

CHAPTER B2

EARLY MATHEMATICS AND ASTRONOMY

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THERE is no generally agreed starting point for the history of Greek mathematics and astronomy. Those scholars who prefer dealing with the fully preserved works begin with Euclid's *Elements* and Autolycus' of Pitane *On the Moving Sphere*, written ca 300 BCE, when Greek *mathemata*—geometry, arithmetic, astronomy, and harmonics—were already fully formed. An awareness that scientific methods and theories known from the works of Euclid and Autolycus are not exactly their own methods and theories, but very often originate from the 4th, the 5th, and even the 6th centuries, leads other scholars to search for the earliest written text in *mathemata*. Such a text is represented by a long fragment from the writing of Hippocrates of Chios (ca 440/30 BCE) on the squaring of lunes (moon-shaped areas between circular arcs). This takes us almost 150 years back, to a period when Greek mathematicians and astronomers systematically started to reveal their theories in writing and arrange previous discoveries. Indeed, Hippocrates was the author of the first *Elements* (Euclid's *Elements* were the fourth such work), where geometrical theorems were systematically expounded in a deductive though not yet entirely axiomatic way. The first systematic work in astronomy was written most probably by Hippocrates' compatriot Oenopides of Chios (ca 450 BCE). Archytas of Tarentum, a generation younger than Hippocrates, was the author of the first writings specifically on arithmetic and harmonics known to us.

1. EUDEMUS: THE MILESIAINS OF THE 6TH CENTURY BCE

Hippocrates' fragment came to us as a quotation from the *History of Geometry* by Eudemus of Rhodes (ca 330 BCE), a student of Aristotle and the author of the first

histories of science. As well as the *History of Geometry*, he wrote the *History of Astronomy*, beginning these sciences with Thales of Miletus (fl. ca 585 BCE)—the famous Sage, whom Aristotle regarded as the founder of natural philosophy. Thales wrote nothing; the other famous mathematician of the 6th century BCE, Pythagoras of Samos, also left nothing in writing. One has to concede that to write the history of pre-Euclidean mathematics on the basis of contemporaneous texts is impossible: there are no such texts for the time from Thales to Hippocrates (and almost none for that from Hippocrates to Euclid). What is known about the earliest period of Greek *mathemata* amounts very often to the fragmentary evidence provided by Eudemos; in some cases it can be augmented by the independent *testimonia* but never by a preserved though fragmentary text. Nevertheless, if one does not want to overlook the century and a half preceding Oenopides and Hippocrates, a period in which geometry and astronomy came into being and took shape, the best thing to do is to follow Eudemos' reports, while subjecting them to critical scrutiny (Zhmod 2006; cf. Netz 2004).

Most modern histories of Greek science bear some important features inherent in Eudemos' histories. One of them consists in regarding the ancient Orient as a source of Greek mathematics and astronomy. Geometry, says Eudemos, was discovered by the Egyptians as a result of the practical needs of land surveying, and arithmetic, in turn, was discovered by the Phoenicians, who were employed in trade. Thales, first having traveled to Egypt, brought geometry to Greece; he discovered much himself and instructed his successors in the principles of the other things (fr. 133 W.). The Egyptian origin of geometry is already attested in Herodotus (2.109), and after him in Aristotle (*Metaphysics* A 1.981b23), meaning that Eudemos simply reflected the widespread *egyptophilia* of the Greeks, especially in their approach to the past. The prestige of Egyptian geometry was so great that the gifted mathematician Democritus boasted that nobody excelled him in the construction of lines with proofs, even the Egyptian "rope stretchers," that is, land surveyors (DK 68 B 299). After more than a century's investigation of Egyptian mathematics, however, there is no basis to assume the presence in it of anything resembling theory or proof. It is more probable that in the Archaic period the Greeks borrowed from Egypt practical knowledge needed for land surveying, building, and the like, the more so as early Greek architecture and sculpture bear obvious traces of Egyptian influence. All available evidence on Egyptian borrowings relates to practical mathematics, moreover to arithmetic rather than geometry. Thus, late scholia to Plato's *Charmides* (163e) refer to Egyptian methods of multiplication and division and also to operations with fractions (Heath 1921, 14, 41, 52).

As the most conventional histories of science still do, Eudemos focused in his works on specific discoveries in *mathemata* and their authors, the "first discoverers." His list of Thales' discoveries runs as follows. Thales: (1) was the first to prove that the diameter divides the circle into two equal parts (Euclid 1.def.17); (2) was the first to learn and state that the angles at the base of any isosceles triangle are equal (1.5), calling them, in the archaic manner, similar, not equal; (3) was the first to discover that if two straight lines intersect, the vertical angles are equal (1.15); and (4) knew the theorem about the equality of the triangles that have one side and two angles equal (1.26), which he must have used

to determine the distances of ships from the shore. Obviously, Thales' theorems of angles and triangles cannot have originated in Egyptian geometry, since the Egyptians neither engaged in comparing the size of angles nor the similarity of triangles. In Egyptian and Babylonian mathematics there was no notion of the angle as a measurable magnitude.

As Kurt von Fritz (1971, 568 n. 79) observed, "All theorems ascribed to Thales are either directly related to the problems of symmetry and can be 'demonstrated' by the method of superposition, or such that the first step of the demonstration is evidently based on considerations of symmetry while the second, which brings the argument to conclusion, is simply an addition or subtraction." Indeed, Thales' propositions can be reduced to the symmetries of the so-called Thalesian basic figure (Becker 1966, 37), that is, a rectangle with the diagonals inscribed in a circle, the center of which is on the intersection of the diagonals (see figure B2.1).

Thales appealed in his demonstrations to the visualizability of the geometrical drawing but certainly went beyond this. Aristotle (*Analytica Priora* 41b13–22) refers to an archaic-looking proof of a theorem that the angles at the base of any isosceles triangle are equal (Euclid, 1.5), which might well go back to Thales (Heath 1926, 1:252–253; Becker 1966, 38–39). It is based on the equality of mixed angles, in particular angles in a semi-circle and angles of a segment of a circle, which could be proved by using only the superposition method. The proof in Aristotle can be re-established in figure B2.2.

ABC is an isosceles triangle with its vertex in the center of the circle. Prove that its base angles are equal. $\angle 1$ is equal to $\angle 2$, since they are angles of a semicircle; $\angle 3$ is equal to $\angle 4$, since they are angles of a segment of a circle. Taking equal angles from equal angles, we obtain that angles CAB and ACB are equal. Thus, the proof demonstrates the normal procedure of deductive reasoning.

The idea of proof is vital for the history of Greek mathematics, for this is what both distinguishes it from the earlier mathematical cultures, like Egypt and Babylon, and makes it akin to modern mathematics. (Recent history of mathematical proof, Chemla 2012, sheds new light on the methods of proving the correctness of algorithms and computations in the East Asian cultures but does not change the traditional view on the Greek origin of deductive proof.) The systematic application of deductive proof was

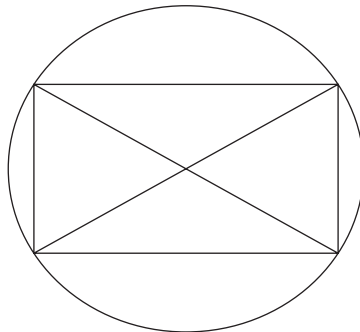


FIGURE B2.1 Thalesian basic figure.

Drawing by W. Sinelnikow based on O. Becker, *Das mathematische Denken in der Antike*. Göttingen, 1966.

