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THE PYTHAGOREAN DOCTRINE

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The article on "Pythagoras: Mathematician and Philosopher" in the Spring issue of this journal¹ discussed philosophy and Pythagoras as a philosopher, made frequent references to the "Pythagorean Number Philosophy," mentioned philosophy of mathematics, and seemed to regret the fact that little has been written on the influence of mathematics on philosophy.

The present article is concerned, principally, with revealing the mathematical nature of Pythagorean philosophy. It is also concerned with doing justice to Pythagoras. The thesis to be advanced here is that the Pythagorean Doctrine, in its arithmo-geometric aspects, was an attempt, perhaps the first, to bring mathematics to the help of philosophy in the study of nature. It is pertinent to insert here that the word mathematics was first used by Pythagoras. It is derived from *mathema*, that which is learnt, a lesson.

Consider then, briefly, the views of the cosmogonists preceding Pythagoras. Anaximander of Miletus (611-547 B.C.) held² that from an endless, boundless, and shapeless mass, the *apeiron*, which is subject to a circular movement, a flat disc, the earth, is formed at the center, then rings of water, air, and fire are thrown out, spreading away from the center in ever thinning layers, but not without limit for an infinite mass cannot rotate. The universe thus formed, however, is not stable; the celestial fire devours and dissipates the center and the outflung layers and thus, in the course of time, everything returns to the original state. But there is an end to this period of dissipation and the same causes that once formed the universe reform it. There is thus an endless succession of worlds and the only

¹H. H. Moorman, "Pythagoras: Mathematician and Philosopher," *The Pentagon*, Vol. 8, pp. 79-84 (Spring, 1949).

²Paul Tannery, *Pour l'histoire de la science hellène*, 2nd ed. Paris, 1930.

thing that remains immortal and imperishable is the circular movement.

Anaximenes of Miletus, a younger contemporary of Anaximander and, by report, a pupil and friend of his, added the elaboration that the boundless air, subject to an eternal movement, is the source of everything. Expanding under the influence of heat, or contracting under that of cold, it has formed all the phases of existence.

In contrast with this we find the Pythagorean Doctrine proceeding more cautiously, from within the cosmos, and trying to build up a theory by abstraction, logical construction, and generalization. The geometric point is defined as "unity in position" and, conversely, the unit of number as "a point without position."³ This identification of the point and the unit of number led immediately to the well-known practice of representing numbers by figures (*schematographien*). For if a number is a plurality of units, and a unit is a point, then a number may be represented by an aggregate of points, arranged into such a figure as the nature of the number might suggest. In this practice a method was developed for the successive generation of numbers of a given type; this was the use of gnomons.⁴ Thus Proclus, Diogenes Laertius,⁵ and Plutarch⁶ attribute to Pythagoras the method of forming successive square numbers by the addition of equilateral gnomons to unity (Fig. 1), while Lucian and Aristotle mention the formation, by the Pythagoreans, of the triangular and the oblong numbers by the addition of the corresponding gnomons (Fig. 2).

That this practice was not a fruitless pastime is shown by the many arithmetic discoveries they made by its use. Thus, from the fact that the gnomons of the squares are the odd numbers after unity, in succession, and the resulting figures are the next higher squares, they deduced, by successive additions,⁷

³Proclus, *Commentary on Euclid's Elements I*, ed. Friedlein, 1873, p. 95.

⁴In geometry, a gnomon is the figure which, added to any figure, preserves the original shape.

⁵Diogenes Laertius, ed. Hubner, Leipzig, 1831.

⁶Plutarch, *Opera Moralia*, Symposium 6, ed. D. Wytttenbach. Oxford, Clarendon Press, 1796-1821.

⁷Nicomachus of Gerasa, *Introduction to Arithmetic*, tr. by M. L. D'Ooge, with studies in Greek arithmetic by F. E. Robbins and L. C. Karpinski. New York, Macmillan, 1926.

