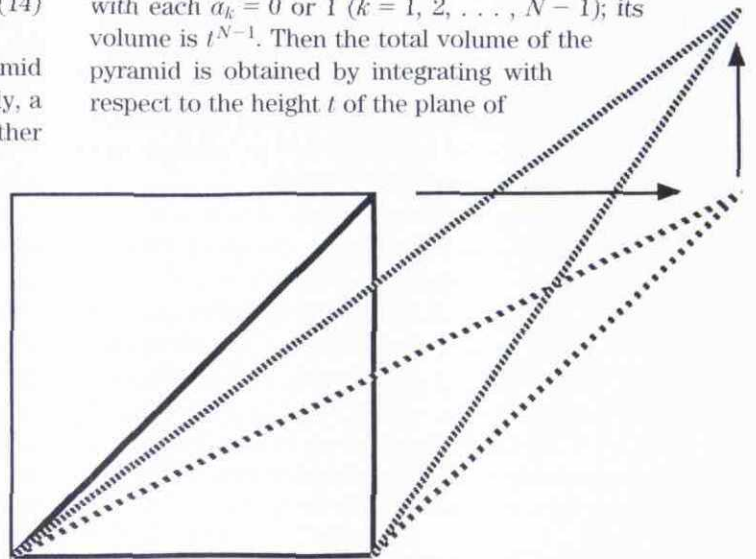


Figure 4. The area of a triangle, again.

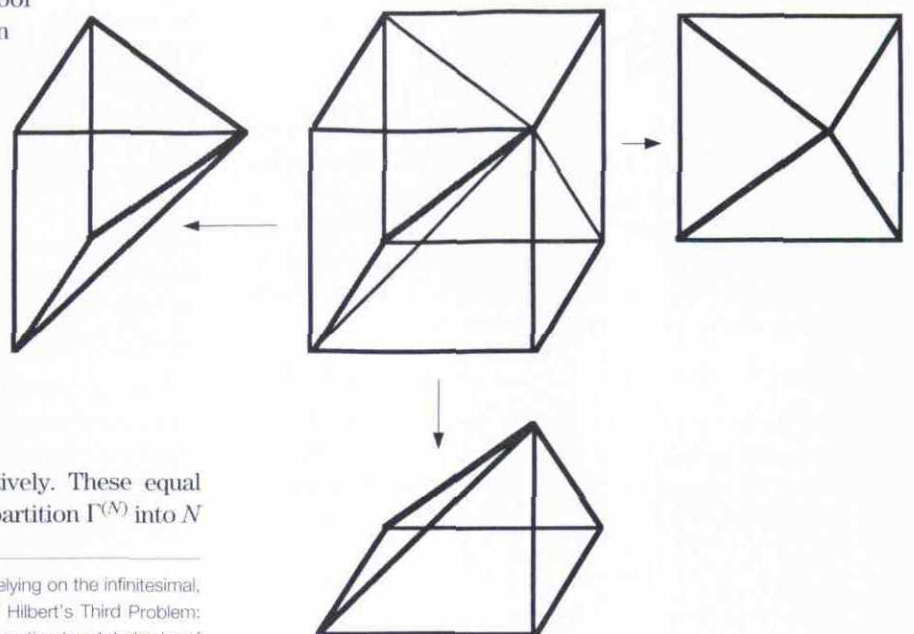


Figure 5. The dissection of the cube and the volume of a pyramid.

⁵By shameless use of "physicist's reasoning" implicitly relying on the infinitesimal, this argument totally bypasses the negative solution of Hilbert's Third Problem: the impossibility (unlike the case of 2 dimensions) of dissecting two tetrahedra of the same volume into congruent elements.

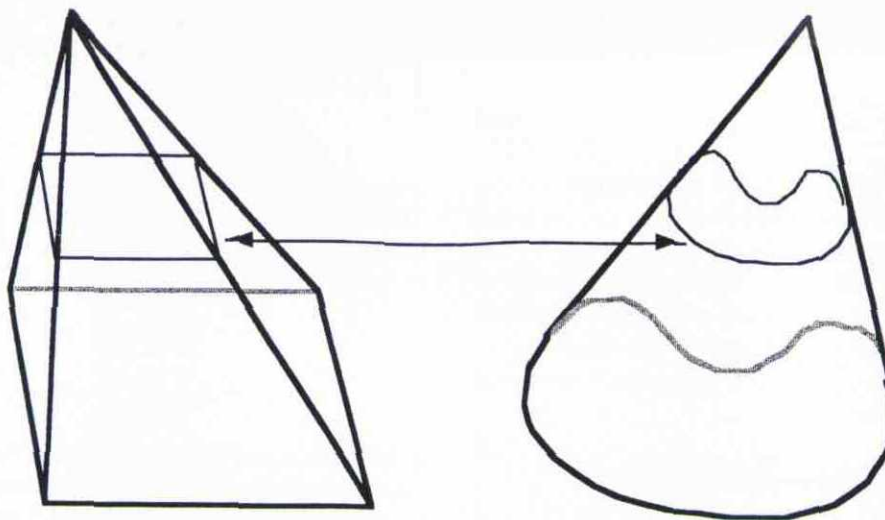


Figure 6. The volume of an arbitrary pyramid.