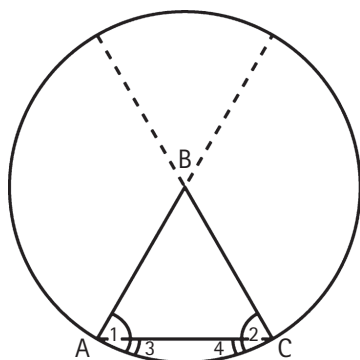


FIGURE B2.2 Aristotle's proof of equality of base angles in isosceles triangle.

Drawing by W. Sinelnikow based on O. Becker, *Das mathematische Denken in der Antike*. Göttingen, 1966.

the most important factor in the formation of theoretical mathematics on an axiomatic basis; this led to the formulation of theorems valid for any numbers, and consequently ousted the empirical, computational methods from mathematical science. Further, it stimulated the search for the axiomatic bases of mathematical theory, since deductive constructions in order to be true and noncontradictory must of necessity rest on initial propositions accepted without proof. Some scholars believe that deductive proof appeared at the very beginning of Greek geometry; others insist that mathematics developed empirically until the early 5th century BCE, whereas deductive proof was borrowed from the Eleatic philosophy or was gradually developed in geometry itself. The problem with the extra-mathematical origin of the deductive proof is that in philosophy it does not possess the logical cogency and irrefutability that it does in mathematics (cf. section 2). Yet the intra-mathematical origin of the deductive method is also not without problems, insofar as this method is not something inherent in dealing with numbers and figures: for thousands of years mathematics developed without it in the ancient Orient, including India and China. Could mathematics of the practical and computational kind, as it existed in archaic Greece, give rise *of itself* to a striving for strict proof? Hardly: Thales in geometry and Pythagoras in arithmetic began by proving things of no practical use that were also too simple to be demonstrations of technical virtuosity. (Høyrup [1994] regards the demonstration of technical virtuosity as one of the chief stimuli in the development by Babylonian scribes of increasingly complex types of calculation.) If mathematics did not of itself give rise to deductive proof, or adopt it from outside, then, most probably, it came into being in mathematics under the influence of external impulses.

As distinct from Babylonian and Egyptian scribes versed in computation, Thales was not a professional: he was a wealthy and politically influential aristocrat. Why did he decide to prove that the angles at the base of an isosceles triangle are equal? And why did he achieve public recognition in this pursuit? Two centuries after Thales' birth an Athenian audience knew him as a famous geometer (Aristophanes, *Birds*, 1009; *Clouds*, 180), which would be impossible if their attitudes to fame and geometry did not

partially overlap. The problem is more general than Thales' geometry, it relates to how Greek science was born and what distinguishes it from similar pursuits in other ancient cultures. A comparison of Greek and Chinese intellectual traditions, offered by G. E. R. Lloyd, emphasizes a conspicuous feature of Greek science: its highly competitive character, which reflects, in turn, an agonistic character of Greek society and culture revealed by Jacob Burckhardt (1898–1902). “The competitiveness of Greek intellectual life” was the decisive factor in the formation of Greek science and, in particular, axiomatic-deductive mathematics (Lloyd 2004, 133, 140, 144). This spirit of pure competition arose in Greek agonistics and then spread to areas of intellectual creativity, multiplying tenfold the force of those striving for truth (Zaicev 1994). A second important factor was that, in the Greece of the 8th to 5th centuries, for the first time in human history, all aspects of productive cultural activity, including those lacking a *direct utilitarian purpose*, gained public approval. The social climate of the time encouraged any and all creative achievements, independent of the extent of their practical value, thus establishing the most powerful stimuli for new investigations. Once set on the path of free research, unconstrained by narrow practicality and corporative ethos, the mathematicians quickly realized that to apply strict, logical proof makes it possible in this pursuit to achieve irrefutable and hence universally recognized results (Zaicev 1994, 167).

“Thales seems by some accounts to have been the first to study astronomy, the first to predict eclipses of the sun, so (says) Eudemus in his *History of Astronomy*” (fr. 144). The prediction of the solar eclipse was the most famous “discovery” made by Thales, and it was reflected in many early sources, among them in his younger contemporary Xenophanes (DK 21 B 19). Successful prediction captured the imagination of the Greeks and made Thales the “father of astronomy,” but what is meant by “prediction,” and how can it be explained? Thales could not have had a theory offering a *correct* explanation of solar eclipses—such a theory only appeared in the mid-5th century BCE. Since Greek tradition before Thales does not know of any predictions of eclipses, the very idea could only have been of Babylonian origin. In the early 6th century BCE, Babylonian astronomy was the only one capable of making predictions that concerned all potential lunar and solar eclipses for a given year, without trying to explain them. Until the mid-20th century, the predominant opinion was that Thales' prediction could have relied on the so-called Saros, a period of 223 synodic months (≈ 18 years), used by the Babylonians to predict lunar and solar eclipses. Later it became known that the Babylonians were unable to *reliably* predict solar eclipses for a given point either in the 6th century or later (Neugebauer 1957, 142–143). It is quite probable, however, that Thales having known about one of the Babylonian schemes, boldly used it to fix the date of the next solar eclipse and thus by lucky coincidence “predicted” the eclipse of May 25, 585 BCE, almost full in Miletus.

Thales' prediction, no matter how famous it was, left no traces in Greek astronomy, which later on was concerned with explanations of the celestial phenomena, including eclipses, not their predictions. Another discovery of Thales, “that the sun's period with respect to the solstices is not always the same” (Eud. fr. 145 W.), had a more durable effect. It seems that Thales tried to estimate the solstices' dates and, hence, the length of the solar

year more accurately than known before. Such an activity became a part of the “calendaric astronomy” that tried to find the best ratio between the solar year and the lunar month for the luni-solar calendar. The results of these investigations, however, were never applied to the civic calendar, remaining a matter for astronomers and philosophers. In the 6th to early 5th centuries among them were the astronomers Cleostratus of Tenedos, Harpalus and Matricetas, and the philosophers Xenophanes and Heraclitus. Cleostratus, in particular, suggested the first intercalation period for a luni-solar calendar, *octaëteris* (eight years), which presumed the year to be $365\frac{1}{4}$ days long; it was further improved by Harpalus and others engaged in astronomical observations (DK 6 A 1, B 4).

In contrast to Babylonian astronomy that was in principle ageometric, the most important stream of Greek astronomy was the creation of geometrical models representing and explaining the apparent motion of the heavenly bodies. It started with Anaximander of Miletus (fl. ca 570 BCE), the first Greek thinker who revealed his theories in writing; he also created the first geographical map of the earth. The system of Anaximander was a peculiar combination of bold speculations, geometrical and spatial imagination, and astronomical observations (he used the gnomon to determine the solstices and equinoxes and set up a sundial in Sparta: DK 12 B 1). The earth in this system has the shape of a column’s drum, and its depth is a third of its width. The earth is freely suspended, supported by nothing, and is afloat in the center of the cosmos. It is this counterintuitive idea, unprecedented in the preceding astronomy, that became a cornerstone of the specifically Greek conception of the universe (Couprie 2011, 99). Anaximander imagined further that the earth is enclosed by three wheels, which consist of thick air (and thus are invisible) and are full of fire: the sun, the moon, and the stars being the holes in the wheels. The sun is the same size as the earth (another revolutionary insight!), and the wheel of sun is the highest of all, followed by the moon and stars; distances from the earth to the stars, the moon, and the sun are equal to 9, 18, and 27 radii of the earth. Solar and lunar eclipses occur when the openings in the rim of the wheel are stopped, which means the moon shines with its own light.

