

IF PYTHAGORAS HAD A GEOBOARD . . .

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THE theorem of Pythagoras is an old favorite of junior and senior high mathematics teachers, and it lends itself to much interesting discovery. This article describes an investigation into the famous theorem, using geoboards.

First, a word about the geoboard. The invention of the geoboard has been credited to Dr. Caleb Gattegno, the British mathematician who developed the Cuisenaire-Gattegno system of colored rods. The geoboard (or nailboard) is a flat board into which nails or panel-pins have been driven in a regular pattern. Geometric figures can then be formed, using rubber bands.

There are three types of geoboards in use: (1) the square grid geoboard, (2) the circular geoboard, and (3) the isometric grid geoboard. This investigation used (1) and (3), which are analogs of squared paper and isometric paper, respectively. However, unlike paper, the geoboard is (in a manner of speaking) a dynamic device, and this I have found to be a strong point in its favor in the classroom.

I do not know of any other teaching device at once so simple and so versatile. At one end of the scale, it is attractive to kindergarten children (Liedtke and Kieren 1970), and at the other, it can challenge the senior mathematics major in college (Buckeye, Ewbank, and Ginther 1971).

By far the best-known geoboard is the square grid model. It is inexpensive and obtainable from several educational suppliers (see listing at the end of this article) or can easily be homemade. However, for this investigation, it is best to

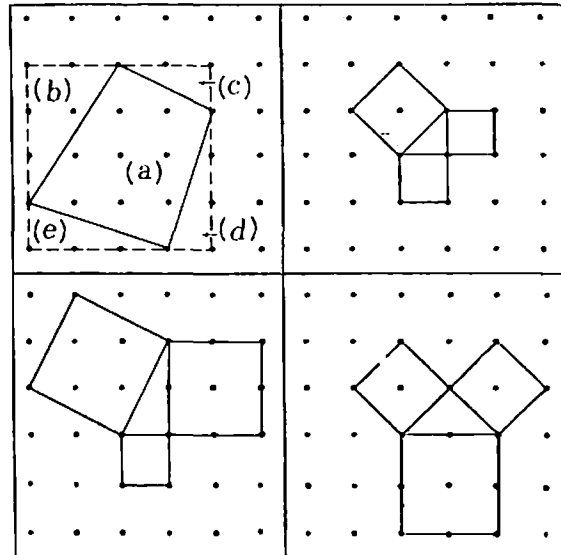


Fig. 1

On this type of geoboard the area of any simple closed curve can easily be found (that of a nonsimple closed curve is a much harder task). It is best done on the subtractive principle: by surrounding the figure with a rectangle (or a polygonal figure with each angle one or three right angles), as shown in fig. 1. The area of quadrilateral (a) is given by subtracting from the area of the square surrounding it (4×4 , or 16), the sum of the areas of triangles (b), (c), (d), and (e). These are $\frac{1}{2}(3 \times 2)$, $\frac{1}{2}(2 \times 1)$, $\frac{1}{2}(3 \times 1)$, and $\frac{1}{2}(3 \times 1)$, respectively, or a total of 7 units. Thus the quadrilateral (a) has area $16 - 7$, or 9 units. A facility with calculations like this is essential for a satisfactory investigation into the theorem of Pythagoras.

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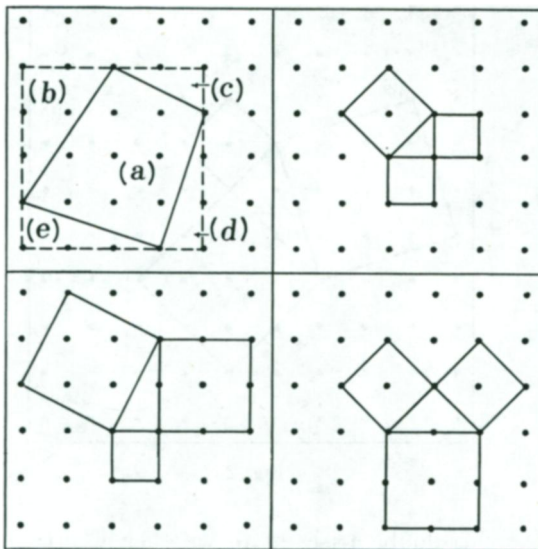