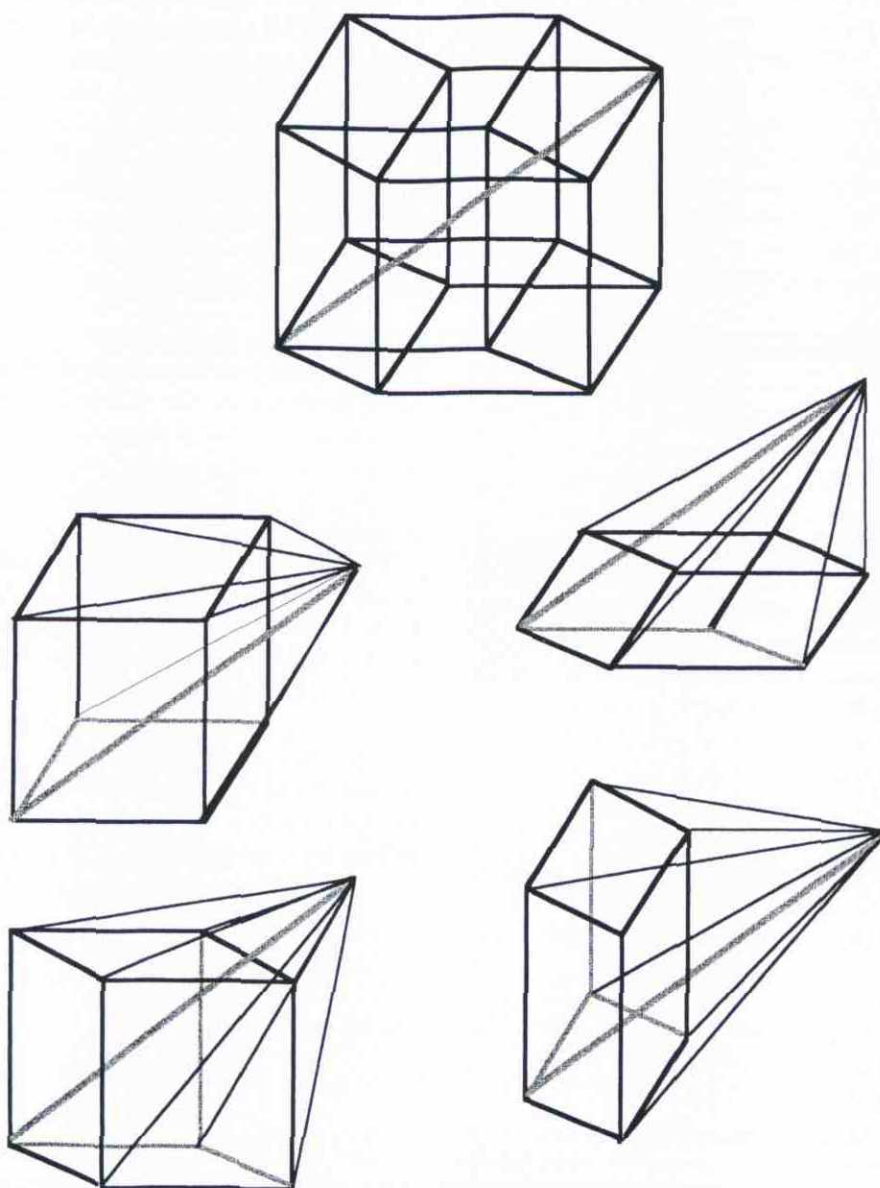


Figure 7. The dissection of the hypercube and the volume of a hyperpyramid.

the section. In conclusion, this shows by purely geometric methods that

$$\int_0^1 dt t^{N-1} = \frac{1}{N}. \quad (16)$$

The calculation of the volume of the pyramid and obtaining the factor $1/3$ by dissection of the cube into pyramids is not exactly the method Euclid gives. According to an earlier proof of Eudoxus, whose idea is generally attributed to Democritus,⁶ the result is obtained by trisecting a prism into three pyramids (not necessarily equal) of the same volume. I leave to the reader the pleasure of verifying that this method also generalizes to N dimensions.