Though Anaximander’s model of the cosmos was not yet purely geometrical, but also physical, with time this physical component receded into the background, whereas geometry became the basis of what Aristotle called “mathematical astronomy.” The other conspicuous features of this system, instrumental in shaping Greek astronomy, are that it is devoid of any divine presence and influence, which are typical, for example, of Babylonian and Chinese astronomy, and that it rests on the assumption of a concealed order of the world that can be revealed both geometrically and numerically. Pythagoras and his school shared this assumption. In a new epistemological situation in the 5th century, the idea of the invisible things and/or regularities in nature was expressed in the pregnant dictum of Anaxagoras of Clazomenae (ca 500–ca 428 BCE): “appearances are a sight of the unseen” (DK 59 B 21a). Developing this line of thought, Eudoxus of Cnidus (ca 390–ca 337 BCE) put forward the principle of “saving (preserving) the appearances” that was to underlie the whole subsequent history of Greek astronomy: to explain the apparently irregular movement of the sun, moon, and planets along the ecliptic by attributing uniform circular movement to them.

The circular motion of the sun and moon around the earth, most probably by analogy with the *visible* circular motion of the stars around the North Pole, was postulated already by Anaximander's system, though his wheels related rather to the diurnal motion of the two luminaries than to their motion along the ecliptic. Unlike Anaximander, his student Anaximenes of Miletus (fl. ca 550 BCE) introduced no new geometrical concepts; he only "moved" the stars beyond the moon, sun, and planets, to the outer place, where they have a fixed position on the celestial vault (DK 13 A 14). As a separate group, distinct from the fixed stars, the planets ("wandering stars") appeared for the first time in Anaximenes, but he did not say anything specific about them. On the whole, the Greeks learned about planets rather late and slowly. In the 6th century, they had no fixed names for them, with the exception of Venus, which was called the Evening and Morning Star, depending on the time of its appearance. That it is the same planet was first attested, according to Theophrastus, in the learned poem of Parmenides (ca 475 BCE). The word "planet" and the fixed number of the planets appear in the late 5th century, and still later, in the mid-4th century, their names, borrowed from Babylon: the stars of Hermes, Aphrodite, Ares, Zeus, and Kronos. The sun and moon were also regarded as planets, for they, too, have independent movement along the ecliptic.

2. THE CONTRIBUTIONS OF THE PYTHAGOREANS

In the late 6th century the Ionian tradition of geometry and astronomy was transferred to Magna Graecia by Pythagoras, who circa 530 BCE left his native Samos because of Polycrates' tyranny and moved to Croton. Pythagoras taught metempsychosis, and many of his ethical rules were supported by belief in his god-like nature. The dual nature of this figure was attested by Aristotle: "Pythagoras, the son of Mnesarchus, first dedicated himself to the study of *mathemata*, especially numbers, but later could not refrain from the wonder-working of Pherecydes" (Arist. fr. 191). This combination of the rational and the religious is not unique among the pre-Socratics: the natural philosopher Empedocles pretended to be a wonder-worker and was a proponent of metempsychosis. However peculiar Pythagoras' personality was, in *mathemata* he continued the work of Thales and Anaximander, and none of the Pythagoreans known to us by name are linked with anything remotely supernatural or miraculous. The Pythagorean school existed until the mid- 4th century BCE, having in almost every generation significant mathematicians and astronomers: Hippiasus of Metapontum (fl. ca 500/490 BCE), Theodorus of Cyrene (active ca 440-ca 400 BCE), Philolaus of Croton (active ca 440-ca 400 BCE), Archytas (active ca 410-360 BCE), Ecphantus of Syracuse (first part of the 4th century BCE). The names of the other Pythagorean mathematical scientists remain unknown.

Pythagoras' contributions to astronomy are hard to discern, partly because the late antique tradition ascribes too much to him: discovering the sphericity of the earth, the obliquity of the ecliptic, the planets' motion along the ecliptic, dividing the celestial and terrestrial spheres into zones, and so on. Early sources are much more reticent. Even if they do connect Pythagoras with astronomy, as with Aristotle's *Protrepticus* (fr. 18, 20 Düring), they do not refer to any specific discoveries. Eudemos, in particular, mentions his followers, rather than Pythagoras: "Anaximander was the first to find an account of the sizes and distances (of the planets), as Eudemos says, adding that the Pythagoreans were the first who found the order of their position" (fr. 146 W). Although Eudemos' fragment does not indicate the number and order of the heavenly bodies, he clearly had in mind their "correct" arrangement, which was accepted in the astronomy of his time: moon—sun—Venus—Mercury—Mars—Jupiter—Saturn—celestial sphere. It was established that, relative to the stars, Mercury and Venus moved the fastest (their sidereal period was equated to that of the sun), Mars more slowly, Jupiter more slowly still, and Saturn extremely slowly. These observations, together with data on the relative brightness of some of the planets (Venus being brighter than Mercury), formed the basis of their order.

Which Pythagoreans did Eudemos mean? In Philolaus' system, five planets were located between the moon and the sun on one side and the stars on the other. Philolaus, however, radically transformed the order of the planets by introducing the Central Fire (Hestia), which he situated in the center of the universe, and around which he made the earth, the invisible counter-earth and all other celestial bodies revolve (DK 47 A 16–17). We should look, then, at the earlier stage of Pythagorean astronomy. Alcmaeon of Croton (ca 500/490 BCE), the Pythagorean natural philosopher, thought, according to the late doxographer Aëtius, that the planets move from west to east in a direction opposite to the movement of the fixed stars (DK 24 A 4). If we believe this evidence, Alcmaeon was aware that the planets, sun and moon, apart from their diurnal movement, also have an annual movement along the ecliptic from west to east, which is to say that they rise each day further to the east in the zodiacal constellations. Though Alcmaeon was not an astronomer (his other astronomical views look rather naive), he might have gained this knowledge from the other Pythagoreans. The evidence of Aëtius implies that the motion of the planets along the ecliptic is circular, as we see later in Oenopides, Hippocrates, and Philolaus. Aristotle says that Alcmaeon taught that the soul was immortal because, like all divine celestial bodies—the sun, moon, planets, and the whole heaven—it is in constant motion (DK 24 A 12). This kind of motion also had to be circular. Transferring the circular motion from Anaximander's model to the motion of the sun, moon, and planets along the ecliptic, the Pythagoreans must have proceeded both from observations and from considerations of symmetry as they attempted to regularize the motion of all the celestial bodies following a single principle. Since a circle was at that time the only possible method of geometrical presentation of planetary motion (only a circular motion is continuous, says Aristotle, *Physics* 264b9–28), the planets' numerous deviations from circular orbits were simply ignored.

The revolution of the celestial bodies around the earth is attested in the other early Pythagorean theory (prior to Philolaus), the famous “harmony of the spheres” that was borrowed by Plato in the *Republic* (616b–617d) and acknowledged but refuted by Aristotle:

The theory that music is produced by their (*sc.* planets and stars) movement, because the sounds they make are harmonious, although ingeniously and brilliantly formulated by its authors, does not contain the truth. It seems to some thinkers that bodies so great must inevitably produce a sound by their movement: even bodies on earth do so, although they are neither so great in bulk nor moving at so high a speed, and as for the sun and moon, and the stars, it is incredible that they should fail to produce a noise of surpassing loudness. Taking this as their hypothesis, and also that the speeds of the stars, judged by their distances, are in the ratio of the musical consonances, they affirm that the sound of the stars as they revolve is concordant. (*De caelo* 290b, tr. W. Guthrie)

Like Anaximander’s model, this theory has a physical component, lacking in Philolaus. There is no sound without movement, said Archytas’ Pythagorean predecessors in harmonics (DK 47 B 1); consequently, there can be no movement without sound, even though we do not hear the celestial harmony. The speed of rotation of the celestial bodies in this system is directly proportional to their distances from the earth, which, according to the late commentator Alexander (Aristotle, fr. 13 Ross), make up the arithmetical progression 1, 2, 3, 4, . . . ($n + 1$). Thus, the ratios of the distances correspond to the ratios of the basic concords: the octave (2:1), the fifth (3:2), the fourth (4:3), and so on.