Fig. 1

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three boards (fig. 1) to illustrate, with the right triangles shown: top right, $1^2 + 1^2 = 2$; lower left, $1^2 + 2^2 = 5$; and lower right, $2 + 2 = 2^2$.

An arrangement with a "skewed" right triangle on a 10×10 geoboard is shown in figure 2. The areas of the squares can be calculated to be 2, 8, and 10.

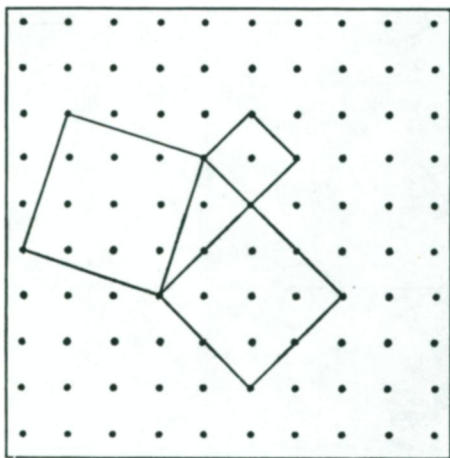


Fig. 2

If doubt arises as to whether a certain triangle is, in fact, a right triangle, and if the students have had enough experiences establishing the principles as illustrated above, they can use the converse of the theorem of Pythagoras to determine what sort of triangle they have made. Thus, if the area of the square on the longest side of a triangle is equal to the sum of the areas of the squares on the two shorter sides, the triangle must be a right triangle. Figure 3 illustrates how a situation of inequality can be determined, showing that because $1 + 2 < 5$, the first triangle is obtuse, and because $2 + 5 > 5$, the second triangle is acute.

The investigation now moves to the isometric geoboard. The isometric board is not nearly so well known. At the time of writing I know of only one supplier in the United States (Math Media), but it is fairly easy to make a board for yourself. As it is important that the nails be as accurately lined up as possible and to