Archimedes and the N -Dimensional Sphere

The double jump of π

Among the many mysteries afforded by the number π ,⁷ not the least is the following: not content to relate the circumference and the area of the circle,⁸ the very same number appears in the formulas for the area of the sphere and its volume.⁹ But better yet, the expressions for area and volume of the N -dimensional sphere get along without requiring any other transcendental number—as they perfectly well might, after all, at least before one has done any calculations. These formulas all appeal only to π , raised to powers which jump in every other dimensionality, thus generalizing the behavior observed in dimensions 2 and 3. Table 1 displays this strange phenomenon:¹⁰

Table 1 Area and Volume of the N -Dimensional Sphere							
N	0	1	2	3	4	5	6
A_N	0	2	$2\pi R$	$4\pi R^2$	$2\pi^2 R^3$	$\frac{8}{3}\pi^2 R^4$	$\pi^3 R^5$
V_N	1	$2R$	πR^2	$\frac{4}{3}\pi R^3$	$\frac{1}{2}\pi^2 R^4$	$\frac{8}{15}\pi^2 R^5$	$\frac{1}{6}\pi^3 R^6$

The most-used explicit calculation throws little light on these results. The classical “dodge” for calculating the area of the sphere in N dimensions consists of integrating over Euclidean space the Gaussian function, which combines the properties of factoring into functions (Gaussian) of a single variable and of having spherical symmetry.

The first of these properties allows one to write

$$\begin{aligned} \int_{\mathbb{R}^N} d^N r \exp(-r^2) &= \int_{\mathbb{R}} \dots \int_{\mathbb{R}} dx_1 dx_2 \dots dx_N \exp(-x_1^2 \dots -x_N^2) \\ &= \left[\int_{\mathbb{R}} dx \exp(-x^2) \right]^N = [\Gamma(1/2)]^N = \pi^{N/2} R^N. \end{aligned} \quad (17)$$

The second gives the expression

$$\begin{aligned} \int_{\mathbb{R}^N} d^N r \exp(-r^2) &= \int_0^\infty a_N r^{N-1} dr \exp(-r^2) = \frac{1}{2} a_N \Gamma(N/2) R^N. \end{aligned} \quad (18)$$

This yields the final result (for the volume, one integrates the surface of the sphere of radius r , from 0 to R):

$$\begin{aligned} a_N &= 2 \frac{[\Gamma(1/2)]^N}{\Gamma(N/2)} \\ v_N &= 2 \frac{[\Gamma(1/2)]^N}{N\Gamma(N/2)} = \frac{[\Gamma(1/2)]^N}{\Gamma(1 + N/2)}. \end{aligned} \quad (19)$$

Now let us make this more explicit, distinguishing according to whether the dimensionality is even or odd. In the former case, $N = 2p$, the denominator is a factorial of an integer, and in the numerator there are N factors $\sqrt{\pi}$, so p factors π . In the latter case, a factor $\sqrt{\pi}$ comes into the denominator which takes care of one of the $(2p + 1)$ factors $\sqrt{\pi}$ in the numerator, finally leaving only p factors π . This compensation, which in this account seems perfectly accidental, gives rise to the “double jump” of π , without giving any understanding of its geometric necessity.

I would like to present now two other methods of calculation which clear up the mystery of the double jump of powers of π ; one of them is based on a simple and elegant recursion which has the virtue of generalizing some results more than two millennia old.

In 3 dimensions (Archimedes)

To begin with, recall the classical calculation of the area of the ordinary sphere (the sphere in *our* space). Archimedes was the first to show¹¹ that the area of a sphere is exactly equal to the lateral area of the circumscribed cylinder of the same radius R and with altitude equal to the sphere's diameter (Fig. 8).

The simplest proof consists in considering on the sphere of radius R the parallel of latitude determined by the polar angle θ . The infinitesimal zone between this parallel and another distant dl from it (on the sphere) has area $dA = 2\pi R \sin \theta dl$, because the small circle at that latitude has radius $r = R \sin \theta$. But the parallel planes containing the two circles cut off on the cylinder circumscribed at the equator a band of altitude $dh = \sin \theta dl$. The area of the band is therefore $dA' = 2\pi R dh = dA$, equal to that of the infinitesimal spherical zone. The sphere and the cylinder thus have the same total area:

$$A = 2\pi R H = 4\pi R^2. \quad (20)$$

⁶See the long note of Thomas L. Heath in *The Thirteen Books of Euclid's Elements*, Dover, New York, vol. III, pp. 365–368.

⁷See Jean-Paul Delahaye, *Le fascinant nombre Pi*, Belin, 1999.

⁸I know, I know; I said it that way when I was a child, but nowadays we are supposed to say “the area of the disk.” Fashions of the times.

⁹Beg pardon, The volume of the ball.

¹⁰... seeing that the circumference of a circle is $2\pi R$ while the surface of a sphere is $4\pi R^2$, we might be tempted to expect the hypersurface of a hypersphere [in 4 dimensions] to be $6\pi R^3$ or $8\pi R^3$. It is unlikely that the use of analogy, unaided by computation, would ever lead us to the correct expression, $2\pi^2 R^3$.” H. S. M. Coxeter, *Regular Polytopes*, Macmillan, 1963, p. 119. In his course on Mathematical Methods of Physics in the 1960s, Laurent Schwartz scored a success with his expressions of regret at not living in a space of 6 dimensions, where the formula for surface area is “so beautiful.”

¹¹Archimedes, *Oeuvres complètes*, vol. 1: *La sphère et le cylindre*, ed. Ch. Mugler, Les Belles Lettres, Paris, 1970.