The doctrine of heavenly harmony does not lend itself to detailed reconstruction, especially in its musical part. What is important for us is to state that it is based on Pythagoras’ discovery of a link between music and number, which led to the inclusion of harmonics in the *mathemata*. Late antique tradition about how Pythagoras discovered the ratios of concords, such as Nicomachus’ story about an experiment with the hammers (*Harmonics*, 6), is unreliable, but the discovery itself is attested by Plato’s student Xenocrates (ca 395–313 BCE), who left behind numerous works on mathematical sciences: “Pythagoras discovered also that the intervals in music do not come into being apart from number, for they are an interrelation of quantity with quantity” (fr. 87 Isnardi Parente). That Pythagoras found the numerical expressions of the octave, the fifth, and the fourth is indirectly confirmed by the evidence of the famous musicologist Aristoxenus (active ca 340—ca 300 BCE), a student of the last Pythagoreans and then of Aristotle. He says that Hippasus fashioned four bronze discs of the same diameter, with thickness in the ratios 2:1, 3:2 and 4:3; when struck they produced harmonic concordance (Aristox. fr. 90 W.). Hippasus of Metapontum was a student of Pythagoras, and his experiment was conducted to confirm what Pythagoras had already discovered, most likely by observations and experiments with a stringed instrument. (Though the Greeks knew no regular practice of experimentation, sporadic experiments were performed.) The ratios of the basic concords are closely bound up with arithmetic $b = (a + c) / 2$

and harmonic $b = 2ac/(a+c)$ means,, which, according to information that goes back to Eudemus, were known to Pythagoras and Hippasus (Zh mud 2006, 173– 175). Thus, the fifth (3:2) is the arithmetic mean between the terms of the octave (2:1), and the fourth (4:3) is the harmonic mean between them; taken together, they form a “musical” proportion (12:9 = 8:6). The only preserved fragment of Eudemus’ *History of Arithmetic* deals with the Pythagorean ratios of the three concords (fr. 142 W.). During the 5th century BCE, arithmetic and harmonics as related sciences remained a monopoly of the Pythagorean school: whereas the Ionias Oenopides and Hippocrates studied only geometry and astronomy, Hippasus (DK 18 A 12– 15), Theodorus (DK 43 A 4), Philolaus (DK 44 A 26, B 5–6), and Archytas (DK 47 A 16– 19, B 1– 2) were engaged also in two other sciences of the quadrivium.

“Pythagoras more than anybody else seems to have valued the science (or theory) of numbers and to have advanced it, separating it from the merchants’ business and likening all things to numbers,” says Aristoxenus in his *On Arithmetic* (fr. 23). This is close to what Aristotle noted about Pythagoras’ study of numbers (fr. 191), but is more specifically related to the origin of arithmetic as a theoretical science, distinct from the art of calculation. The arithmetic known to us from the three books of Euclid’s *Elements* (books 7–9) is the theory of *arithmoi*, which is to say whole numbers greater than one, and their properties. “A unit is a beginning of a number” (and thus not a number), and “a number is a multitude consisting of units”—these definitions from the same fragment of Aristoxenus are likely to have opened an early Pythagorean arithmetical treatise. The next definitions introduce two basic kinds of number: even numbers are divisible into equal parts, odd number are divisible into unequal parts and have a middle. (Philolaus, following the arithmetic of his time, also mentions the division of numbers into even, odd, and even-odd: DK 44 B 5). The latter assertion indicates that the early Pythagoreans represented numbers not by line segments, as Archytas (DK 47 A 19) and later Euclid did, but by *psephoi*, counting stones. (Hence there is no “middle” in Euclid’s definition of the odd number: 7.def.7). If you add or subtract a *psephos* to or from an even number, you get an odd number (DK 24 B 4), says a character from the comedy of the Sicilian writer Epicharmus (ca 480 BCE), alluding most probably to Pythagorean arithmetic. (Practical arithmetic does not need and, thus, does not know odd and even numbers. It is Epicharmus’ fragment, where “even” and “odd” in their mathematical meaning first occur in Greek literature, whereas the practical and computational mathematics of Mesopotamia and Egypt did not have special terms for odd and even numbers.) The simplest example of this arithmetic is a summation of odd and even numbers, represented by pebbles; such arithmetical series produce the so-called figurate numbers (figure B2.3). The added number, called the *gnomon*, preserves the form of that to which it is added.

square number $1 + 3 + 5 + \dots + (2n - 1) = n^2$;

oblong number $2 + 4 + 6 + \dots + 2n = n(n + 1)$.

The presence of definitions in the early Pythagorean arithmetic implies that it contained some deductively proved propositions. The high standard of Archytas’

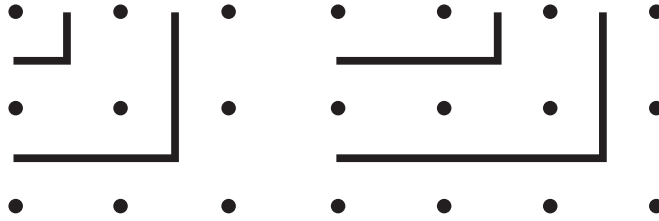


FIGURE B2.3 Gnomon for square and oblong numbers.

Drawing by W. Sinelnikow based on T.L. Heath, *A History of Greek Mathematics*. Oxford, 1921.

arithmetical proofs (DK 47 A 19) shows that by the late 5th century BCE arithmetic was established as a demonstrative science. In Archytas' opinion, it even surpassed geometry in clarity and exactness, accomplishing proofs where geometry failed (DK 47 B 4). An early specimen of the axiomatic-deductive method in arithmetic is the theory of even and odd numbers, preserved at the very end of the last arithmetical book of the *Elements* (Becker 1966, 44–49). This theory, consisting of propositions 9.21–34, based only on definitions of even and odd numbers (7.def.6–11), is of an elementary character and lacks any intrinsic connection with the material of other arithmetical books. Here are its first five propositions in abridged form:

21. The sum of even numbers is even.
22. The sum of an even number of odd numbers is even.
23. The sum of an odd number of odd numbers is odd.
24. An even number minus an even number is even.
25. An even number minus an odd number is odd.

Becker showed that both the propositions and their proofs retained by Euclid are easily illustrated through the use of *psephoi*. Meanwhile, four of these propositions (9.30–31, 33–34) are proved by *reductio ad absurdum*, one of the powerful tools of Greek mathematics, which allows the establishment of a proposition by showing that its contradictory involves impossible consequences, for example that the same number is both even and odd. We see again how very simple mathematical problems lead to nontrivial results. It is hard to establish whether indirect proof originated in arithmetic or earlier in geometry (proposition I, 26, attributed by Eudemus to Thales, is proved indirectly). Judging by the preponderance of *reductio ad absurdum* in the theory of even and odd, one can reasonably infer that deduction, which is to say a formal proof technique, was shaped by the early Pythagorean *psephoi*-arithmetic, which appealed not to the (then nonexistent) lettered diagram (cf. Netz 1999) but to pebbles arranged in such a way as to give an ocular demonstration.

