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Author(s): Solomon Gandz

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The Origin of Angle-Geometry

I. INTRODUCTORY. GEOMETRY OF LINES AND GEOMETRY OF ANGLES.

1. — *Ambiguity of the Terms.*

There is a great deal of ambiguity and lack of clearness in the use of the terms line-geometry, angle-geometry, (1) and trigonometry. It is, however, a good old habit, among mathematicians in particular, to commence their treatises by giving clear definitions of the terms to be employed. In following this example the writer desires to set forth at the outset the sense and meaning in which he is going to use these words in the subsequent discussion.

2. — *Geometry of Lines and Geometry of Angles.*

Under geometry of lines this phase of geometry will be understood where *lines only*, not angles, are used for purposes of mensuration. In the beginning, of course, the lines are the sole *objects of mensuration*. Later on, with the progress of civilization, people proceeded to measure surface and solids; but even then, lines were the sole *means of mensuration*, areas and volumes being measured by their help. The sides, the length and breadth, the base and perpendicular, the diagonals, and the diameter and circumference are considered both as objects and means of mensuration, but no cognizance is taken of angles. It is very likely that there is a tacit addition and a subconscious understanding that the figures are rectangular, but this is never expressly mentioned or clearly stated. When the human mind first realizes the importance of angles, formulating in a distinct and scientific manner definitions, classifications and theorems relating to them,

(1) Some prefer the word *goniometry*. This word was first used by THOMAS FANTEL DE LAGNY (c. 1710); see SMITH, *History of Mathematics*, II, p. 612.

and using them as objects and means of mensuration, then a new stage of mathematics is inaugurated, *the geometry of angles*.

3. — *The Geometry of Angles. Angle-Geometry and Trigonometry.*

In general historians of mathematics use the words trigonometry and angle-geometry promiscuously to indicate any kind of geometry of angles. In this discussion, however, it seems convenient to distinguish between a geometry of angles where the angles are the *object of mensuration* and another kind of geometry where the angles are also used as a *means of mensuration*, to ascertain the length of the sides, or the areas and volumes of figures. Angle geometry and trigonometry will be used as two different terms for these two different conceptions. Under angle-geometry this branch will be understood where angles are primarily the object of mensuration and comparison, while trigonometry will be used to signify this large field of geometry where angles are primarily the means of mensuration. Trigonometry is a well established term, universally used to mean this branch of mathematics which deals with the ratios of the sides as certain functions of an angle (sine, cosine, etc.) (2).

The term «angle-geometry» will be used in this paper to signify all those theorems and propositions, relating to angles and to the theory of angles, which are not contained in the science of trigonometry. For example, it will include the definition of angles and parallels (3); the classification of angles into right and oblique, acute and obtuse; theorems on the equality of right angles, or of oblique angles in the isosceles and equilateral triangles, in parallelograms, etc.; propositions

(2) It was fitly discussed by D. E. SMITH (*History*, II, p. 600) as follows : « If we take trigonometry to mean the analytic science now studied under this name, we might properly place its origin in the 17th century, after the development of a satisfactory algebraic symbolism. If we take it to mean the geometric adjunct to astronomy in which certain functions of an angle are used, we might look for its real origin in the works of HIPPARCHUS (c. 140 B. C.), although there are earlier traces of its use. If we take it to mean literally « Triangle measurement » the origin would naturally be placed much earlier, say in the second or third millennium B. C. Since this third phase is considered under geometry, we may properly confine our work to the development of the idea of the functions of an Angle ».

(3) Parallels being understood as straight lines in a plane, but not forming an angle.

on the sum of the angles in a triangle or polygon, on the angles in the semicircle, or with their vertices anywhere on the circumference or at the centre. In short all the theory of angles which constitutes the subject-matter of most of *Books* I-III, IV, VI, and XI of EUCLID, and which together with the theory of proportions might be declared as the foundation of the scientific geometry of the *Elements*. Trigonometry like the geometry of lines was a practical science (**), an adjunct of astronomy and mensuration (*geodesy*); whereas angle-geometry was an abstract science, one of the foundations of the theory of geometry.

II. THE ORIGIN OF ANGLE-GEOMETRY.

ANGLE-GEOMETRY AMONG EGYPTIANS, BABYLONIANS AND GREEKS.

4. — *Greek tradition. Greek origin.*

The question as to who were the founders of angle-geometry receives a clear answer from Greek tradition. The Greek historians of mathematics EUDEMUS (c. 335 B. C.) and PROCLUS (c. 410-485 C.E.) (4) name THALES as the inventor of the following propositions: (1) The angles at the base of an isosceles triangle are equal; (2) When two lines intersect, the vertical angles are equal; (3) Two triangles are congruent, if they have two angles and a side respectively equal; (4) An angle in a semicircle is a right angle. After THALES the theory of angle mensuration was further developed by the Pythagoreans. « The Pythagoreans, before... 450 B. C., had practically completed the subject-matter of Books I, II, IV, V, (and perhaps III) of EUCLID's *Elements* » (5).

5. — *The modern historians.*

Egyptian origin advocated by German school.

In contrast with the clear statement of Greek tradition some of the modern historians of mathematics would like to credit

** The editor does not agree with this. Trigonometry is neither less theoretical nor more practical than any other branch of mathematics; it all depends upon one's understanding of it (G. S.)