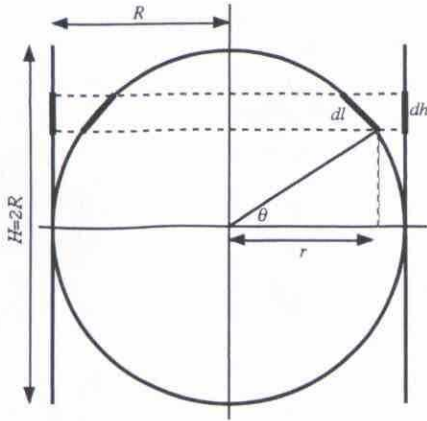


Figure 8. The area of the sphere.

As to the volume of the interior of the sphere, it suffices to consider it as cut up into infinitesimal pyramids having vertex at the center and bases on the surface: for each of these pyramids, of course, $dV = \frac{1}{3} R dA$, hence for the whole sphere one has also $V = \frac{1}{3} RA$, so

$$V = \frac{4\pi}{3} R^3. \quad (21)$$

Archimedes expressed this result by declaring equal the volume of the sphere and that of the cylinder after removing from it the pyramids with vertex at the center and bases the bases of the cylinder.¹² It is said that Archimedes was so proud of these results—rightly—that he wanted the key figure of sphere with circumscribed cylinder carved on his tombstone. Three centuries after his assassination at the siege of Syracuse, Cicero recognized his grave by this insignia; it has not been rediscovered in recent centuries, alas. The correspondence which comes in here between the sphere and the circumscribed cylinder is just the cartographer's "cylindrical parallel" (or "normal") projection, which, we have seen as an incidental extra, preserves area locally, not only globally; but, as we knew, it distorts angles, worse and worse as one approaches the poles, where it is singular.¹³

In N dimensions

In euclidean N -space $\mathcal{E}^N$ let us consider the sphere $\mathcal{S}_N$ of radius R , denoting its area by $A_N(R)$, and its interior ball $\mathcal{B}_N$ of volume $V_N(R)$. Again denoting by a_N and v_N the area and volume of the sphere of unit radius, we have for evident reasons of homogeneity

$$\begin{cases} A_N(R) = a_N R^{N-1} \\ V_N(R) = v_N R^N. \end{cases} \quad (22)$$

Finally, simple integration just as in the 3-dimensional case, allows us to write

$$V_N(R) = \int_0^R A_N(r) dr = \int_0^R a_N r^{N-1} dr = a_N \frac{1}{N} R^N, \quad (23)$$

whence

$$v_N = \frac{1}{N} a_N, \quad V_N(R) = \frac{1}{N} R A_N(R). \quad (24)$$

Now we can get to the heart of the argument, establishing a generalized Archimedean cylindrical projection. To this end, consider the variable point $r = (x_1, x_2, \dots, x_N) \in \mathcal{E}^N$, and introduce polar coordinates for its first two coordinates:

$$\begin{cases} x_1 = \rho \cos \varphi \\ x_2 = \rho \sin \varphi \\ (x_3, x_4, \dots, x_N) = r' \in \mathcal{E}^{N-2} \end{cases} \quad (25)$$

In the new coordinates, the volume element becomes

$$d^N r = dx_1 dx_2 \dots dx_N = \rho d\rho d\varphi d^{N-2} r'. \quad (26)$$

Now we set up a bijection between the sphere $\mathcal{S}_N$ (omitting its poles to avoid the singularity of the parametrization at $\rho = 0$) and the Cartesian product of the ball $\mathcal{B}_{N-2}$ (parametrized by r' , with $|r'| < R$) with the sphere $\mathcal{S}_2$ (that is, the circle parametrized by angle φ), in other words, a torus:

$$\{r = (\rho, \varphi, r'); |r| = R\} \in \mathcal{S}_N \leftrightarrow \begin{cases} \varphi \in \mathcal{S}_2 \\ r' \in \mathcal{B}_{N-2}; |r'| < R \\ \rho = \sqrt{R^2 - |r'|^2} \end{cases} \quad (27)$$

This really does generalize the usual cylindrical projection: in the case of 3 dimensions, the sphere ($\mathcal{S}_3$) is projected to the cylinder, which is the Cartesian product of the circle ($\mathcal{S}_2$) and the segment ($\mathcal{B}_1$), which we can just as well regard as a torus. In the general case too, the projection preserves measure; for the integral over the sphere $\mathcal{S}_N$ (with its uniform measure) of an arbitrary function F , can be written in terms of the Dirac δ -function as

$$\begin{aligned} \int_{\mathcal{S}_N} d^{N-1} \sigma F(r) &= \int_{\mathbb{R}^N} d^N r \delta(|r| - R) F(r) \\ &= \int \rho d\rho d\varphi d^{N-2} r' \delta(\sqrt{|r'|^2 + \rho^2} - R) F(\varphi, r') \\ &= R \int d\varphi \int_{|r'| < R} d^{N-2} r' F(\varphi, r'), \end{aligned} \quad (28)$$