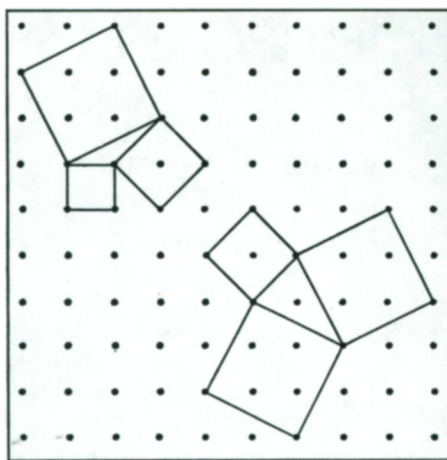
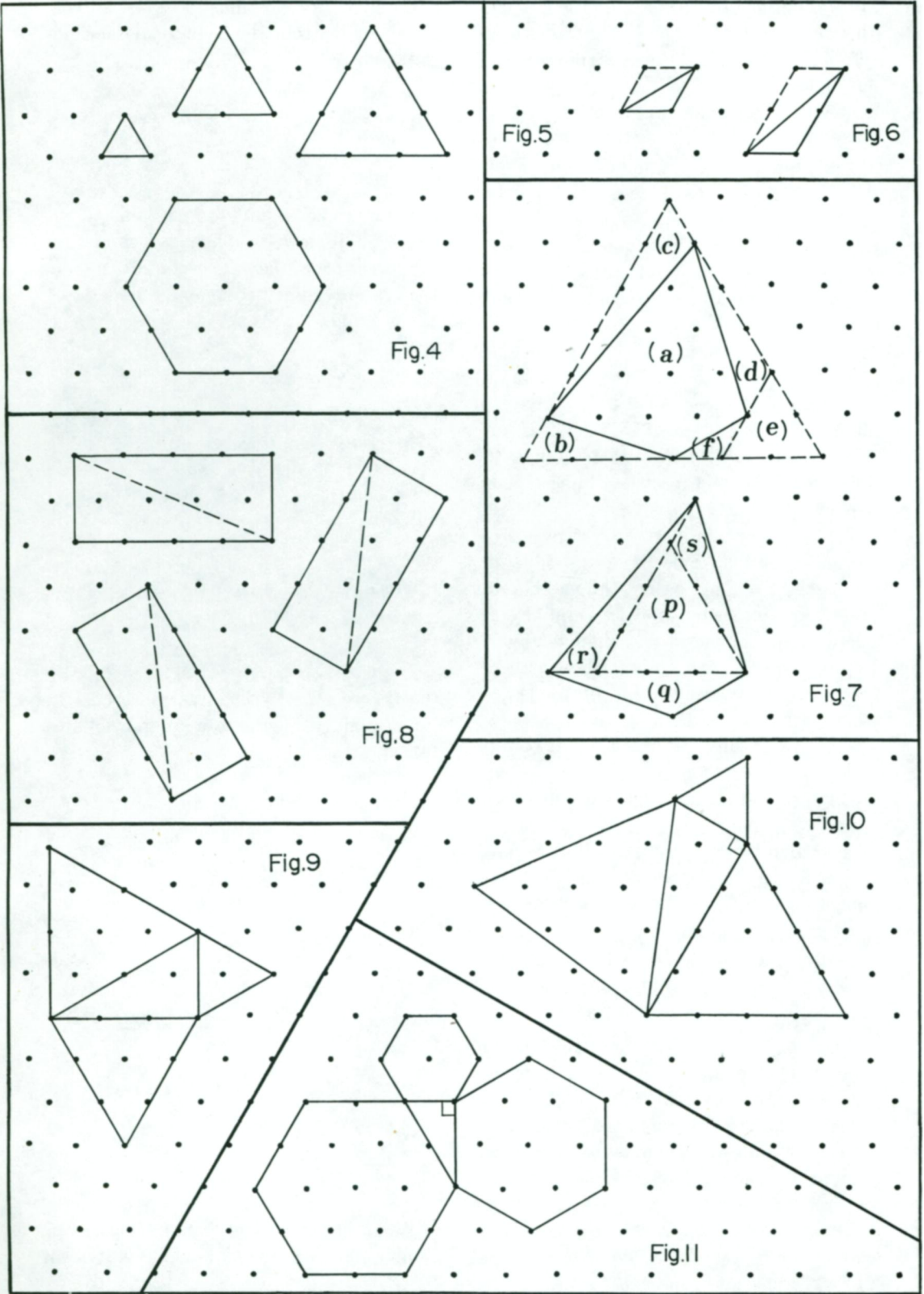


Fig. 3

avoid the labor of marking out the lines, it is recommended that a piece of isometric graph paper be taped to the board. With this, the nails can be driven through the intersections at desired spacing ($\frac{7}{8}$ " is convenient), and the paper then removed.

With the board in hand, a snag arises which, at first sight, seems rather formidable: You cannot make a square on the isometric geoboard! This makes it impossible to check on the area of the square on the side of any triangle, and difficult to work out *any* area in square units.

Now the basic unit of shape on the isometric board is the equilateral triangle. So why not use the area of the smallest equilateral triangle as our unit of area? Figure 4 shows, with triangles of area 1, 4, and 9 and a hexagon of area 24, that this system works well with simple figures. In figure 5, the isosceles obtuse triangle has area 1, because it is half of a rhombus of area 2. In figure 6 the obtuse triangle has area 2 because it is half of a parallelogram of area 4. In this way the area of any simple closed curve on the isometric geoboard can be found. For example, the area of quadrilateral (a) in figure 7 can be found by subtracting from the area of the outside equilateral triangle (side 6, area 36), the sum of the areas of triangles (b), (c), (d), (e), and



(f). This gives $36 - (3 + 5 + 3 + 4 + 1)$, or 20.