(4) Cf. HEATH, *Greek Mathematics* I, p. 130; SMITH, *History*, I, p. 67; CANTOR, *Geschichte der Mathematik*, I, 4th ed., pp. 135 *seqq.*

(5) HEATH, *Greek Mathematics*, I, p. 2.

the ancient Egyptians, or also the Babylonians, with the knowledge of those theorems. The German historians, in particular, show this tendency to antedate many of the Greek inventions and to ascribe them to the Egyptians and Babylonians. C. A. BRETSCHNEIDER, one of the first to represent this theory, says : (6) « Dass gerade diese sogenannte ionische Schule in der Geometrie nur mässiges geleistet, im Allgemeinen sich nur darauf beschränkt hat, das aus Ägypten überkommene zu erhalten und weiter zu geben. Wir können daher *mit Zuversicht behaupten* (7), dass die getrennte Beweisführung für die Winkelsumme im gleichseitigen, gleichschenkligen und ungleichseitigen Dreiecke noch aus der ägyptischen Geometrie stammt. » Then he continues : (8) « Aus dem Rhind'schen Papyrus scheint so viel *mit Sicherheit* hervorzugehen, dass die Lehre von den Winkeln und Parallellinien... der Hauptsache nach von den ägyptischen Geometern entwickelt und durch *bündige Beweise* festgestellt waren. » According to BRETSCHNEIDER (9) all the geometric discoveries of THALES, and also his measurement of the pyramids by the shadow and his ascertaining of the distance of the boats from the sea shore are of Egyptian origin. He even goes so far as to say (10) : « Es scheint sogar, als ob die Lehre von der *Flächenvergleichung und Messung*, die die Ägypter..... bereits in ziemlichem Grade ausgebildet hatten, dem THALES und seinen Schülern, wenn auch nicht geradezu unbekannt, doch wenig geläufig gewesen sei. Denn alle Probleme, die uns innerhalb dieses ersten Jahrhunderts der Griechischen Geometrie genannt werden, beziehen sich ganz ausschliesslich auf *Construction und Messung von Linien*; es ist kein einziges bekannt, was einen Flächeninhalt beträfe. Würde es nun auch vereilig sein, daraus den Schluss zu ziehen, dass THALES in seinen geometrischen Studien bei den Ägyptern garnicht bis zur Messung und Vergleichung des Flächeninhaltes ebener Figuren gelangt sei (eine Annahme, die an sich gar nichts unwahrscheinliches hat), so möchte doch wenigstens so viel daraus folgen, dass er sowohl wie seine Schüler gerade diesem Teile der Planimetrie keine besondere Aufmerksamkeit zugewendet

(6) *Die Geometrie und die Geometer vor Euklides*, 1870, p. 15.

(7) These and the following italics are due to the present writer.

(8) *Ib.*, p. 17.

(9) *Ib.*, pp. 42 and 44.

(10) *Ib.*, pp. 66-67.

haben.» BRETSCHNEIDER thus, with a great deal of «buts and ifs» intimates that THALES, the disciple of the Egyptians, and with him his school, did not proceed as far as to the *mensuration of areas* and their comparison, but, as a kind of dull and backward pupil of the Egyptian school remained behind in the lower field of construction and *mensuration of lines*. And were we to follow RÖTH, so BRETSCHNEIDER points out (11), then nearly all that the Greeks before EUCLID accomplished was Egyptian property. It seems indeed that BRETSCHNEIDER is influenced, to a certain extent, by the philosopher RÖTH who undertook it with great force (12) to prove that all the early Greek achievements in science and philosophy go back to Egyptian sources.

This theory of RÖTH, as it seems, exercised a great and lasting influence upon the German historians of mathematics. HANKEL (13) believes that the propositions attributed to THALES and OENOPIDES (c. 465 B. C.) are of Egyptian origin. «There is no doubt about it», he writes, «that OENOPIDES learned his constructions from the Egyptians».

CANTOR (14) summarizes the divergent opinions as follows: «So kam es, dass dem einen bewiesen schien, die Ägypter hätten von Winkeln nichts gewusst, und THALES sei der erste gewesen, der eine Winkelgeometrie ersann. (15) Dass ein zweiter ein Verdienst des THALES darin fand, dass er eine Liniengeometrie in dem Sinne schuf, dass er das Verhältnis der Linien einer Figur ins Auge fasste, während den Ägyptern nur die praktische Geometrie der Flächenausmessung bekannt gewesen sei, (16) dass

(11) *Ib.*, p. 18. See also the Preface *ib.*

(12) In his *Geschichte unserer abendländischen Philosophie*, 2nd vol., 2nd ed. Mannheim, 1862.

(13) *Zur Geschichte der Mathematik*, 1874, pp. 91-92.

(14) I, 4th ed., p. 139.

(15) CANTOR does not quote the sources nor does he mention the authors by name. The sources of this theory are unknown to the writer.

(16) This is the theory of ALLMAN, G. J.; see his *Greek Geometry from Thales to Euclid*, 1889, pp. 7-8 and note *ib.* It was also adopted by SMITH, *History*, I, p. 68. Under «line-geometry» these authors seem to understand something similar to trigonometry or angle-geometry, at any rate something higher than mere mensuration of areas. It is interesting to note that BRETSCHNEIDER, who was the first to use the phrase «geometry of lines» (*loc. cit.*, pp. 66-67, quoted above) gave it another meaning. According to BRETSCHNEIDER, «geometry of lines» is something lower, more elementary than mensuration and the comparison of areas. ALLMAN was well acquainted with BRETSCHNEIDER's work,