where the last equality requires the standard result that $\delta[u(x) - u(a)] = [u'(a)]^{-1} \delta(x - a)$. Integrating here the identity function ($F = 1$), one obtains the area of the sphere as the product of the area of the sphere $\mathcal{S}_2$ with the volume of the ball $\mathcal{B}_{N-2}$, that is, the charming formula

$$A_N(R) = 2\pi R V_{N-2}(R), \text{ so that } a_N = 2\pi v_{N-2}. \quad (29)$$

¹²The equality in this form can be demonstrated directly by slicing the two volumes by planes perpendicular to the axis of the cylinder and observing that the two sections (a disk and an annulus respectively) have equal area. The Cavalieri theory of indivisibles (already known before the 17th century), a heuristic form of a rigorous theory of integration, allows one then to assert that the two solids—the sphere and the hollowed-out cylinder—made up of sections of equal areas, have the same volume.

¹³It is commonly but mistakenly said that this projection is Mercator's. That great cartographer invented in 1569 a cylindrical projection which is conformal (preserves angles), much less trivial. It is curious, in view of its appearance in Archimedes's result, that the cylindrical normal projection was apparently not used in cartography until the work of J. H. Lambert. See John P. Snyder, *Flattening the Earth (Two Thousand Years of Map Projections)*, University of Chicago Press, 1993.

This formula is the essential result; despite its simplicity, I have not seen it in any book.¹⁴

In view of (24), one can now give the following recurrence relations:

$$a_N = \frac{2\pi}{N-2} a_{N-2} \quad \text{and} \quad v_N = \frac{2\pi}{N} v_{N-2}, \quad (30)$$

which lead easily to the table above,¹⁵ and also lead back to the formulas (19).

This shows us at once why the power of π jumps at every other step. Namely, the dimensionalities 0 and 1 are trivial and do not involve π at all; after that, one gets from dimension 0 to all the even dimensions, introducing an additional power of π each time (note that eq. (29) does hold for $N = 2$), and similarly one gets from dimension 1 to the odd dimensions.

One π per plane!

Here is another viewpoint on this curious difference between spaces of even dimensionality and of odd dimensionality. The volume of the sphere $\mathcal{S}_N$ can be written

$$\begin{aligned} V_N &= \int_{\mathbb{R}^N} d^N r \theta(R^2 - r^2) \\ &= \int_{\mathbb{R}^N} dx_1 dx_2 \dots dx_N \theta(R^2 - x_1^2 - x_2^2 - \dots - x_N^2), \end{aligned} \quad (31)$$

where θ is the Heaviside function.

First look at *even* dimensionality $N = 2p$. Change variables to polar coordinates in each of the p planes defined by pairs of coordinate axes:

$$x_{2k-1} = \rho_k \cos \varphi_k, \quad x_{2k} = \rho_k \sin \varphi_k \quad (k = 1, 2, \dots, p). \quad (32)$$

Further, let

$$t_k = \rho_k^2 \quad (k = 1, 2, \dots, p). \quad (33)$$

It follows at once by (31) that

$$\begin{aligned} V_{2p} &= \int_{\mathbb{R}^N} \prod_{k=1}^p \rho_k d\rho_k d\varphi_k \theta\left(R^2 - \sum_{k=1}^p \rho_k^2\right) \\ &= \left(\frac{2\pi}{2}\right)^p \int \prod_{k=1}^p dt_k \theta\left(R^2 - \sum_{k=1}^p t_k\right), \end{aligned} \quad (34)$$

and in the last integral the integrand is 1 on the p -dimensional volume interior to the pyramid described by the positive half-axes ($0 < t_k, k = 1, 2, \dots, p$) and the hyperplane cutting them at $t_k + t_2 + \dots + t_p = R^2$. That volume is $(p!)^{-1} R^2$, giving us finally

$$V_{2p} = \frac{\pi^p}{p!} R^{2p}. \quad (35)$$

If on the other hand the dimensionality is *odd*, $N = 2p + 1$, one makes the same change of variables as before in p planes corresponding to p pairs of coordinates, but now one coordinate is left an old maid. Thus we get, instead of eq. (34),

$$\begin{aligned} V_{2p+1} &= \left(\frac{2\pi}{2}\right)^p \int \left(\prod_{k=1}^p dt_k\right) dx_{2p+1} \theta\left(R^2 - \sum_{k=1}^p t_k - x_{2p+1}^2\right) \\ &= \frac{\pi^p}{p!} \int_{-R}^R \left(R^2 - x_{2p+1}^2\right)^p dx_{2p+1} = 2 \frac{\pi^p}{p!} R^{2p+1} \int_0^1 (1 - u^2)^p du, \end{aligned} \quad (36)$$

or, finally,

$$V_{2p+1} = \frac{2^{2p+1} p! \pi^p}{(2p+1)!} R^{2p+1}, \quad (37)$$

in agreement with the general expressions.