Further nontrivial results of Pythagorean arithmetic appeared rather quickly. First was the discovery of the irrationality of $\sqrt{2}$, the classic example of which is the incommensurability of the diagonal of a square with its side. The probable context of the discovery was the search for the ratios of the sides in the right-angled triangle

that corresponded to Pythagoras' theorem (see the end of this section). It was found then that the side and diagonal of a square cannot be expressed as a ratio of two numbers. This theorem was one of Aristotle's favorite mathematical examples: referring to it more than 15 times, he twice alludes to the fact that its indirect proof relies on the theory of odd and even numbers (*Analytica priora* 1.23, 41a24–27, and 1.44, 50a37). It might have been that Archytas had this very proof in mind, saying that arithmetic accomplishes proofs where geometry fails (DK 47 A 4). “The analysis of certain classes of problems in geometry, e.g. the construction of irrational lines, can only be completed by means of arithmetical principles” (Knorr 1975, 311). Plato ascribes to Theodorus a proof of irrationality of the magnitudes between $\sqrt{3}$ to $\sqrt{17}$ (DK 43 A 4), which means that the proof of the irrationality of $\sqrt{2}$ was found earlier. Ancient tradition, probably going back to Eudemus, attributes the discovery of irrationality to the Pythagoreans; the name of Hippasus is mentioned or implied in the legendary stories surrounding it (von Fritz 1974, 545–575; Zhmud 2012, 274–275). The ancient (though not the original) arithmetical proof of the proposition that the diagonal and side of a square are incommensurable in length is preserved at the end of book 10 of the *Elements* (app. 27); it makes use of the Pythagoras' theorem, the theory of even and odd numbers, the method of *reductio ad absurdum*, and the least numbers in a given ratio. This all points to its Pythagorean origin. As we know from Archytas (DK 47 A 17) and Eudemus (fr. 142 W.), the early Pythagoreans took the ratios of the concords in lowest terms (2:1, 3:2, 4:3), which they called “first numbers,” or *pythmenes* (base numbers). Archytas' proof that a superparticular ratio $(n + 1) : n$, and so the concordant intervals represented by it, for example the fifth and the fourth, cannot be divided into equal parts (DK 47 A 19) and have no mean proportional (or geometric mean), also contains *reductio ad absurdum* and the least numbers in the same ratio.

The problems evoked by the discovery of irrationality provided the impulse for the research of Theodorus and his student Theaetetus (discussed later), the author of the general theory of irrational magnitudes (book 10 of Euclid's *Elements*) and led to the development of Eudoxus' theory of proportions, which was applicable to commensurable and incommensurable magnitudes (book 5). In the modern literature, the impact of Hippasus' discovery has often been overrated. Thus, it was widely believed that it was originally motivated by Pythagoras' dogma “all is number” and then had dealt a “fatal blow” to this dogma by demonstrating the existence of incommensurable magnitudes in geometry, which in turn led to the “foundation crisis” in Greek mathematics. All three assumptions are not borne out by the reliable sources. The “foundation crisis” of the 5th century BCE is a retrospective projection of what happened in mathematics at the turn of the 20th century (Knorr 2001). The motto “all is number” is unattested in ancient Pythagoreanism; it was first ascribed to the unnamed Pythagoreans by Aristotle, who mistakenly regarded them as the predecessors of the Platonic number doctrine (Zhmud 2012, 433–452). As for general interaction between mathematics and philosophy, Greek mathematics appeared to have been independent of contemporary philosophy, whereas the latter was frequently influenced by mathematical ideas (Knorr 1981). One of the earliest examples of such an influence was systematic deductive reasoning, including

indirect proofs, employed by Parmenides (DK 28 B 8) and his student Zeno (DK 29 A 15, B 1–2) in attempting to prove their bold theses that contradicted all experience, for example, that there is no movement or plurality. “Parmenides’ reasoning is the extension of the Pythagorean proof . . . Not only in mathematics, where the Pythagoreans had already developed *reductio ad absurdum* proofs in their exploration of quantities, but throughout nature—in philosophy, physics, everywhere—it became possible to show simply by examining their logical consequences that some generalizations cannot be true” (Brumbaugh 1981, 54–55). The Eleatics put deductive proof in a much wider context, but, in contrast to the Pythagorean mathematicians, they succeeded neither in proving any of their basic theses nor even in formulating their indirect proofs in a rigorous form. Their *reductio ad absurdum* proofs are formally incomplete.

Two pieces of early Greek geometry—the theorem of Pythagoras and the theory of the application of areas that Eudemus deemed “ancient” and attributed to the “Pythagorean muse” (fr. 137 W.)—were from the 1930s considered derived from Babylonian mathematics. One of its rediscoverers, O. Neugebauer (1957, 40), believed to find on the tablet Plimpton 322 (18th century BCE) “the fundamental formula for the construction of triples of Pythagorean numbers,” that is, positive integers (a, b, c) for which $a^2 + b^2 = c^2$. The much-repeated idea that the Babylonians knew the Pythagorean theorem became a cliché, and Pythagoras was regarded as the transmitter of Babylonian knowledge (van der Waerden 1961, 92–93). Over recent decades, the leading students of Babylonian mathematics have changed this trend. First, the Babylonians knew not the theorem, but the rule for determining the values numerically, which they did not prove or even formulate explicitly (Høyrup 1998). Secondly, a detailed examination of the tablet has shown that it has nothing to do with number-theoretical problems in general, nor with Pythagorean numbers in particular, but contains a school problem using a list of reciprocal pairs (Robson 2001). As for the Greeks, Proclus (5th century AD) in his commentary on the first book of Euclid (*In Euclid*, 428.7–21) ascribes to Pythagoras the method of defining Pythagorean triples, starting from the odd number, which is based on figurate numbers (Heath 1926, 1:356). The first author to claim that Pythagoras proved the theorem named after him was a certain Apollodorus the Arithmetician (Diogenes Laertius 8.12), who may be identical with the Democritean Apollodorus of Cyzicus (second half of the 4th century BCE); he was followed by virtually all the Greek writers who wrote about it. This evidence, though not irrefutable, is confirmed by the fact that the proof of irrationality of $\sqrt{2}$, associated with Hippasus, is based on Pythagoras’ theorem. Hippocrates already knew the generalized Pythagorean theorem for acute- and obtuse-angled triangles (2.12–13); it comes from book 2 of the *Elements*, which belongs to the Pythagoreans.

The application of areas with excess or defect, “one of the most powerful methods on which Greek geometry relied” (Heath 1926, 1:343), relates to the transformation of areas into equivalent areas of different shape. The propositions of this theory, comprising theorems 1.44–45, the entire book 2 of the *Elements*, and theorems 6.27–29, can be reformulated into algebraic identities and quadratic equations. Thus, the application of areas with defect means the construction on a given line a of the rectangle ax ,

so that by subtracting from it the square x^2 , the given square b^2 is obtained ($ax - x^2 = b^2$). Proposition 2.3 can be presented as the identity $(a + b)a = ab + a^2$ and 2.4 as $(a + b)^2 = a^2 + 2ab + b^2$. Since the late 19th century, these propositions have come to be known as geometric algebra and seen as a geometric reformulation of algebraic problems. When Neugebauer found in Babylonian mathematics corresponding identities and equations, he concluded that the algebra reformulated by the Greeks was Babylonian. That he regarded his interpretation as a working hypothesis, unconfirmed by documentary evidence (Neugebauer 1957, 147), did not prevent it from soon becoming the dominant theory. This theory came under attack from S. Unguru (1975), who claimed that the application of areas was not a reformulation of Babylonian algebra, but arose on Greek soil in the course of solving purely geometric problems. After a lengthy discussion, most historians of Greek mathematics accepted his view. “We have no good reason to believe,” noted Taisbak (2003, 306), “that the Greeks were thinking of quadratic equations in any form when working with the different types of application of areas.” Revealingly, there is no evidence of the practice of mathematics analogous to geometric algebra in Mesopotamia in the 6th–5th centuries: all extant texts relate to the Old Babylonian period. “Old Babylonian mathematics cannot have influenced early Greek developments: it was a part of a scribal culture that all but died out nearly a millennium before the earliest Greek literate culture, 1200 miles away” (Robson 2005, 13). Real or assumed isomorphism between two mathematical theories, formulas, or methods often gives rise to common-origin hypotheses, but only the theories placed in a specific historical setting with identifiable ways of transmission survive the tests.