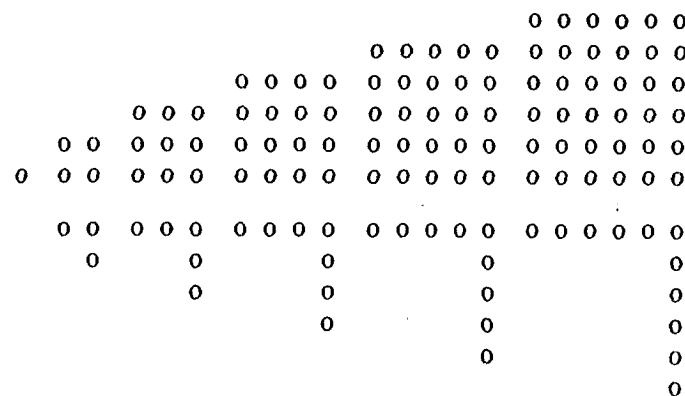


Fig. 1. Square numbers and their gnomons.

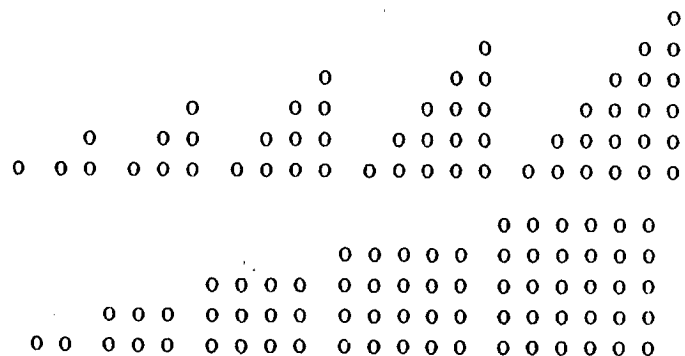


Fig. 2. Triangular and oblong numbers.

$$1 + 3 + 5 + \dots + (2n-1) = n^2,$$

and from the fact that the gnomon $(2n+1)$ added to the square n^2 produces the next higher square, $n^2 + (2n+1) = (n+1)^2$ they found⁸ that if $(2n+1) = k^2$, where k is any odd number, then the numbers k , $\frac{1}{2}(k^2-1)$, $\frac{1}{2}(k^2+1)$ are the sides and hypotenuse of a right triangle or, in present-day terms, they are Pythagorean numbers. Proclus⁹ says:

But there are delivered certain methods of finding triangles of this kind (sc. right-angled triangles

⁸J. Allman, *Encyclopaedia Britannica*, Werner Edition, 1900, Vol. XX, p. 141.
⁹Proclus, *op. cit.*, p. 428.

whose sides can be expressed by whole numbers) one of which they refer to Plato, but the other to Pythagoras, as originating from odd numbers. For Pythagoras places a given odd number as the lesser of the sides about the right angle, and when he has taken the square erected upon it, and diminished it by unity, he places half the remainder as the greater of the sides about the right angle; and when he has added unity to this he gets the hypotenuse But the Platonic method originates from even numbers. For when he has taken a given even number he places it as one of the sides about the right angle, and when he has divided this into half, and squared the half, by adding unity to this square he gets the hypotenuse, but by subtracting unity from the square he forms the remaining side about the right angle.

The reader will find that if two successive gnomons have a sum which is an even square, that is, if $(2n-1) + (2n+1) = m^2$, m even, then, since $(n-1)^2 + (2n-1) + (2n+1) = (n+1)^2$, it follows that the numbers $(n-1)$, m , and $(n+1)$ provide Plato's solution.

These are only a few of the mathematical discoveries of Pythagoras and his school. A complete list cannot be given here; it is enough to say that the first two books of Euclid's *Elements* (on triangles, rectangles, and areas), part of the third (on circles), part of the fourth (constructions of polygons), the seventh (theory of proportions), part of the sixth (applications of proportions), the bulk of the thirteenth (on the regular solids), and the discovery of the irrationals are now known to be due to Pythagoras and his school.¹⁰ These form the bulk of their contribution to the subject matter of mathematics.