The preceding changes of variable can be regarded as projecting, with preservation of Euclidean measure,

- in even dimensions, the ball $\mathcal{B}_{2p}$ onto the Cartesian product of p spheres $\mathcal{S}_2$ (circles) with the interior of a p -dimensional polyhedron;
- in odd dimensions, the ball $\mathcal{B}_{2p+1}$ onto the Cartesian product of p spheres $\mathcal{S}_2$ (circles) as before with the interior of a segment of a $(p+1)$ -dimensional paraboloid (given by $\sum_{k=1}^p t_k + x_{2p+1}^2 \leq R^2, 0 < t_k (k = 1, 2, \dots, p)$).

In both cases, the volume of the polyhedron or paraboloid is rational, so there are as many factors π as there are circles.

The situation is simple in essence: there are as many factors π in the expression for volume (or for area) of the sphere in N dimensions as there are “independent circularities” in the space, meaning by that, the number of independent ways to turn around in space, that is, simply the number of independent planes. That is evidently the integral part of half the dimensionality.¹⁶

The surface effect

It may be interesting to show how the expression for the volume of the sphere in N dimensions illuminates the physical nature of high-dimensional spaces. One might call this the “surface effect”: the higher the dimensionality, the more points there are close to the surface—in a sense to be clarified by the following examples. Naturally this is true for most sufficiently regular N -dimensional bodies; the only advantage of the sphere is in permitting explicit calculations leading to simple results.

First let us ask this question: what is the radius of the concentric ball inside a ball of radius R which contains half

¹⁴Thus Coxeter (op. cit., p. 126) gives the result, in the form of the first of equations (30), but derived from the expressions obtained by the “gaussian” method, not from the geometric meaning.

¹⁵Note in particular the case of the sphere in 4 dimensions: its area is equal to the volume of a torus in 3-space, and I wonder if the corresponding cartography would be a useful geometric tool.

¹⁶We may be permitted to wonder whether the difference in behavior brought out here between even- and odd-dimensional spaces is related to the Euler-Poincaré characteristic, which also distinguishes them.

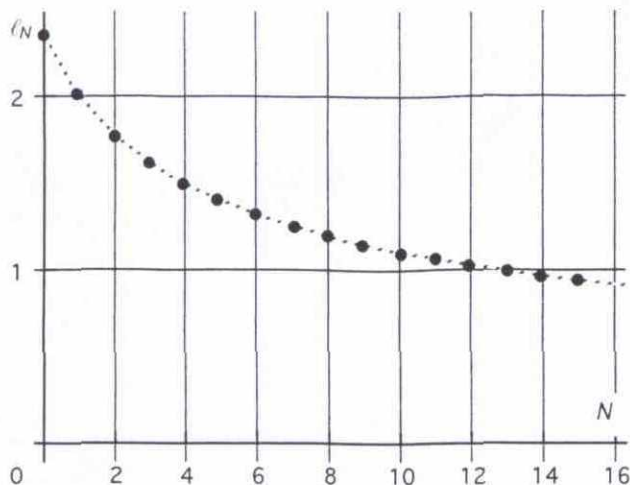


Figure 9. The "effective size" of the unit ball in N dimensions.

its volume? This requires that $V(R') = \frac{1}{2}V(R)$, or $v_N R'^N = \frac{1}{2}v_N R^N$, or

$$R' = 2^{-1/N} R. \quad (38)$$

Now plainly when N grows indefinitely, the radius R' approaches R . It is more striking to think instead of the outer shell between the inner and outer sphere, and containing half the volume of the latter. The higher the dimensionality, the more the thickness $\Delta R = R - R'$ shrinks relative to the radius R , as shown in Table 2.

Table 2. The Surface Effect							
N	1	2	3	4	5	...	$N > 1$
$\Delta R/R$	0.5	0.29	0.21	0.16	0.13	...	$\ln 2/N < 1$

We see that the outer half of the volume is crowded into a thinner and thinner layer. The reason is evidently the availability of a high number—namely $(N - 1)$ —of directions orthogonal to the radial direction, allowing a transversal "spreading," enabling volume to be large with small thickness.

In the same order of ideas, it is worthwhile to compare the surface of the sphere $\mathcal{S}_N$ with the volume of its "principal section," the ball $\mathcal{B}_{N-1}$, by calculating the ratio $A_N/V_{N-1} = \ell_N/v_{N-1}$ (it doesn't depend on the radius because of homogeneity, otherwise the quantity would have no significance). By Eq. (19), it turns out that

$$\ell_N/v_{N-1} = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{N-1}{2}\right)}{\Gamma\left(\frac{N}{2}\right)} = B\left(\frac{1}{2}, \frac{N-1}{2}\right) \quad (39)$$

an Eulerian function, whose first few values are given in Table 3.

Table 3. The Surface Effect (continued)							
N	1	2	3	4	5	...	$N > 1$
ℓ_N/v_{N-1}	2	π	4	$3\pi/2$	$16/3$	...	$2\sqrt{\pi N}$
		(3.14 ...)		(4.71 ...)	(5.33 ...)		(> 1)

It is a monotone increasing function, showing that the area of a sphere is larger (relative to the volume of the section) the larger the dimensionality.