3. THE MILESIAANS, PYTHAGOREANS, AND ATHENIANS: PRODUCTIVE INTERACTIONS

In the mid-5th century BCE, studies of geometry and astronomy were revived in Ionia by two natives of Chios, Oenopides and Hippocrates. Before them we know only Anaxagoras, who taught that the moon received its light from the sun and offered correct explanations for both lunar and solar eclipses (DK 59 B 8, A 76–77). On the whole, however, his astronomy was physical rather than mathematical. Oenopides, mentioned by Eudemus in both the *History of Astronomy* and the *History of Geometry*, attempted to establish closer connections between these two *mathemata*. According to the late evidence, he “was the first among the Greeks who wrote down the methods of (mathematical) astronomy” (Boll 1894, 53–55), which is essentially confirmed by the early sources. Eudemus attributes to Oenopides two elementary geometrical constructions that later entered Euclid’s book 1: to draw a perpendicular to a given straight line from a point outside it (1.12); at a point on a given straight line, to construct a rectilinear angle equal to a given rectilinear angle (1.23). Oenopides considered problem 1.12 useful for astronomy. Proclus says the same about proposition 4.16 (this is the last proposition of book 4, which

the scholia to Euclid 273.3–13, probably on the authority of Eudemos, ascribe to the Pythagoreans), on a regular pentadecagon inscribed in the circle: its side is equal to the angle between the celestial equator and the zodiacal circle, that is, 24° (*In Euclid*, 283.7–10, 269.8–18). Theon of Smyrna's excerpt from Eudemos clarifies the way in which it may be related to Oenopides' astronomy: he "was the first to discover the obliquity of the zodiacal circle" (Eud. fr. 145 W.). This can mean either that Oenopides discovered that the annual path of the sun is inclined to the celestial equator or that he first measured the angle of the obliquity of the ecliptic (Bodnar 2006, 4–6). The latter variant seems more plausible in view of Aëtius' evidence about Alcmaeon and the zodiacal motion of the planets (DK 24 A 4). The ecliptical motion of the sun, moon, and planets against the background of the celestial sphere is attested both in Philolaus (DK 44 A 21) and in Hippocrates (DK 42 A 5), which is hard to explain if Oenopides shortly before them discovered that the annual path of the sun is oblique. Von Fritz (1937, 2258–2259) argued convincingly that the end of Theon's excerpt from Eudemos was originally related to Oenopides: "And others discovered in addition to this that the fixed stars move round the immobile axis that passes through the poles, whereas the planets move round the axis perpendicular to the zodiac and that the axis of the fixed stars and that of the planets are separated from one another by the side of a (regular) pentadecagon" (fr. 145 W.). Though Oenopides' astronomical system defies reconstruction, we can surmise that his work, firstly, incorporated geometrical notions of the structure of the universe developed by the Greeks from Anaximander to Anaxagoras, removing them from the cosmological context to which they belonged in the works of natural philosophers, and secondly, expounded them in conformity with the requirements of the deductive geometry of the mid-5th century.

There is reciprocal influence between the Pythagoreans and the Chians: Oenopides held the same theory of the Milky Way, as being the former course of the sun, as did the Pythagoreans (DK 41 A 10); Philolaus borrowed from him the 59-year luni-solar cycle (Eudemos fr. 145 W.; DK 44 A 22). Hippocrates shared the view of some Pythagoreans that a comet is one of the planets, visible at long intervals and rising low over the horizon (DK 42 A 5). Hippocrates' theory as set out by Aristotle is more complex than the Pythagorean, demonstrating advanced concepts of the geometry of the universe: the celestial sphere is divided into zones by a celestial equator and two tropic circles crossed by the oblique circle of the zodiac; the planets move in circular orbits along the ecliptic; the horizon divides these circular orbits into unequal segments; and the earth, to all appearances, is spherical (Wilson 2008). The sphericity of the earth, safely attested for Philolaus, is related in the Greek tradition alternatively to Pythagoras and Parmenides (Diogenes Laertius 8.48). From what we know about their astronomy, neither appears to be a suitable candidate for this discovery; it is safer to attribute it to the Pythagorean tradition of the 5th century, though certainty is impossible. The discovery of the earth's spherical shape led to the formation of the main astronomical model of antiquity, which consisted of two concentric spheres, the celestial and the terrestrial, divided into zones. In the generation of Philolaus and Hippocrates, this two-sphere model of the cosmos was still in the making (Philolaus' spherical earth was not the center of the cosmos), in a more developed form we find it in Plato's *Republic* and later in his *Timaeus*.

An Athenian astronomer Meton (ca 430 BCE) belonged probably to the same generation as Philolaus and Hippocrates. Meton and his colleague Euctemon made systematic observations in different regions of Greece; created the first astronomical calendars, the so-called *paraegmata*; suggested a new 19-year calendar cycle; and determined the inequality of the four astronomical seasons (according to their calculations, the seasons are 90, 90, 92, and 93 days, starting with the summer solstice). Meton and Euctemon were the earliest of the Greek astronomers whose dated observations are cited by Ptolemy.

By the time of Hippocrates several geometrical problems, such as squaring the circle and doubling the cube, became famous, attracting the attention of audiences far beyond a narrow circle of specialists. Aristophanes ridicules Meton for promising to square the circle (*Birds* 1004–1009); Plutarch describes Anaxagoras as busy in prison squaring the circle (DK 59 A 38); Aristotle and Eudemus record unsuccessful attempts by the Sophists Antiphon of Athens and Bryson of Heraclea to solve the same problem. The agonistic spirit that surrounded the problem of doubling the cube led Greek geometers to continually search for new solutions to the problem long after it had been solved, first by Archytas, and then by his student Eudoxus and by Eudoxus' student Menaechmus (Knorr 1986). Eratosthenes' dialogue *Platonicus*, relying on the Academic legend of Plato as the architect of *mathēmata*, ascribes to the latter an instrumental role in doubling the cube, but this tradition is unreliable (Zhud 2006: 84–86; Kouremenos 2011).

The way to Archytas' solution was paved by Hippocrates, who was the first to reduce the problem of doubling the cube to finding two mean proportionals x and y in continuous proportion between two lines, a (side of the cube) and $2a$, that is, if $a : x = x : y = y : 2a$, then $x^3 = 2a^3$, $x = a\sqrt[3]{2}$. It was suggested long ago that Hippocrates came to this idea by analogy with the planimetric problem, solved by the Pythagoreans, of doubling the square, which is equivalent to the problem of finding the mean proportional x between two lines, a and $2a$, $x^2 = 2a^2$, $x = a\sqrt{2}$ (Heath 1921, 201). In turn, Archytas found a brilliant solution to the problem formulated by Hippocrates, which was reported by Eudemus (fr. 141 W.). Archytas constructed a series of similar right triangles AMI, AIK, AKD and then showed that their sides are in continued proportion, so that $AM : AI = AI : AK = AK : AD$, where AM was equal to the side of the original cube and $AD = 2AM$ (figure B2.4). To prove this, he employed a remarkable stereometric construction, which for the first time introduced movement into geometry (note that the moving point D appears twice). Point K, the key point for the construction of similar triangles, was determined as the intersection of three surfaces of revolution: the right cone, the torus, and the half-cylinder (Knorr 1986, 50–52; Huffman 2005, 342–346).