To check this result, the quadrilateral can be decomposed into the triangles (p) , (q) , (r) , and (s) , which have area $9 + 4 + 4 + 3$, or 20.

Although squares cannot be made on the isometric geoboard, rectangles can be made, also right triangles (fig. 8). The three rectangles in this diagram are congruent, and each is of area 15. The diagonals divide them into congruent right triangles.

Whenever right angles are formed, one arm must lie on one of the three main lines of nails, i.e., one of the axes of symmetry of the system. The reader is invited to attempt to form right angles in other positions, and to satisfy himself that it is not possible.

We are now ready to investigate Pythagoras further. Since it is possible to make right triangles, but impossible to make squares, why not try equilateral triangles on the sides of a right triangle? Figure 9 shows such an arrangement, and calculation gives the areas of the equilateral triangles to be 3, 9, and 12. So far Pythagoras, duly adapted, still holds good!

What about figure 10? Does it work here?

Since equilateral triangles will tessellate to form regular hexagons, perhaps Pythagoras can be adapted as follows:

The area of the regular hexagon on the hypotenuse of a right triangle is equal to the area of the regular hexagons on the other two sides.

Figure 11 shows that this does appear to be true, though of course one illustration does not prove the rule.

Let us assume that Pythagoras holds good for equilateral triangles, squares, and regular hexagons. Does it hold good for any regular n -gon?

Due to the nature of the geoboard, this cannot be checked, as only the above three regular polygons can be formed on

a geoboard. So we must resort to paper and pencil. From the numerical relationship

$$a^2 = b^2 + c^2$$

we can derive:

$$ka^2 = kb^2 + kc^2,$$

where k is a constant.

We shall now establish a formula for the area of a regular n -gon of side a . Using the regular pentagon as model (fig. 12), we find:

$$\text{area } \triangle ABC = \frac{a}{2} \cdot \frac{a}{2} \cot \frac{180}{5} = \frac{a^2}{4} \cot \frac{180}{5}$$

$$\therefore \text{area pentagon} = 5 \left(\frac{a^2}{4} \cot \frac{180}{5} \right)$$

$$\therefore \text{area regular } n\text{-gon of side } a = \frac{na^2}{4} \cot \frac{180}{n} \\ = Ka^2$$

where K is constant for the type of polygon under consideration. Since

$$Ka^2 = Kb^2 + Kc^2$$

the theorem of Pythagoras must be true for any given regular n -gon, and not just for squares.

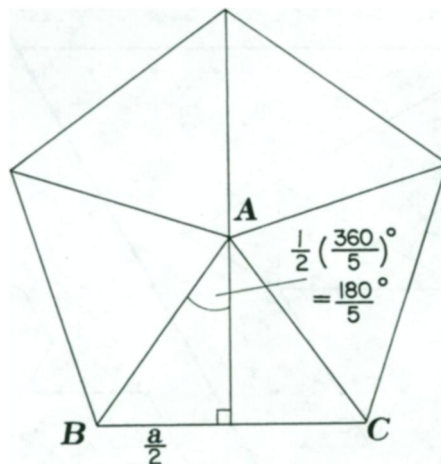


Fig. 12

What about nonpolygonal figures, e.g., semicircles? Is the following statement true?

The area of the semicircle on the hypotenuse of a right triangle is equal to the sum of the areas of the semicircles on the other two sides.