Here is another way to look at this. Consider a uniform probability density on the unit ball in N dimensions. Let us ask what is the mean distance $\bar{r}_N$ between a random point in the ball and the center. This is given, of course, by

$$\bar{r}_N = \frac{\int_0^1 dr r A_N(r)}{\int_0^1 dr A_N(r)}, \quad (40)$$

which becomes, by eq. (22),

$$\bar{r}_N = \frac{\ell_N \int_0^1 dr r^N}{\ell_N \int_0^1 dr r^{N-1}} = 1 - \frac{1}{N}. \quad (41)$$

In other words, the mean distance to the center, for large values of the dimensionality N , is very close to the radius of the ball, which means that most of the points are very close to the boundary.

That means that the radius of the ball gives a misleading idea of its effective size, making it seem smaller than it is, because the abundance of available dimensions somehow compensates for the concentration near the surface.

Consider this rather surprising illustration of this interpretation. If we think that a good idea of the "effective size" of an N -ball is given by the length ℓ_N of the side of a hypercube of volume equal to that of the ball, which we may take to have unit radius, then by eq. (19) we get this length as

$$\ell_N := (v_N)^{1/N} = \frac{\Gamma(1/2)}{[\Gamma(1 + N/2)]^{1/N}}, \quad (42)$$

which turns out to be a *decreasing* function of N (Fig. 9). Notice the obvious special cases $\ell_1 = 2$ (in 1 dimension, the ball of unit radius is the same as the cube of side 2) and $\ell_2 = \sqrt{\pi}$.

Equation (42) is of course not defined in dimension zero. However, its limit as $N \rightarrow 0$ is well defined:

$$\ell_0 = \sqrt{\pi} e^\gamma = 2.3656 \dots, \quad (43)$$

where γ is Euler's constant. This apparently obliges us to regard the curious constant (eq. 43) as the effective size of the 0-dimensional unit ball (?).

Dimensionality and Orthogonality

I close with some elementary geometric considerations on N -dimensional euclidean space $\mathcal{E}_N$, which will bring out the great leeway afforded by letting the dimensionality be high. A simple, naïve statement would be that in a space of high dimensionality vectors tend to be "more and more independent" (which is natural) and even "more and more orthogonal" (which is less so). The point is that the direction cosines of any direction (with respect to a system of or-

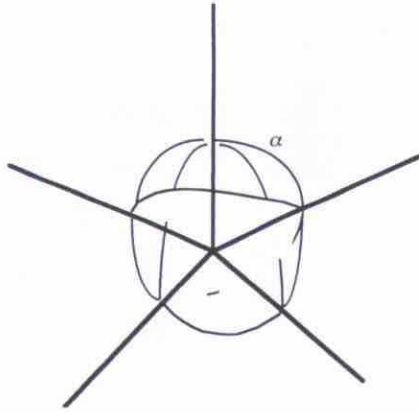


Figure 10. The equiaxial angle.

thogonal axes), having to add to unity, must generically all be small, making the angle between them close to a right angle. Let us check this in some special cases.

Equiaxiality

Consider a system of N orthogonal axes in $\mathcal{E}_N$, and call "equiaxes" referred to them the 2^{N-1} lines making equal angles with all of them; in 2 dimensions, the equiaxes are the two bisectors of the angle between the axes. Denote by α_N the (acute) angle between an equiaxis and a coordinate axis; call it the "equiaxial angle" (Fig. 10). Thus the unit vector along an equiaxis projects onto each coordinate axis to a segment of $\pm \cos \alpha_N$ so that the angle α_N is given by $N \cos^2 \alpha_N = 1$, or

$$\cos \alpha_N = \frac{1}{\sqrt{N}}. \quad (44)$$

The numerical values of the equiaxial angle for some low dimensionalities are given in Table 4. Indeed, the equiaxes are closer and closer to orthogonality to the coordinate axes.

Table 4. The Equiaxial Angle							
N	1	2	3	4	5	...	$N > 1$
α_N	0	45°	54.7°	60°	63.4°	...	$\approx \frac{\pi}{2} - \frac{1}{\sqrt{N}}$

Isogonality

Just as natural is the figure which we may call "isogonal" formed in $\mathcal{E}^N$ by $N + 1$ half-lines making equal angles with each other. In 2 dimensions this gives the Mercedes trigon; in 3 dimensions, the directions to the vertices of a regular tetrahedron from its center—in general, the directions of the vertices of a regular simplex seen from its center. Let us denote by β_N the "isogonal angle" between any two of these lines (Fig. 11). By symmetry, the $N + 1$ unit vectors $\mathbf{u}_k$ ($k = 1, 2, \dots, N + 1$) on isogonal rays are linearly dependent by