ein dritter nicht Anstand nahm THALES und die älteren Griechen überhaupt fast jeden Erfinderrechtes für verlustig zu erklären und ihr ganzes geometrisches Wissen für Ägypten zuruckzufordern (17); dass ein vierter an die entgegengesetzte Grenze streifend es für gleichgültig hielt, ob THALES überhaupt Ägypten besucht habe oder nicht, weil er Geometrisches in nennenswerter Menge von dort nicht habe mitbringen können » (18). CANTOR, himself, is inclined to adopt a theory corresponding to the third view with certain modifications. He credits the Egyptians with the knowledge of those propositions which Greek tradition ascribes to THALES and the early Pythagoreans. He attributes to the Egyptians the theorems on the equality of the angles at the base of the isosceles triangle and on the bisection of the circle by the diameter. He sees in the *segt* exercises of the Egyptians a « theory of proportions » or a « chapter in trigonometry », and he believes in general that the whole difference between Greek and Egyptian geometry lay only in the method. The Egyptians used the inductive method, while the Greeks introduced the deductive method (19). In the same way CANTOR likes to credit the ancient Babylonians with a considerable amount of geometric knowledge. The Babylonians, in CANTOR's opinion, knew of parallel lines and of the radius, of the division of the circle into six parts by applying the radius as a chord; they knew of the 360 degrees of the circle and of angle degrees (20).

which he highly esteemed (*Greek Geometry*, pp. 1-2 note). He took up the phrase « geometry of lines » apparently in deference to BRETSCHNEIDER, but employed it with quite a different meaning as something more advanced and more abstract than the Egyptian mensuration of areas. ALLMAN, *loc. cit.*, p. 7, points out: « THALES brought with him from that country (Egypt) the knowledge of Geometry and astronomy. He added the capital creation of the geometry of lines, which was essentially abstract in its character. The only geometry known to the Egyptian priests was that of surfaces, together with a sketch of that of solids... THALES, on the other hand, introduced *abstract* geometry, the object of which is to establish precise *relations*, between the different parts of figure, so that some of them could be found by means of others in a manner strictly rigorous.» In this sense of the term « geometry of lines » his theory comes much nearer to the truth. For the sake of clearness and precision, however, the writer thought it more apt to use the terms « geometry of lines » and « angle-Geometry » in a different way.

(17) This probably refers to RÖTH.

(18) Author unknown to this writer.

(19) CANTOR, I, 4th ed., pp. 109, 112-113, 140.

(20) CANTOR, I, 4th ed., pp. 45-47, 50.

In the footsteps of CANTOR follows TROPFKE. The theory of the qualities of parallel lines, the construction of the regular hexagon and various propositions of the theory of the circle and the triangle have their origin in Babylonian mathematics (21).

EUCLID's definition of the perpendicular (22), the division of angles into right, acute and obtuse, and the theorem of the equality of the vertical angles are to be traced to Egyptian origin (23).

Thus we might properly speak of a scientific tradition, represented mainly by the German school of RÖTH, BRETSCHNEIDER, HANKEL, CANTOR and TROPFKE, which is opposed to the Greek tradition and teaches the Oriental origin of Greek geometry.

6. — *Refutation in general.*

The fact, however, is that this tradition is mostly based upon vague conjectures and generalizations. In the mathematical documents of the Egyptians and Babylonians there is no evidence whatever to support it. There is no mention made of a theory of angles or of a mensuration of angles in Egyptian or Babylonian mathematics. There are even no terms for right and oblique angles, and the Egyptian word for the angle, *Knbt*, seems rather to signify a « corner » or « gate » (24) and not an angle, as a measurable quantity, in the modern or Greek sense of the word. No attempt is made to set up an equality between oblique angles, and even the equality of right angles is not expressly mentioned. Nothing is said about the postulate that the sides of the squares and rectangles, or that the altitude and the base form a right angle. The Egyptian formula for the area of the isosceles triangle is $\frac{b a}{2}$, where b is the base and a is one of the equal sides.

The formula for the trapezium is $a \cdot \frac{b_1 + b_2}{2}$ where $b_1 + b_2$ are the two parallel lines, and a is one of the sides. Here they take the side forming an oblique angle instead of the perpendicular forming a right angle. Thus we might properly speak of direct

(21) TROPFKE, IV, second ed., p. 4.

(22) *Book I*, def. 10.

(23) TROPFKE, IV, 2nd ed., pp. 48, 50-51.

(24) See hereafter, § 7, note of A. B. CHACE.

evidence for the statement that the Egyptians did not distinguish between right and oblique angles, or rather that they often neglected the difference between them. This is also the opinion of HERMANN SCHNEIDER, in his new book *Die Kulturleistungen der Menschheit*, p. 64, that the Egyptian did not discover the idea of the height.

Not being familiar with Egyptian language and hieroglyphics, and drawing his knowledge only from modern translations of and treatises on Egyptian mathematics, the writer thought it fit to ask an authority on Egyptian language and mathematics for information. After this article was written some of its findings and results were submitted to Chancellor A. B. CHACE, the latest editor of the Rhind Mathematical Papyrus, who devoted many years to the study of Egyptian language and mathematics, with the request that he would advise the author on their admissibility. Dr. CHACE was kind enough to send the subsequent note for which the writer wishes to express his deep appreciation.

7. — *Note of Chancellor A. B. CHACE.*

«I do not think that they had any conception of an angle as a thing to be measured, and it may be not even of one angle as being larger or smaller than another. In the building of a pyramid each layer of stone was set back a certain amount; that is what «*seked*» means. The word «*ifd*» means four. Applied to structures, or pieces of land, it may mean having four sides or it may mean having four corners. They had the idea of height  _i written with a man holding up his hands), but we can think of a line measuring height as drawn upwards without thinking of it as forming a right angle with anything, and so, even if «*merit*» did mean altitude, or if the triangles of the Rhind papyrus were right triangles (halves of rectangles), it would not mean that they had a conception of angles and angle-measurement.

Of course there may be evidence of the idea of angle in Egyptian writings yet to be discovered, but so far as I know now they did not have it.»