The problem of squaring the circle arose in the first part of the 5th century, after the Pythagoreans had found how to square a rectangle (Euclid 2.14). Being equivalent to constructing a line segment whose length is $\sqrt{\pi}$ times the radius of the circle, the problem is unsolvable using compass-and-straightedge techniques, or even algebraic equations, as was established in the late 19th century. (It does not seem, however, that in the pre-Euclidean period Greek mathematicians consciously restricted the means allowable to

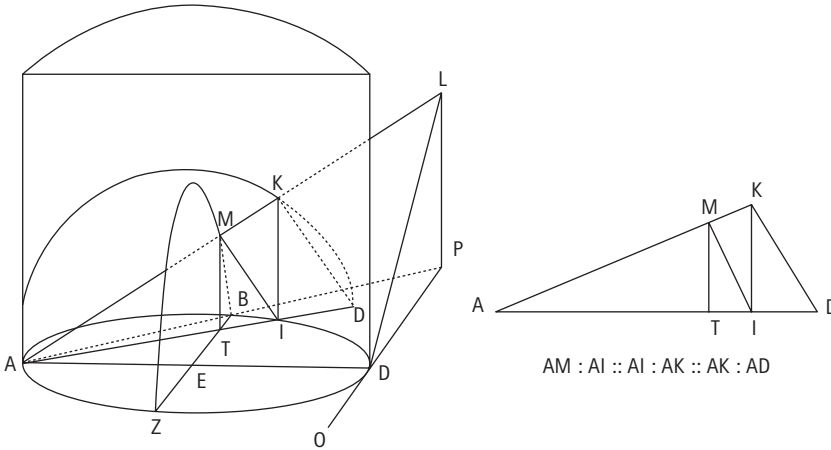


FIGURE B2.4 Archytas' stereometric construction, and the similar triangles whose sides are in continued proportion.

Drawing by W. Sinelnikow based on C. Huffman, *Archytas of Tarentum*. Cambridge, 2005.

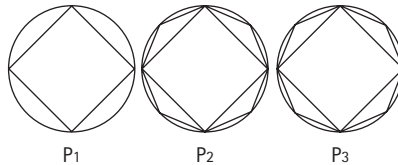


FIGURE B2.5 Squaring the circle by inscribed polygons.

Drawing by W. Sinelnikow based on W. Knorr, *The ancient tradition of geometric problems*. Boston: Birkhäuser, 1986.

their constructions to compass and straightedge; see Knorr 1986, 40–41). It is unknown, whether Anaxagoras came up with a solution of the problem. The solutions of Antiphon and Bryson, says Aristotle, were “eristic” (*Sophistic Refutations* sec. 11, 171b16–18, 172a2–7; *Physics* 1.2, 185a14–17), which is to say unscientific, since they proceeded not from geometrical principles. Eudemus passes over Bryson in silence but specifies Antiphon’s procedure (fr. 140 W.): the latter started by inscribing a regular polygon in a circle; then, by doubling the number of its sides repeatedly, he obtained an inscribed polygon whose sides coincided with the circumference (see figure B2.5).

Thus, concludes Eudemus, Antiphon did not admit the basic principles of geometry, in particular, that geometrical magnitudes are infinitely divisible. This criticism, which reflected a position of the mathematicians, applies to Bryson as well. We know from late sources that, squaring the circle, he added circumscribed polygons to the inscribed ones and claimed that by multiplying their sides he could obtain an intermediate polygon equal to the circle. T. L. Heath, the author of the still-standard history of Greek mathematics, believed that Antiphon’s and Bryson’s procedures anticipated the famous method of exhaustion, discovered by Eudoxus (Heath 1921, 222), but this idea did not find much support.

Whereas Aristotle believed that Hippocrates pretended to have solved the problem of squaring the circle, but had committed a logical mistake (*Sophistic Refutations* sec. 11, 171b12–16; *Physics* 1.2, 185a14–17), Eudemus disagreed with his teacher:

The quadratures of lunes, which were considered to belong to an uncommon class of propositions on account of the close relation (of lunes) to the circle, were first investigated by Hippocrates, and his exposition was thought to be in correct form. (Fr. 140 W., tr. T. Heath)

The opinion of specialists, to which Eudemus refers, implies that though originally squaring the lunes was most probably intended to lead to squaring the circle, Hippocrates did not claim to have solved the last problem, so Aristotle's interpretation was incorrect (Lloyd 1987). But Hippocrates succeeded in squaring three out of the five lunes that are possible in plane geometry (two others were found in the 18th century), namely, with the outer circumference equal to a semicircle (see figure B2.6a), greater than a semicircle (see figure B2.6b), and smaller than a semicircle, the most elaborate case. He also squared a figure that consisted of a lune and a circle.

In his problem-solving attempts, Hippocrates did not proceed axiomatically. Thus, he started his quadrature of the lunes not from definitions or unproved principles, but by proving two theorems: first, similar segments of circles have the same ratio as the squares on their bases (12.2), which he then reduced to the second theorem, that the squares on the diameters have the same ratio as the circles. But Hippocrates' *Elements*,

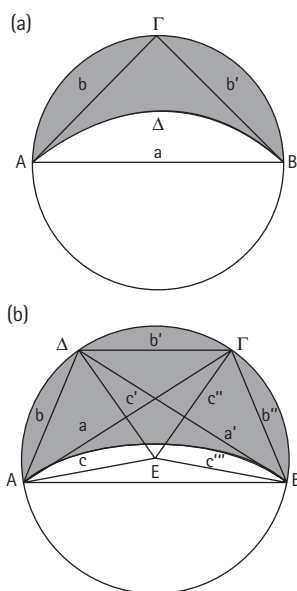


FIGURE B2.6 Two of the lunes of Hippocrates.

Drawing by Paul A. Whyman.

whose goal was to organize interrelated mathematical propositions in their logical sequence, must have built them on the explicitly formulated definitions and axioms. A papyrus text, most probably dating back to a Platonist of the 4th century BCE, asserts that in Plato's time "the theory of proportions (μετρολογία) and research on definitions reached their peak, as Eudoxus and his students completely revised the old theory of Hippocrates" (Zhmd 2006, 87–89). Whereas Eudoxus created a new theory of proportions applicable to commensurable and incommensurable magnitudes, Hippocrates, working around 75 years before him, applied Pythagorean theory of proportions to a new field—solid geometry—and worked out the axiomatic basis for his *Elements*. It is generally believed that his compendium contained much of books 1–4 and 6 of the Euclidean *Elements* and that most propositions of book 3 belonged to Hippocrates himself.

The authorship of arithmetical books 7–9 is a tricky question. Eudemus' *History of Geometry* did not touch on this, and from his *History of Arithmetic* only one fragment is preserved. Many scholars believed that an arithmetical compendium analogous to Hippocrates' *Elements* in geometry existed before Archytas, but what did it comprise? Archytas obviously relied on the basis of book 7, which may have then belonged to Theodorus, a contemporary of Hippocrates, though this is no more than conjecture; Knorr (1979, 244) attributed book 7 to Theaetetus. Book 8 is usually related to Archytas; the end of book 9 to the early Pythagoreans. What is certain is that Theaetetus' theory of irrational magnitudes is based on these arithmetical books. The first significant geometer who was born in Athens, Theaetetus was, as mentioned, a student of Theodorus and belonged, according to Eudemus (fr. 133 W.), to the generation of Archytas and Plato. This places his birth around 435/425 BCE, but since Plato depicts him in the *Theaetetus*, whose dramatic date is 399 BCE, as an adolescent, his birth date is usually given as 415/413 BCE. It is known, however, that Plato sometimes changed the age of his personages depending on the dramatic situation in the dialogue, so that it may be safer to stick to the dating provided by Eudemus, who was particular about chronology. Theaetetus' main achievements in mathematics, the theory of irrational lines (book 10), and the theory of the regular solids (book 13) show him as a successor of the Pythagoreans. He proved that there is an infinite number of straight lines, which are incommensurable in length or both in length and in square; and introduced three particular kinds of such lines, medial, binomial, and apotome, associating them with three known means, the geometric, the arithmetic, and the harmonic (Eudemus, fr. 141–I W.).

