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THE ORIGIN OF THE PYTHAGOREAN "APPLICATION OF AREAS"

We are told in Proclus' commentary to Book I of Euclid's Elements (F, pp. 419-20) that the 'application of areas' was a discovery of the old Pythagoreans. They distinguished the following three kinds of this construction.

I. The simple application of a parallelogram to a given straight line as one of its side, is the so-called parabole. This is dealt with in Euclid's proposition I 44: "To a given straight line to apply, in a given rectilineal angle, a parallelogram equal to a given triangle." We may conjecture that the particular form of this task -- the application of a given rectangle to a given straight line -- was the original one. Euclid deals with the same problem in a more generalized form.

This is valid for the two other kinds of 'application of areas' as well. That is, we find in Euclid in these cases, too, only the more generalized forms of both problems.

II. The elliptic form, or: the application of an area with defect is, as we read in Book VI of the Elements (28): "To a given straight line to apply a parallelogram equal to a rectilineal figure and deficient by a parallelogrammic figure similar to a given one, etc." -- On the other hand the hyperbolic form (or: application of an area with excess) reads in the next proposition (VI 29):

III. "To a given straight line to apply a parallelogram equal to a given rectilineal figure and exceeding by a parallelogrammic figure similar to a given one."

It is more than probable that the original particular form of the problem was, in both last cases as well, the application of a rectangle (of given area) to a given straight line, and the 'given parallelogrammic figure', to which the falling short or the exceeding area had to be similar, was a square. The particular cases are not dealt with in Euclid, but they can be solved by means of Book II of the Elements, since both propositions II 5 and 6 are but the somewhat simpler and modified forms of the problems VI 28 and 29, respectively.

Earlier research tried to explain these

constructions as geometrical solutions of quadratic equations. Thus the simple application of a rectangle to a straight line corresponds to the equation $y^2 = px$, while the corresponding equations to the applications with 'defect' and 'excess' are: $ax - x^2 = b^2$ and $ax + x^2 = b^2$, respectively.

Now I do not contest that the said equations can be taken -- particularly in the cases of 'application with defect' and 'application with excess' -- as more or less adequate interpretations of ancient mathematical ideas by modern means. But all the same, these algebraic equations do not reveal the origin of the geometrical constructions in question at all. So far as I know, there is absolutely no historical evidence for the conjecture, as if the pythagorean 'application of areas' would ever have been a method for solving quadratic equations. In fact, nothing is known about any pre-euclidean Greek algebra that could have given a geometric formulation. In my mind, it was some problems of the arithmetic that gave rise to the interesting pythagorean discovery of the 'application of areas'. I shall try to explain here first of all the origin of the simple application of a rectangle to a given straight line.

Let us consider the following three arithmetical propositions from the Elements:

IX 16: "If two numbers be prime to one another, the second will not be to any other number as the first is to the second." (I.e., If a and b are numbers relatively prime, then there is no number x such as to be $a:b = b:x$.) -- A variant to this proposition is:

IX 18: "Given two numbers, to investigate whether it is possible to find a third proportional to them." (I.e., to find the conditions on numbers a and b which are necessary and sufficient to ensure the existence of an x such as to be $a:b = b:x$.) The third proposition I should like to mention here, reads:

IX 19: "Given three numbers, to investigate when it is possible to find a fourth proportional to them." (I.e., to find the conditions on a , b and c which are necessary and sufficient to ensure the existence of an x such as to be $a:b = c:x$.)

These propositions were obviously written down at a time when it was known that the problem of finding a third proportional to two given numbers (or a fourth proportional to three given ones) could only

be solved under certain conditions.

But what about, if a , b and c are taken to be arbitrary lines instead of numbers? -- In such a case the geometrical solution of a third or of a fourth proportional can always be found. This is proved by Euclid in Book VI of the Elements (VI 11 and 12). His constructions, however, frequently yield incommensurable magnitudes. Hence it cannot be claimed that they always provide correct solutions, unless the eudoxian definition of proportionality (V def. 5) is assumed.