8. — *The Seqt.*

All that we know about the so called *seqt* or *sqd* of the Egyp-

tians (25) is that it is a ratio between two lines of the pyramid, but we have not yet a certain knowledge of the nature of these lines, and of the use that was made of this ratio. So far we know that one of the lines was in vertical direction, forming the height, slant height or edge of the pyramid, and that the other one was in horizontal direction forming the half side or half diagonal of the base. The most probable theory is that advocated by REVILLOUT (26) and BORCHARDT (27) that the *sqd* is the ratio between half the side of the base ($\frac{b}{2}$) and the height of the pyramid (h); or in modern terms the cotangent of the angle of the slope. The pyramids were built in decreasing layers or stories of stones. In order to obtain an even slope and a smooth surface in filling out the corners the inlaying stones must have the same slope or the same ratio between the base and the height. This was in all likelihood the *seqt* (28). But it would be a great mistake to assume that the ancient Egyptians actually measured angles or computed their cotangents. They only had the practical rule and some vague idea, as all the primitive people engaged in building would have, that the slope of a pyramid is determined by the ratio of half the side of the base to the height.

9. — *Instinct and Consciousness, Intuition and Science.*

CANTOR, however, conceives of the *seqt* as a method to measure the abstract angle, and, since we have the choice to take the one or the other half of the side of the base, he further draws the conclusion that the equality of the angles at the base of the isosceles triangle *doubtless must* have been known to the Egyptians (29). In a similar way he credits the Egyptians with the knowledge of the bisection of the circle by the diameter, and the Babylonians with the knowledge of the qualities of parallel lines, basing his belief upon the evidence of certain Egyptian wall-pictures showing a circle divided into many equal parts, and of certain drawings

(25) See the *Rhind Mathematical Papyrus*, problems, No 56-60, ed. PEET, pp. 97-102; ed. CHACE, pp. 37-38, 154; CANTOR, I, 4th ed., pp. 99 *seq*; HEATH, *Greek Mathematics*, I, pp. 126 *seq*.; SMITH, *History*, II, pp. 600 *seq*.; 619, note 3; TROPFKE V, 2nd ed., p. 4; IV, 2nd ed., p. 155.

(26) In *Revue Egyptologique*, II, pp. 308, *seqq*.

(27) *Zeitschrift für ägyptische Sprache*, XXXI, (1893), pp. 9-17.

(28) See CANTOR, I, 4th ed., p. 100; TROPFKE, PEET and CHACE. *loc. cit*.

(29) I, 4th ed., pp. 139-140.

of two parallel lines on a Babylonian inscription. (30) By such a procedure we could perhaps credit Egypt and Babylon with the content of all the thirteen books of EUCLID's *Elements*. Furthermore, we know « that the bees built their hexagonal wax cells according to the laws of maxima and minima, that the birds observe the principle of symmetry in the structure of their nests », and “ that no animal is so foolish as to follow a curved instead of a straight line between two points in its path ” (31). Are we going to credit the bees with the *knowledge* of the differential calculus or the animals with the *definition* of the straight line as the shortest path between two points? Men were thinking and reasoning logically many thousands of years before they had the slightest idea of the science of logic, they were computing correctly before they knew of the common notions of EUCLID, they built their tents with two equal sticks on both sides having an equal inclination (32) before they had the slightest idea of angle-measurement or of the theorem of the equality of the angles at the base of the isosceles triangle, and in the same way it was also that the Egyptians built their great pyramids.

The theory of parallelism is closely connected with angle-geometry. But there is not the slightest trace of the knowledge of such things among the Babylonians (33). CANTOR thinks that the Babylonians knew the radius and recognised the rule that the radius drawn six times as a chord would give the inscribed regular hexagon, and that this was the reason that they took $\pi = 3$. In the writer's opinion, however, the fact that they took $\pi = 3$ proves to the contrary. If they would know that $6r = 3d$ is the perimeter of the inscribed hexagon they certainly would not identify this perimeter with the circumference of the circle and the arc of 60 with its chord against the clear evidence from sense perception. Apart from this, such concepts as the centre and radius of a circle were entirely unknown to the Egyptians, Babylonians and Hebrews ; They had no words for them,

(30) *Ib.* pp. 140, 109, 50, 46.

(31) SMITH, *History*, I, p. 5.

(32) See HANKEL, *Zur Geschichte der Mathematik*, p. 90.

(33) MEISSNER B., *Babylonien und Assyrien*, vol. 2, (1925), pp. 389-394, gives a complete summary of Babylonian geometry. But he knows nothing of angle-geometry.

and even the Greeks had no special term for the radius (34 .

10. — *Historic and Prehistoric Phases in Mathematics.*

It is interesting to note that CANTOR (35), himself, laid down the principle that the real history of mathematics commences, for him, with the first written documents relating to computation and to the comparison of geometric figures, and that it is not his intention to begin, like the historians of mathematics from the former centuries, with the creation of the world. In the course of his investigations, as it seems, he soon realized the importance of the prehistoric phase in mathematics, and found himself compelled to deviate from his principle and to include some of the manifestations of the earliest stages of the arts and crafts; a procedure which, as we have seen, created some confusion. It seems, therefore, more appropriate and consistent to devote a chapter or two in the beginning to the prehistoric stage of mathematics, a method adopted by DAVID EUGENE SMITH in his *History of Mathematics*. Here the mathematical laws as manifested in the cosmos, in the organic and animal life of the earth may be discussed. Here belongs also the prescientific phase of mathematics as it is developed in architecture and decorative art, in religious mysticism and ritualism, and in commerce and the observation of stars.