Just as above we found Pythagoras was true for any given regular n -gon, since the areas of regular n -gons are proportional to the squares of their sides, so here, all we have to do is to show that the areas of semicircles are proportional to the squares of their diameters. This is clearly so, since the area of a semicircle of diameter a would be $\frac{\pi a^2}{8}$. Thus we have

$$\frac{\pi a^2}{8} + \frac{\pi b^2}{8} = \frac{\pi c^2}{8}$$

So far the adaptations of the Pythagorean theorem have been true only for figures that are *similar*. Erect non-similar rectangles on the sides of a right triangle, and obviously there will be no area relationship, except by chance. Even if the rectangles chosen were similar, e.g., with ratio of sides 2:1, their areas may not satisfy the Pythagorean relationship. This can easily be shown on a geoboard (fig. 13). This situation arises when the Pythagorean theorem is adapted to figures which, although similar, are not “similarly placed” or “positioned” on the triangle. However, Euclid stated in Proposition 31 of Book VI of *The Elements* that “in right-angled triangles, the rectilinear figure described on the side opposite the right angle is equal to the similar and similarly described figures upon the sides containing the right angle.”

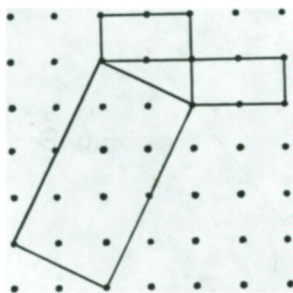


Fig. 13

An example of similar rectangles similarly placed is given in figure 14, and of scalene triangles in figure 15.

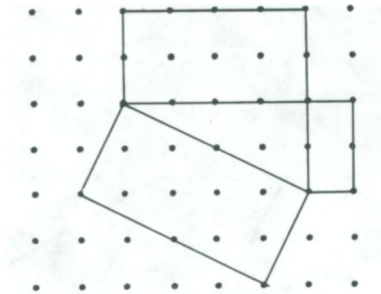


Fig. 14

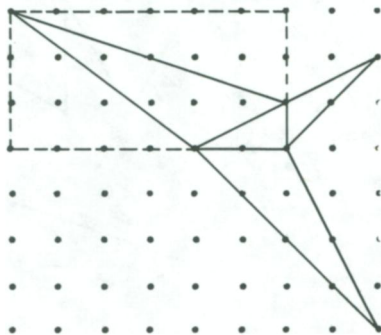


Fig. 15

It is not easy to make a series of similar nonregular polygons on a geoboard of limited size. In figure 16 an attempt has been made to construct similar trapezoids on the sides of a right triangle on an isometric board, but since