But of at least equal importance to these must be counted certain methodological contributions they made to mathematics. One of these is the method of definition and logical proof essentially as we use it today. This finds its greatest expression in the many and varied uses made of

¹⁰G. J. Allman, *Greek Geometry from Thales to Euclid*. Dublin, Longmans, 1889.

the theory of proportions as a mathematical tool. This theory, begun by Pythagoras and brought to its highest development by his successor, Archytas of Tarentum,¹¹ became the chief tool and most striking characteristic of all later Greek mathematics. It was skillfully used by Euclid, Aristarchus, Archimedes, Apollonius, Ptolemy, and many others; it survived through all later periods and reappeared as late as the days of Galileo (*Two New Sciences*, 1638) and Newton (*Principia*, 1687).

Latent in this theory lies Pythagoras' deep insight of the oneness of magnitude. Lengths, numbers, all magnitudes in fact can, as magnitudes, be represented by the same symbols. This was a profound abstraction and a broad generalization. In the theory of proportions lines represent numbers, and numbers lines. It is then a theory applicable both to geometry and arithmetic; it effects a fusion of the two. "In this respect," says Allman, "Pythagoras is comparable to Descartes to whom is due the combination of Algebra and Geometry."

We have seen, in the practice of *schematographiein*, the early attempts to represent numbers by points. In the theory of proportions this develops into the conception of the line as a series of juxtaposed points. It was easy to pass then to the conception of the plane as a series of juxtaposed lines, and the solid as a series of juxtaposed planes. We are told by Diogenes Laertius¹² that

Pythagoras taught that the principle of all things is the monad, or unit; arising from this monad is the infinite dyad . . . from the monad and the infinite dyad arise numbers; from the numbers points; from points lines, from lines planes, from these solid figures and from these sensible bodies . . .

One may see in this the attempt to supply to the study of geometry, and physical bodies, the necessary abstract background for their study by means of numbers. The Cantor-Dedekind Axiom, which postulates a one-to-one cor-

¹¹P. Cajori, *A History of Mathematics*. New York, The Macmillan Company, 1938, p. 20.

¹²Diogenes Laertius, *op. cit.*: *De Vit. Pyth.*

respondence between the points of a line and the real numbers, is indispensable in present-day geometry. The corresponding Pythagorean Axiom seems to have assumed (not explicitly of course) a one-to-one correspondence between the points of a line and the rational numbers. And they seem to have looked upon the physical particle as analogous to the geometric point. This would make the last sentence in the quotation from Diogenes Laertius (above) comprehensible. And since, as we have seen, points and numbers were identified, it would also lend meaning to Aristotle's statement¹³ that

The Pythagoreans seem to have looked upon number as the principle and, so to speak, the matter of which beings consist,

and to Philolaus' assertion¹⁴ that

Number is perfect and omnipotent, and the principle and guide of divine and human life.

For, these utterances of later followers and commentators express but crudely and sensuously the tenets of the school and must therefore be taken with a certain degree of freedom of interpretation. The qualifying phrase, "so to speak," in the quotation from Aristotle should be noticed.

It is instructive now to survey briefly the theories of the cosmogonists after Pythagoras; they are the group of philosophers now known as "The Atomists."

Anaxagoras of Clazomenae (500-428) held that a chaotic mass existed from the beginning and contained within it, in infinitesimally small fragments, the seeds of things.

Leucippus (circa 480 B.C.), pupil and friend of Zeno of Elea believed in an infinity of atoms (*atoma* = indivisibles) as the ultimate constituents of things.

Democritus (circa 460 B.C.), pupil and friend of Philolaus and Leucippus, elaborated the theory of the latter and applied it to geometry.

¹³A. Seth, *Encyclopaedia Britannica*, Werner Edition, 1900. Vol. XX, p. 138.

¹⁴Philolaus of Thebes (496-396), the Pythagorean who gave the first written exposition of the Pythagorean doctrine.

Epicurus (341-270) adopted the views of Democritus and systematized them into a broader philosophy.

Lucretius (94-55) gave the best written exposition of the Epicurean philosophy.

The difference between these views and the earlier cosmogonies is noticeable. From chaotic, boundless masses of amorphous substance the emphasis passes to considerations of the structure of matter from an infinity of indivisible particles, arranged according to law, and with due regard to form, size, internal properties, and numerical relations. The atomic theory of the *De Rerum Natura* of Lucretius¹⁵ is a far cry indeed from the vague conceptions of the older cosmogonists. But, whence came it?