11. — *The Origin of the Geometry of Lines and Geometry of Angles.*

All we possibly may claim for the Babylonians and for the Egyptians are some practical rules of a primitive trigonometry. There is definite proof that the ancient Babylonians and Egyptians had the *gnomon* or sun-dial and also some primitive astrolabe.

(34) The Hebrew word for the centre *mirkaz* is of Arabic origin. It cannot be older than the ninth century. There is also a Hebrew word *muššaq* for the centre; it was used mostly by IBN EZRA; see the Hebrew Dictionary of BEN JEHUDA, VI, p. 2859. (BEN JEHUDA, *ib.*, did not understand that the word in SAVASORDA's *Hibbūr ha-Meshīḥah* means a « conic figure ». BEN JEHUDA's explanation of the word « *šūrā thisbartīth shawā ha-šela'ōth* » is meaningless). The word *mirkaz* and the idea of the centre are not found in the geometry of AL-KHOWĀRIZMĪ (c. 820), which forms a special chapter (*Bāb al Misāḥah*) of his algebra... For the history of the word *radius* see TROPFKE, IV 2, p. 106 seq. and SMITH, *History*, II, p. 278.

(35) I, 4th ed., p. 16.

Some trigonometric rules might have been known and used for the shadow reckoning and the observation of the stars. On their cuneiform tablets of the time short before HIPPARCHUS (c. 150 B. C.) astronomical data have been discovered which are more exact than those given by the latter (36).

Thus we might be justified in saying that the *geometry of lines*, as a practical science, originated among the Babylonians and Egyptians. They accomplished the mensuration of areas and solids with the help of lines only. In *trigonometry* too, as a practical science, we might perhaps look for its earliest traces, its prehistoric phase, to the Egyptians and Babylonians. While *angle-geometry*, as an abstract science, is of Greek origin. THALES introduced the abstract *geometry of angles*, which in turn gave rise to a more advanced, scientific trigonometry, using the angles in addition to the lines for the purposes of mensuration and astronomy.

Trigonometry as a practical science, was further developed by the Hindus and Arabs in the Middle Ages, (37) while the abstract angle-geometry hardly received any further development through the Oriental races, being confined, up to the modern times, almost exclusively to the achievements of the ancient Greeks.

The Greeks frankly acknowledged the fact that the Egyptians were the founders of arithmetic and geometry in general (38). There is no reason to doubt the truth of Greek tradition when it claims one branch, the angle-geometry, as their invention and as their own field and dominion. Indeed nothing was more congenial to the Greek spirit than the angle-geometry which is the very foundation of the abstract and logical system of Greek mathematics. Apparently it is not a mere coincidence that EUDEMUS (c. 335 B. C.), the first historian of Greek mathematics, attached so much importance to this subject. He wrote a book *On the Angle*, and most of the fragments preserved from his

(36) See SMITH, *History*, I, p. 49; II, p. 351, 600; WEIDNER, E. F., *Handbuch der Babyl. Astronomie, Lieferung I* p. 62, TROPFKE, *V*, 2nd ed., p. 12; L. BORCHARDT *Altägyptische Zeitmessung* pp. 27 seqq. (on the gnomon), 54 (on dioptra).

Cf. however, BORCHARDT, *ib.*, p. 34, denying that the Egyptians had any knowledge of even the simplest trigonometric computations. «da wir auch einfache trigonometrische Rechnungen bei den alten Ägyptern nicht als bekannt voraussetzen dürfen.»

(37) Compare also CARL SCHOY, *Die trigonometrischen Lehren des.... Al-Birūnī*, Hannover, 1927, and the review by TROPFKE in *Islam*, XVII (1928), pp. 299-300.

(38) See the numerous passages quoted by CANTOR I, 4th ed., pp. 86, 102-103.

history of mathematics discuss the origin of theorems of angle geometry (39).

With the change of one word only (40) the writer would like to conclude this chapter with the fitting words of D. E. SMITH (41). « In the history of mathematics, as in the history of civilization in general, it is the setting forth of a great idea that counts... The great contribution of THALES lay in suggesting a *geometry of angles* and in making the subject abstract... Without THALES there would not have been a PYTHAGORAS,... and without PYTHAGORAS there would not have been a PLATO. »

III. THE ANGLE-GEOMETRY AMONG THE HEBREWS AND ARABS.

12. -- *The Mishnath ha-Middoth* (42) and the *Geometry of* AL-KHOWÂRISMÎ (c. 825) (43).

The *Mishnath ha-Middoth* does not show the knowledge of angle-geometry that the Greeks possessed, but apparently it shows more than the Egyptians had. Thus the *Mishnath ha-Middoth* represents, in a certain sense (44) a phase of transition from Egyptian to Greek geometry. All that the *Mishnath ha-Middoth* contains about angles is that there are right (45) and oblique angles,

(39) See HEATH, *Euclid*, I, pp. 34, 38, 181, 294, 296.

(40) « Geometry of angles » instead of « geometry of lines. »

(41) *History*, I, p. 68.

(42) This is the oldest Hebrew geometry (c. 150 C. E.). The writer has already had occasion to refer to this ancient treatise in *The American Mathematical Monthly*, vol. 33, pp. 263 note 5; and vol. 34, p. 81, note 3, and to give his view concerning the time in which it was written. A recent investigation which is to be published soon as an introduction to a new edition of the *Mishnath ha-Middoth* being prepared by the writer for the publications of The Alexander Kohut Memorial Foundation, has confirmed his conviction and will furnish evidence that the *Mishnath ha-Middoth* was written about 150 C. E. by RABBI NEHEMYAH, the famous teacher of the Mishnah. This introduction will soon appear in the *Hebrew Union College Annual*, vol. 6, 1929.