On the other hand, it is possible to solve these same problems without considering proportions or incommensurability, if they are first transformed in the following way. The problem of finding the third proportional becomes that of finding a rectangle (ax) which has one side of length a and the same area as a given square, b^2 . Similarly, instead of trying to find a fourth proportional, one looks for a rectangle which has one side of length a and the same area as a given rectangle, bc . [In transforming so the problem of proportionality, we make use of the well-known arithmetical theorem, VII 19: "If four numbers be proportional, the number produced from the first and fourth will be equal to the number produced from the second and third, etc." Besides, one has to keep in mind that the Pythagoreans illustrated the product from a multiplication of two factors, indeed, as a so-called 'plane number', the sides of a rectangle being the factors in question.]

I think therefore, that the simple application of a rectangle to a given straight line is just the general solution of a problem of proportionality, that in arithmetic could not be solved but under certain conditions.

Now let us recall here, how the geometrical construction to apply a rectangle (of given area) to a given straight line, might have been found. -- If two lines which are parallel to the sides of a given parallelogram are drawn through an arbitrary point of its diagonal, they will split the area of the parallelogram into four parts. Two of these parts, the so-called "parallelograms about the diameter" will be similar to each other and to the original parallelogram (VI 24), whereas the other two, the so-called "complements" (parapleromata) will be equal to one another (I 43). Just these facts made possible the application of a given rectangle to a given

straight line. The given rectangle was taken as one "complement" (shaded part of the diagram), while the given straight line was taken as the continuation of one of its sides. Then they constructed, with the help of these data, first the one "parallelogram about the diameter," the diameter itself, the "big parallelogram" and at last the other "complement." This other "complement" was just the application they were looking for.

Now if we want to understand, how in geometry the application with "defect" and "excess" came about, we have to make a little detour. -- Consider the following definition in Euclid (II 2):

"In any parallelogrammic area let any one whatever of the parallelograms about its diameter with the two complements be called gnomon."

As you see, the definition of the concept "gnomon" presupposes that the area of the parallelogram has already been split into four parts in the manner described previously. The "gnomon" consists of three of these parts, namely the two "complements" together with either one of the "parallelograms about the diameter." The three components are indicated in my illustration of the "gnomon" of a square by different kinds of shading. The figure also shows that the original square (x^2) can be obtained as the sum of a smaller square (y^2) and the gnomon. In other words, subtracting a smaller square from a bigger one leaves a gnomon as the remainder. This fact is important because the gnomon itself can be very easily transformed into a rectangle. There are two ways of carrying out this transformation, as I illustrate it by means of two diagrams.

In one case the area consisting of one of the parallelograms about the diameter (the smallest square, which for convenience I denote now by "o") and one of the "complements" "c" is, in a manner of speaking, joined onto the left hand side of the other "complement"; whereas in the other case it is only the "complement" c which is joined onto the left hand side of the area consisting of the smallest square and the other "complement." The resulting rectangle is the same in both cases; it has sides $(x + y)$ and $(x - y)$ -- these letters refer to the same lengths in both figures. The only difference between the two constructions has to do with the square "o", a component of the original gnomon. In one case this small square seems to fall short, while in the other

one it seems to exceed.

Euclid deals with both these ways of transforming a bigger square into the sum of a smaller square and a rectangle in his rather clumsily formulated theorems II 5 and 6. Since it is in fact the application of a rectangle with defect, which reads in Euclid in the following form (II 5):

"If a straight line (AB) be cut into equal (at C) and unequal segments (at D), the rectangle contained by the unequal segments of the whole (shaded on the figure), together with the square on the straight line between the points of section (y^2) is equal to the square on the half (x^2)."

On the other hand, the application with "excess" has the euclidean wording (II 6):

"If a straight line be bisected (at C) and a straight line be added to it in a straight line (DE), the rectangle contained by the whole with the added straight line (that is the shaded rectangle in the figure) together with the square on the half (y^2) is equal to the square on the straight line made up of the half and the added straight line (x^2)."

Now we understand the historic role of these last propositions only if we realize that they are necessary lemmas for the solution of two other very important geometrical problems.