We have seen that the Pythagoreans had conceived of sensible bodies as consisting of particles that corresponded in a one-to-one way with the geometric points. From this it would follow that, since points are the indivisibles of space in the Pythagorean Doctrine, particles would be the indivisibles of matter. Atomism seems to have been forecast by Pythagoreanism. Further, it has been pointed out that Leucippus and Democritus, the founders of the Atomism, had had contacts with and received instruction from men acquainted with the Pythagorean Doctrine. For Zeno is well-known as the propounder of the famous paradoxes against the Pythagorean Doctrine, and Philolaus was, after Archytas, the chief exponent of Pythagoreanism. It displays, of course, the fundamental error involved in the Pythagorean Axiom as to the correspondence of material points, geometric points, and the rational numbers. Consider, for example, the following dilemma to which Democritus was led by his adherence to the indivisibles. In the letter to Eratosthenes prefixed to *The Method of Archimedes*¹⁶ we find the following statement:

This is a reason why, in the case of the theorem the proof of which Eudoxus was the first to discover, namely, that the cone is a third part of the cylinder,

¹⁵Lucretius, *On the Nature of Things*, tr. by H. A. J. Munro. London, 1932.

¹⁶T. L. Heath, *The Method of Archimedes*. Cambridge, University Press, 1912.

and the pyramid of the prism, having the same base and equal height, we should give no small share of the credit to Democritus who was the first to make the assertion with regard to the said figure though he did not prove it.

And Plutarch¹⁷ presents Democritus as saying:

If a cone were cut by planes parallel to its base, what must we think of the surfaces of the sections, that they are equal or unequal? For, if they are unequal, they will show the cone to be irregular, as having many indentations like steps, and unevennesses; and if they are equal, the sections will be equal and the cone will appear to have the property of a cylinder, namely, to be composed of equal and not unequal circles, which is very absurd.

It appears then that Democritus was led to assert that the volume of a cone is one third that of a cylinder having the same base and equal height, but was puzzled by the dilemma he mentions and could not prove the theorem.

The root of the difficulty lay, of course, in the conception of matter as composed of indivisible and juxtaposed particles. This difficulty does not arise today (the irregularities and unevennesses are smoothed out) because the Cantor-Dedekind Axiom has replaced the Pythagorean Axiom. Eudoxus proved the theorem, and others similar to it, by ignoring the question of the structure of matter, that is, whether it is discrete or continuous, and using instead the well-known Eudoxian Axiom (erroneously attributed by some to Archimedes) which asserts that:¹⁸

Of unequal lines, unequal surfaces, and unequal solids, the greater exceeds the less by such a magnitude as, when added to itself, can be made to exceed any assigned magnitude among those which are comparable with it and with one another.

This lemma marked the beginning of a new trend in Greek mathematics. With its help Eudoxus created a new

¹⁷Plutarch, *De Communiibus Notitiis*, Vol. IV, ed. by Didot, Paris, p. 1321.

¹⁸T. L. Heath, *The Works of Archimedes: On the Sphere and Cylinder*. Cambridge University Press, 1897.

number system (essentially as given in the fifth book of Euclid) and a new method (the method of exhaustions) for handling problems of the types that balked Democritus. The "indivisibles" were banished from mathematics.



"Jacobi states that at various times he had tried to persuade a young man to begin research in mathematics, but this young man always excused himself on the ground that he did not yet know enough. In answer to this statement Jacobi asked this man the following question: Suppose your family would wish you to marry, would you then also reply that you did not see how you could marry now, as you had not yet become acquainted with *all* the young ladies?"

—G. A. MILLER.



CURIOSUM

$$\begin{aligned}(A + \sqrt{B}) + (A - \sqrt{B}) + (A + i\sqrt{B}) + (A - i\sqrt{B}) &= 4A \\ (A + \sqrt{B})^2 + (A - \sqrt{B})^2 + (A + i\sqrt{B})^2 + (A - i\sqrt{B})^2 &= 4A^2 \\ (A + \sqrt{B})^3 + (A - \sqrt{B})^3 + (A + i\sqrt{B})^3 + (A - i\sqrt{B})^3 &= 4A^3\end{aligned}$$

—DR. ALFRED MOESSNER.¹

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