(43) The earliest Arabic geometry, so far as known, is the so-called *Bâb al-Misâḥah*, « the chapter on mensuration », as it appears in AL-KHOWÂRISMÎ's *Algebra*, ed. ROSEN, London, 1831; Arabic text, pp. 50-64; English translation, pp. 70-85. It is almost a verbal translation of the *Mishnath ha-Middoth*. Anything said about the *Mishnath ha-Middoth* will therefore apply to AL-KHOWÂRISMÎ's geometry, unless otherwise stated.

(44) As far as the stage of its knowledge is concerned, but not in regard to the time of its origin.

(45) They are called « upright » angles; see hereafter.

and that the latter are divided into acute and obtuse, also called narrow and broad angles. (46) It also infers that all right angles are equal, though this is not expressly stated. It defines square and rectangle merely as having equal angles, without saying that they are right angles. (47) The third chapter of the *Mishnath ha-Middoth* is devoted to definitions and propositions concerning quadrilaterals. We are justified in assuming that the author tells us all he knows about these figures. The square is defined as a quadrilateral with equal sides and angles, and the rectangle as a quadrilateral with unequal sides and equal angles, but it is not mentioned that these angles are right angles, which we certainly would regard as an important characteristic. There is no doubt that the author of the *Mishnath ha-Middoth* knew the fact, however, for we see it from paragraph IV, 2, where the right triangle is said to be the half of the rectangle and the angle is characterised as an upright one. We see it also from III, 4-5, where the rhombus and the parallelogram are said to have oblique angles, which implies that the former figures (the square and rectangle) have right angles.

On the other hand the *Mishnath ha-Middoth* shows no knowledge of the measurement of the oblique angles. The author did not know that the opposite angles of the rhombus and rhomboid are equal, although he mentions that the sides of the rhombus are equal and that there are in the rhomboid two pairs of equal sides (lengths and breadths). All that the author knows about the angles of the rhombus was that there is one pair of narrow and one pair of broad angles. He does not mention (48) that the two angles at the base of the isosceles triangle, as well as all the angles of the equilateral triangle, are equal, although mention is made of the equality of angles in the square and rectangle. Indeed in the whole book no mention is made of the equality of any two oblique angles. Hence the conclusion is justified that the mensuration of oblique angles, with the possibility of establishing an equality between two such angles, was entirely unknown to the author. That angles are equal, can, in his opinion,

(46) See *Mishnath ha-Middoth*, IV, 2, 4, 9, 10, III, 5, 4.

(47) III, 1-3. AL-KHOWÂRIZMÎ, however, expressly says that they have right angles (p. 55 of the Arabic text).

(48) See the fourth chapter dealing with trilaterals.

be said only of right angles; equal angles in his view, is only a different name for right angles. This is perhaps the reason that he omits to define square and rectangle as having right angles; equal angles, in his judgment, being necessarily the same as right angles (49).

This poor development of the theory of angles is quite natural as a consequence of the fact that the phenomenon and theory of parallel lines was entirely unknown to the author of the *Mishnath ha-Middoth*. There is not even a word relating to parallelism in the whole Hebrew literature of that time. The Hebrew terms *nokhehî* or *maqbil* are new words belonging to the Arabic period, beginning at about 900 C. E. The *Mishnath ha-Middoth* has no word for the parallelogram. This is defined either as a quadrilateral with equal sides and unequal angles (the rhombus), or as a quadrilateral with unequal sides and angles, (the rhomboid), having two pairs of equal sides and having oblique angles. The area of the parallelogram is obtained by dividing it into two triangles and computing the area of each triangle for itself as is the case with the trapezoid. (50) The *Mishnath ha-Middoth* does not show any knowledge of the fact that the parallelogram is bisected by the diagonal and that its area is *bh*.

IV. THE THEOREM OF PYTHAGORAS (c. 540 B. C.).

13. — *The Hebrew Form of the Pythagorean Theorem.*

It is of great interest to note the way in which the *Mishnath ha-Middoth* gives form and expression to the theorem of PYTHAGORAS. The fourth chapter deals with triangles in the following way: (51) « There are three kinds of trilaterals, to wit: the upright, acute, and obtuse ones. Which is the upright one? When the sum of the squares of its two short sides is equal to

(49) A vague reminiscence of this primitive conception might be found in EUCLID's definition of the right angle (*Elements*, I, def. 10): « When a straight line set upon a straight line makes the *adjacent angles equal to one another*, each of the equal angles is right »

(50) See III, 4-5.

(51) The translation is not complete, nor strictly literal. It is rather a free and abbreviated rendering of the content stressing the points relating to the problem in discussion. These points are given literally.

the square of the long side. The area is obtained by the multiplication of one of the short sides in the half of the other one. And the angle standing between the short sides is an upright one... How is the acute trilateral? When the sum of the squares of the two short sides is greater than the square of the long side. Thus you find the angles to be acute... How is the obtuse one? When the square of the long side is greater than the sum of the squares of the short sides... You find thus one of the angles to be obtuse and broad ». This chapter is thus primarily concerned with trilateral figures and with their classification, definition and mensuration. The right, literally « the upright standing », acute and obtuse trilateral are not defined by their right, acute and obtuse angles. The author is apparently more familiar with lines and surfaces, than with angles, and it would be absurd for him to define the known thing by the unknown one. (52) He only incidentally, in a parenthetic, passing remark, mentions that the angles contained by the short sides are respectively right, acute or obtuse. Thus he defines the right angle as standing between the short sides of the upright trilateral, not the upright trilateral as having a right angle (53). The real definition of the upright, acute and obtuse trilateral consists in the relation of the sum of the squares of the two short sides to the square of the hypotenuse or long side. In the upright trilateral we have $a^2 + b^2 = c^2$, in the acute one $a^2 + b^2 > c^2$ and in the obtuse one $a^2 + b^2 < c^2$.