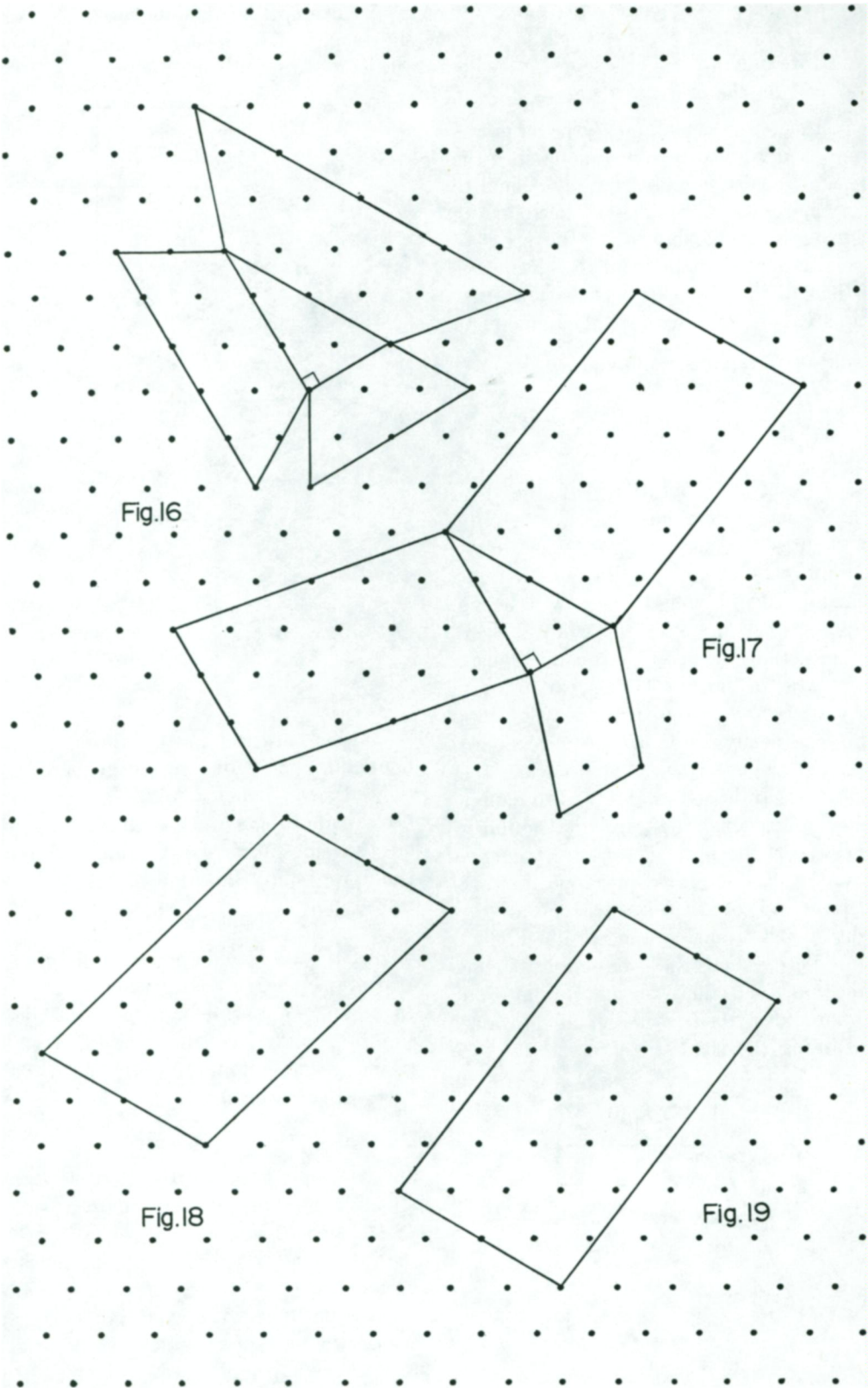
$$16 + 9 \neq 24,$$

one of the trapezoids must be dissimilar. It is fairly easy to see which one it is. In figure 17 parallelograms have been similarly placed on the sides of a right triangle, again on an isometric board. This time we can see that

$$36 + 10 \neq 44$$

and attempts to alter the largest parallelogram in shape only result in areas of 48 (fig. 18) or 52 (fig. 19).

It is probably easier to pursue investigations such as these on isometric dot paper, and those with expansive ideas can stick two or more sheets together. Isometric dot paper is quite easily made by



placing a piece of isometric graph paper on top of a ditto mat and pricking through vertices at regular intervals. Square grid dot paper can be made, with tolerable accuracy, with a typewriter and a ditto mat.

The writer feels that with these simple materials an interesting field of discovery is open and that great value is to be found in evolving simple rules to cope with strange and novel situations.

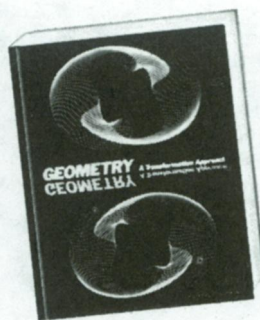
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- Liedtke, Warner, and Thomas E. Kieren. "Geoboard Geometry for Preschool Children." *ARITHMETIC TEACHER* 17 (February, 1970): 123-27.

SOME GEOBOARD SUPPLIERS

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