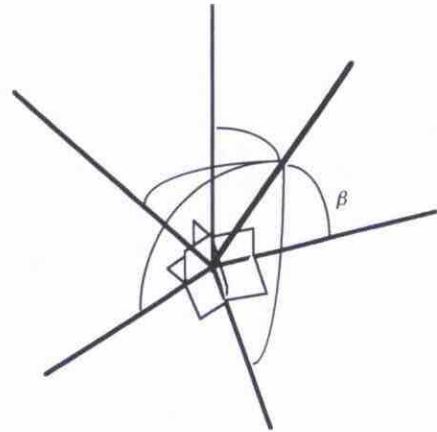


Figure 11. The isogonal angle.

$$\sum_{k=1}^{N+1} \mathbf{u}_k = \mathbf{0}. \quad (45)$$

Projecting this relation onto any one of the vectors, one gets the relation defining the isogonal angle, namely

$$\cos \beta_N = -\frac{1}{N}. \quad (46)$$

These values may be tabulated as shown in Table 5

Table 5. The Isogonal Angle ¹⁷							
N	1	2	3	4	5	...	$N > 1$
β_N	180°	120°	109.4°	104.5°	101.5°	...	$\approx \frac{\pi}{2} + \frac{1}{N}$

Again, the increase of dimensionality brings with it a tendency toward orthogonality.

Uniformity

Given the sphere $\mathcal{S}_N$ in N dimensions, what is the mean angle between two vectors taken randomly on the sphere, given its uniform measure? A simple symmetry argument shows that it is a right angle. But the probability distribution of this angle θ , call it $p_N(\theta)$, rewards attention. Let us parametrize the sphere in generalized spherical coordinates:

$$\begin{aligned} x_1 &= \cos \theta_1, x_2 = \sin \theta_1 \cos \theta_2, x_3 = \sin \theta_1 \sin \theta_2 \cos \theta_3, \dots \\ &\dots x_{N-1} = \sin \theta_1 \sin \theta_2 \dots \cos \theta_{N-1}, \\ x_N &= \sin \theta_1 \sin \theta_2 \dots \sin \theta_{N-1}. \end{aligned} \quad (47)$$

In these coordinates the uniform measure on the sphere appears as

$$d^{N-1}\sigma = d\theta_1 d\theta_2 \dots d\theta_{N-1} \sin^{N-2} \theta_1 \sin^{N-3} \theta_2 \dots \sin \theta_{N-2}, \quad (48)$$

Of course, integration on all of the angles would give back the area of $\mathcal{S}_N$ calculated above. Here we are look-

¹⁷Note that the familiar relation $\beta_3 = 2\alpha_3$, which determines several features of spatial symmetries in our world (for example in crystallography), is altogether confined to the 3-dimensional case. The equation $\text{Arccos}(N^{-1}) = 2 \text{Arccos}(N^{-1/2})$, whose only solution is $N = 3$, must be added to the other particularities of our space, along with the fact that $N = N(N - 1)/2$ (whence the exterior product of two vectors can be a "vector product") and the fact that $N + 1 = 2^{N-1}$ (whence two equiaxial lines can also be isogonal).

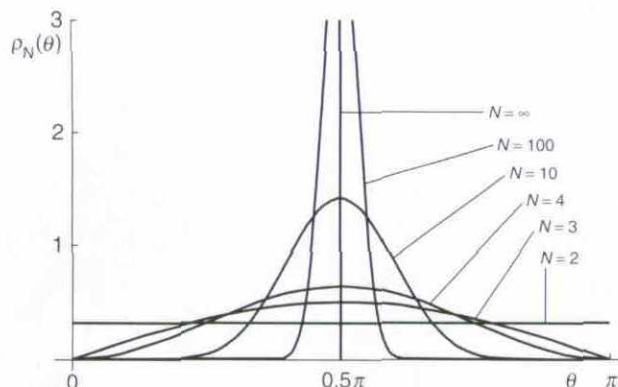


Figure 12. The probability density of the angular distance of two random points on an N -dimensional sphere.

ing at the angle between a randomly chosen variable direction and a fixed direction. Let us take the zenith axis Ox_1 as reference direction; so it will be the random variable θ_1 , which is at issue. Now integrating over all the other angular variables, we see that the desired probability density, which determines the random distribution of the angular distance between two arbitrary directions, is given by

$$\rho_N(\theta) = K \sin^{N-2} \theta. \quad (49)$$

This is a distribution with mean value $\bar{\theta} = \pi/2$, more and more "sharply peaked" as the dimensionality N grows (Fig. 12); for large values of N its width is $\delta\theta \approx O(N^{-1/2})$, and it approaches the Dirac distribution $\delta(\theta - \pi/2)$ in the limit $N \rightarrow \infty$.

That is to say, the higher the dimensionality of a Euclidean space, the more likely arbitrary directions are to be close to orthogonal.

This idea may have some effect on the intuition we can have (or can't have!) about an infinite-dimensional space like the Hilbert space of quantum mechanics.¹⁸

¹⁸It is my pleasure to extend my deep thanks to Chandler Davis for translating this article into English.

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