In this enlarged form the Pythagorean theorem was already known to HIPPOCRATES OF CHIOS (c. 460 B. C.) (54). However, the Hebrew form of the theorem from the square of the hypotenuse differs from the Pythagorean theorem in two regards. (1) In the *Mishnath ha-Middoth* this theorem does not refer to angles at all, but to solid, massive trilaterals, and the term angle does not even occur in the Hebrew proposition. (2) It does not appear as an abstract theorem, but rather as a characteristic, or as a practical rule for finding how to construct the upright, acute

(52) The same is the case with the definitions in I, 2-3. The elements of the quadrilateral are side, diagonal and area. The elements of the trilateral are the side-pair, the base, the altitude, and the area. No mention at all is made of the angles.

(53) Cf. hereafter, § 15-18.

(54) See CANTOR, I, 4th ed., pp. 208-209. Cf. also EUCLID I, prop. 47; II, propp. 12-13. TROPFKE, IV, 2nd ed., p. 146.

and obtuse trilateral. It is the so-called converse of the Pythagorean theorem, stating that an angle is a right angle when $a^2 + b^2 = c^2$ but applied to trilateral figures instead to angles (55).

14. — *The Origin of the Pythagorean Theorem.*

Here again the *Mishnath ha-Middoth* seems to represent the ancient, primitive stage of mathematics. It is now generally assumed that the proposition from the square of the hypotenuse was well known long before the time of PYTHAGORAS. Yet the ancient Oriental races did not know it in its Greek form as an abstract theorem of geometry of angles, but as a practical rule of geometry of lines for construction of perpendicular lines or rectangular edges in their architecture and mensuration. The Pythagorean numbers 3, 4, and 5, or 5, 12 and 13 etc., were known and used by the Hindu ropestretchers to construct rectangular altars, the Chinese availed themselves of these numbers for construction of the *Kun*, (gnomon), and the Egyptians for the erection of their temples or for the making of the right angled instruments of their artisans (56). And in this ancient original form, as a characteristic and practical rule for the construction of upright triangles, it is still preserved in the *Mishnath ha-Middoth*, and in the converse given by EUCLID and HERO.

(55) Cf. EUCLID I, propp. 47-48; II, propp. 12-13; HEATH, *Euclid*, I, pp. 349, 368, 403-408; HERO, *Opera Omnia*, vol. III, p. 209 and in *al-Nairizi*, ed. CURZE p. 109; CANTOR, I, 4th ed., p. 371; TROPFKE, IV, second ed., p. 149. The Pythagorean theorem in its enlarged form is given by EUCLID, I, prop. 47; II, propp. 12-13; the converse of it is given by HERO, *loc. cit* and by EUCLID I, prop. 48. H. SCHAPIRA, in his edition of the *Mishnath ha-Middoth*, pp. 26-27, note 4, has already noticed this second difference, that the theorem is brought in the form of the converse. SCHAPIRA also referred to the fact that the Arab mathematicians, AL-KHOWÂRIZMÎ, AL-KARKHÎ (c. 1020) and BEĤÂ EDDÎN (c. 1600) bring this theorem also in its converse. The truth, however, is that AL-KHOWÂRIZMÎ also brings the Pythagorean theorem in its usual Greek form with a proof (*Algebra*, Arabic text, pp. 53-54). On p. 57 he brings the version of the *Mishnath ha-Middoth* defining the upright, acute and obtuse trilateral as having $c^2 \begin{smallmatrix} \geq \\ < \end{smallmatrix} a^2 + b^2$. Then again he changes his terminology calling them « the one with an upright angle » (p. 57, line 13), « the one with an acute angle » (p. 58, line 7; p. 59, line 10), and « the one having an obtuse angle » (p. 61, line 5).

(56) See CANTOR, I, 4th ed. pp. 104-106, 636-637, 679-680; SMITH, *History*, II, p. 288, HEATH, *Greek Mathematics*, I, pp. 145-47; TROPFKE, IV, pp. 139 seq. There is also an article by O. NEUGEBAUER entitled *Zur Geschichte der Pythagoräischen Lehrsatzes* in *Göttinger Nachrichten, Math. nat. Kl.*, 1928, p. 45 ff., but I was unable to see it.

From the historical point of view it would, therefore, be more proper to speak of the Pythagorean theorem as of the *converse* of the old Egyptian-Chinese-Hindu rule. Here again (57) we might properly say that the great discovery of PYTHAGORAS lay in making the subject abstract, in converting the old practical rule of the artisans to a fundamental theorem of abstract geometry, in developing the primitive geometry of lines to the science of angle-geometry.

V. THE PERPENDICULAR AND THE GNOMON.

15. — *Terminology of the height.*

In the preceding two chapters the poor development of the knowledge of angles in the *Mishnath ha-Middoth* was shown. This is further demonstrated by the theory of the height.

There are different terms for the height or altitude in the various figures. (1) The solids with an even thickness, like prisms, cylinders and spheres, have « depth » ('*Omeq*) (58) Arabic '*umq*, Greek *βάθος*. (2) The decreasing solids, like pyramid, cone and frustum, have « height, altitude », *qōmah* (59) ; (*ὑψος*). (3) In the triangle the height is called '*ammūd*, « pillar, column » (60). (4) The height of the segment is represented by the « arrow », *hēṣ* drawn from the mid-point of the arc to the mid-point of the chord (61). AL-KHOWÂRIZMÎ uses '*umq*, « depth », for parallelepipeds, and the Hebrew word '*ammūd* for triangles and decreasing solids (62). It is worth while to note that both the *Mishnath ha-Middoth* and AL-KHOWÂRIZMÎ have no term, nor the idea, of an altitude of the quadrilaterals. Squares and rectangles have length and breadth; the other quadrilaterals are divided into triangles.