According to a scholion on book 13 (*Scholia in Euclid*, 654.3), which very likely derives from Eudemus, the Pythagoreans constructed three regular solids, pyramid, cube, and dodecahedron, to which Theaetetus added the octahedron and icosahedron (see figure B2.7).

Though the construction of the octahedron, a combination of two pyramids on a square base, is much simpler than that of the dodecahedron, they are ascribed respectively to Theaetetus and Hippasus, who lived a century before him (von Fritz 1945;

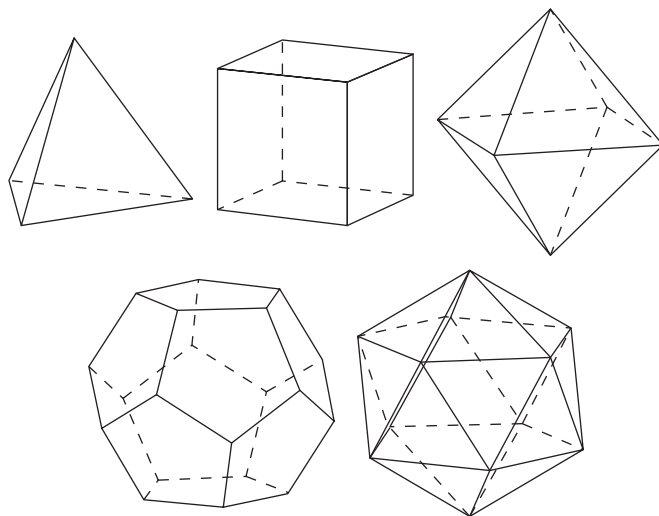


FIGURE B2.7 The five regular solids.

Drawing by W. Sinelnikow.

Zhmud 2012, 275). To divide the theories of regular polyhedra into two stages—the investigation of individual polyhedra and their general theory—helps clarify why the more complex polyhedron was constructed before the simpler one (Waterhouse 1973). Hippiasus studied not the theory of regular solids as such, but the dodecahedron itself. On the other hand, Theaetetus, having posed the question of which regular solids could be constructed, easily discovered the octahedron. He wrote a systematic treatise, in which he set forth methods for constructing the five regular solids and for inscribing them in a sphere; he also described the relations between the edges of the regular solids and the diameter of the sphere. The last book of Euclidean *Elements* is based on this treatise.

The five regular solids became famous outside of mathematics, after Plato used them in his *Timaeus* to impart a geometric structure to the four physical elements traditional for Greek philosophy. Creating the world, the Platonic demiurge makes fire from pyramids, air from octahedra, water from icosahedra, earth from cubes, and he uses the dodecahedron to decorate the whole universe. In the Hellenistic era, the five regular solids were called “Platonic bodies,” and Proclus even claimed that Euclid belonged to the Platonic school “and this is why he thought the goal of the *Elements* as a whole to be the construction of the so-called Platonic figures” (*In Euclid*, 68.20–23). Proclus’ teleological view of the history of mathematics is typically Neoplatonic but is akin to Plato’s own “appropriative” approach to mathematics. Since the geometricians and astronomers do not know how to make use of their discoveries, asserts Plato in his early *Euthydemus* (290c), those of them who are not utter blockheads must hand these discoveries over to the dialecticians, who will find proper use for them—just as hunters and fishermen give what they catch to cooks!

4. MATHEMATICS: THE BEGINNING OF SELF-REFLECTION

The successes of *mathemata* during the 5th century BCE made their methods of attaining true knowledge highly attractive, especially against the background of the endless debates of the natural philosophers about basic principles, as well as doubts and denials that the truth is attainable, expressed by the Sophists. Philolaus became one of the first pre-Socratics to introduce *mathemata* into a philosophical work and to make its results and methods an object of discussion and analysis. (Parmenides and Zeno took from mathematics the technique of deductive proof, but in them we find no reflection on the subject of their borrowed methods.) The Pythagoreans, involved in *mathemata*, were the first to look at mathematics from an epistemological point of view. In his treatise *On Nature*, Philolaus declares: “And indeed all the things that are known have number. For without it we can neither understand nor know anything” (DK 44 B 4). This fragment of Philolaus often has been taken as evidence of the Pythagorean doctrine that “everything is number.” But “to have number” does not mean “to consist of numbers,” it means “to be countable,” since “number or that which has number is countable” (Nussbaum 1979). Thus, number in Philolaus makes a knowable thing countable, for example, by representing the octave as a ratio 2:1, the fifth as 3:2, and the fourth as 4:3 (fr. 6a Huffman). “Fr. 6a suggests that the whole-number ratios which govern musical scales served as the model of the kind of mathematical account which should be supplied for all phenomena” (Huffman 2012).

Archytas started his *Harmonics* by praising his Pythagorean predecessors, “those concerned with the mathematical sciences” (*hoi peri ta mathemata*), for their, one might say, great epistemological successes. They showed true insight, and it is not strange that they have a correct understanding of particular things as they really are:

For since they exercised good discrimination about the nature of the universe (*peri tas tōn holōn phusios*), they were likely also to get a good view of the way things really are taken part by part. They have handed down to us a clear understanding of the speed of the heavenly bodies and their risings and settings, of geometry, of numbers, and not least of music. For these sciences seem to be sisters. (DK 47 B 1, tr. A. D. Barker, slightly modified)

In Archytas, the word *mathemata* acquires its terminological character and designates a particular group of four mathematical sciences, all of which he regards as related. (This quadrivium soon appears in Plato’s *Republic*.) It is these sciences, claims Archytas, that give us real understanding of the world and everything in it. This claim is, firstly, very un-Platonic, for Archytas obviously did not need any intermediary to interpret results of scientific research; and secondly, it is quite unusual, for in antiquity claims to true understanding of reality were usually raised by philosophers rather than by mathematicians.

There were other exceptions, too (Feke 2014). Of all the *mathemata*, Archytas clearly preferred arithmetic, declaring in particular that it surpassed all other arts, including geometry, in clearness, evidence, and obviousness, which makes it, in comparison, more demonstrative (DK 47 B 4). Apart from the fact that arithmetic is more exact than geometry, it is also socially useful. In the introduction to *On Mathematical Sciences*, Archytas relates important social changes, such as an increase of concord and an advance toward greater equality, the to the discovery of calculation. Moreover, calculation proves capable of improving people's moral qualities, keeping them from greed and injustice or, at any rate, exposing these vices (DK 47 B 3). Archytas' conviction that mathematical knowledge makes a man and, accordingly, the society in which he lives better, was shared by his friend Plato.

In the same fragment, Archytas again tackles epistemological issues, presenting different ways of acquiring knowledge:

To know what was heretofore unknown, one has either to learn it from another, or to discover oneself. What one has learnt, he has learnt from another and with another's assistance, what one has found, he has found himself and by his own means. Discovery without research is difficult and rare, by research easy and practicable, but without knowing (how) to research it is impossible to research. (DK 47 B 3)

To make a discovery, conscious research is needed because one cannot conduct research without knowing how to do it. What, then, must the researcher know? To all appearances, he must know *what* and *how* to seek—in other words, he must know the object and method of his research. It follows, then, that the method, which is to say the art of correct research, becomes for Archytas a prerequisite for success in science, although he did not altogether rule out the chance, small as it might appear, of an accidental discovery. Thus, by the beginning of the 4th century BCE, Greek exact sciences not only succeeded in creating new powerful methods and in solving many difficult problems but began also to look narrowly at themselves: What did they achieve, and why did this become possible? We can only regret that the results of this self-analysis are so seldom available to us.

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