The euclidean proposition II 14 deals with the problem: "To construct a square equal to a given rectangle."*

This problem can be solved on the basis of the following consideration. -- The application of an area with 'defect' (Elem. II 5) states, that: If we subtract a smaller square from a bigger one, it remains a rectangle (more precisely: it remains a gnomon which can easily be transformed into a rectangle). Therefore we can make with the two sides of a given rectangle (a and b) a bigger square:

$\left(\frac{a+b}{2}\right)^2$ and a smaller one: $\left(\frac{a-b}{2}\right)^2$. Their

difference will be of course the rectangle itself. If we now make the same bigger square to be the square on the hypotenuse of a right-angled triangle, and the smaller square to be the square on its one

*In fact Euclid deals with this problem in a more generalized form: "To construct a square equal to a given rectilinear figure."

perpendicular side, then their difference will be -- according the so-called theorem of Pythagoras -- the square on the other perpendicular side. It is in this way, that any rectangle can be squared, following both Euclidean theorems II 5 and II 14.

We must not forget however, that the squaring of a rectangle is nothing else but the more generalized form of the problem: how to find a mean proportional between two numbers or magnitudes. The Pythagoreans were of course aware of the fact, that this arithmetical problem could not be solved but under certain conditions. I.e., there is no mean proportional number between two given numbers a and b, unless their product ab is a perfect square. Therefore we get, in squaring a rectangle, sometimes a linearly incommensurable magnitude as the side of the square equal to the given rectangle, in all those cases namely, in which there is no mean proportional number between the two numbers representing the sides of the given rectangle. As you see, the application of an area (of a rectangle) to a given straight line with "defect" was -- as a necessary lemma to the squaring of a rectangle -- in all likelihood closely connected with the discovery of incommensurability: it enabled the geometers to solve a problem of proportionality, which often could not be solved otherwise.

The other construction, the application of a rectangle to a given straight line with "excess" had a quite similar role in the Pythagorean science as well. -- It was namely a particular case of the mean proportional the so-called "extreme and mean ratio." As we read in Euclid (VI 30): "To cut a given finite straight line in extreme and mean ratio." That is, the given straight line a is to be cut in x and (a - x), so as to be $a:x = x:(a - x)$. The solution of this problem is necessary to that of inscribing a regular pentagon in a circle, and so to that of getting the sacred symbol of the Pythagoreans the pentagram. (The diagonals of the pentagram are intersecting each other according to the extreme and mean ratio.) In the same manner as the common case of the mean proportional, this particular case has been transformed, too into the equation of a rectangle and a square. As Euclid formulated II 11):

"To cut a given straight line so that the rectangle contained by the whole and one of the segments is

equal to the square on the remaining segment." That is the rectangle $a \cdot (a - x)$ had to be equal to the square x^2 . The solution was obtained by coupling the application of a rectangle with "excess" (II 6) and the so-called Pythagoras' theorem. It was particularly important to this solution, since the cutting "in extreme and mean ratio" never had a numerical solution. The application of area was in that case beyond any doubt connected with the discovery of incommensurability.

I think therefore that the Pythagorean application of areas was a powerful method of the early Greek geometers to overcome those difficulties which they encountered in discovering the existence of linearly incommensurable magnitudes that were at the same time commensurable in square. (Later on these same problems were dealt with by means of the eudoxian theory of proportions.)

Blackboard figures

(I) parabole I 44

$$y^2 = px$$

(II) elliptic (with "defect") VI 28 ~ II 5

$$ax - x^2 = b^2$$

(III) hyperbolic (with "excess") VI 29 ~ II 6

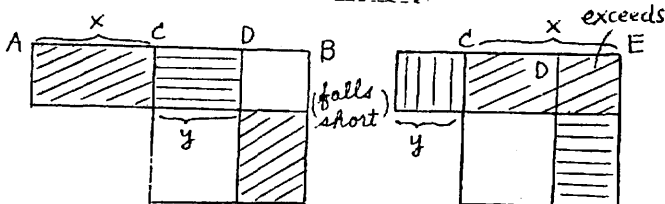
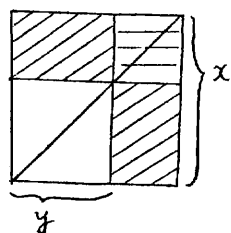
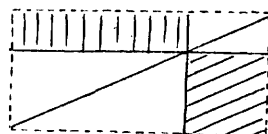
$$ax + x^2 = b^2$$

IX 16

IX 18

IX 19

$$\left. \begin{array}{l} a:b = b:x \\ a:b = c:x \end{array} \right\} \begin{array}{l} ax = b^2 \\ ax = bc \end{array}$$



$$\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = ab$$

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