16. — *Definition of the height.*

Now it is very instructive and characteristic for the stage of

(57) Cf. end of chapter II.

(58) II, 5, 6; V, 1.

(59) II, 7, 9, 10.

(60) I, 3; II, 2; IV, 1-10. The form *ômêd* in II, 2, is apparently a corruption of *ammūd*.

(61) I, 5; II, 4; V, 5-7. Cf. the term *sagitta*.

(62) H. SCHAPIRA, ed. of *Mishnath ha-Middoth*, p. 21, note 5, justly remarked already that the *Mishnath ha-Middoth* could not possibly use the term '*ammūd* for the height of pyramid and cone, because it employs the same word as name for the frustum.

mathematical knowledge of the author of *Mishnath ha-Middoth*, that not once in all these cases does he remark that the height forms a right angle with the basis (63). He certainly knows it (64), but, apparently, does not deem it proper to define the height by this characteristic. Of the « depth » of the solids it is required « that it be straight and nice » (65) which means that the thickness be equal through the whole height. The altitude of the trilateral is defined as follows: (66) « And the pillar, that is the common thread which runs from between the side-pair down to the base. And it is in the corner 'To the corners of the Tabernacle' (67) The whole definition is unclear. In speaking of the trilateral, the *Mishnath ha-Middoth* most probably thinks of the isosceles (68); hence the « side-pair » and the « base ». The « common thread » seems to be the diagonal of the parallelogram, whereof the triangle is the half. The isosceles is completed to a rhombus by constructing an equal isosceles at the base, and the height is demonstrated as the thread (diagonal) « common » to both triangles, namely as the diagonal of the rhombus generated by doubling of the isosceles (69). The area of the rhombus is given in the *Mishnath ha-Middoth* (70) as half the product of the diagonals; hence it must know that the diagonals are perpendicular to one another. In the constructed rhombus the base of the isosceles trilateral is the one diagonal, and by drawing of the second diagonal the perpendicular is obtained.

In the same way the height of the segment, the so-called arrow,

(63) Here again AL-KHOWÂRIZMÎ represents the more advanced stage by expressly saying that the perpendicular forms a right angle; see *Algebra.*, Arabic text, p. 58, line 11; p. 61, lines 1-2.

(64) Cf. IV. 2, where a right angle is said to be contained by the two short sides of the upright trilateral, and the two short sides are called the two « pillars » or « heights ».

(65) II, 6.

(66) I, 3.

(67) *Exodus*, XXVI, 23. The last sentence and the quotation from the scriptures are rather obscure. « In the corner » means perhaps that it comes from the corner, but certainly not that it forms an angle with the base, as SCHAPIRA in his edition, pp. 15, 54, thinks.

(68) The same is the case in Egyptian geometry; see CANTOR, I, 4th ed., p. iii.

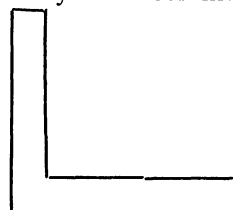
(69) Cf. COLEBROOKE, p. 59, note 2: « A thread... is the diagonal or diameter of a tetragon... Or in the case of a triangle, it is the diagonal of the parallelogram, whereof the triangle is the half ».

(70) III, 4.

is not defined as forming a right angle with the chord, but as « the straight line drawn from the mid-point of the arc to the mid-point of the chord » (71). Thus the author of the *Mishnath ha-Middoth* gives rather instructions how to construct the height (72), than a definition. EUCLID, however, clearly defines the height of any figure as the perpendicular drawn from the vertex to the base (73), the perpendicular as forming a right angle, and the right angle as one of two adjacent angles being equal to one another (74).

17. — *The definition of OENOPIDES OF CHIOS* (c. 450 B. C.).

A similar phenomenon in Greek geometry might perhaps help us to a better understanding of the *Mishnath ha-Middoth*. In his commentary to EUCLID, I, *proposition* 12; « To draw a perpendicular to a given straight line from a point outside it », PROCLUS says (75): « This problem was first investigated by OENOPIDES (of Chios, c. 450 B. C.)... He, however, calls the perpendicular in the « Archaic manner » (a straight line drawn) *gnomon-wise* (κατὰ γνῶμονα) because the gnomon is also at right angles to the horizon ». The gnomon was an upright style or column erected perpendicularly to a horizontal arm. It was used originally as an instrument of the artisans, a carpenter's square for drawing right angles, and also as an astronomical instrument for the measuring of time by the measuring of the shadow which the column cast on the horizontal arm or plane (76). It is now evident that OENOPIDES, too, did not define the perpendicular as forming a right angle, but as a « gnomon-like » line, as a line



(71) I, 5.

(72) Cf. EUCLID, *Elements*, I, propp, ii, 12; HEATH, I, p. 271.

(73) *Elements*, VI, def. 4.

(74) *Elements*, I, def. 10.

(75) *Proclus on Euclid*, I, p. 283, 7-10; HEATH, *Euclid*, I, pp. 271-272; *Greek Mathematics*, I, p. 175.

(76) See HEATH, *Greek Mathematics*, I, p. 78 seq.; SMITH *History*, II, pp. 601 seq., 670 seq.; CANTOR, I, 4th ed., pp. 161, 190, 536. For further details see the writer's article « On the Origin of the Gnomon », which will perhaps appear in the new issue of the *Bibliotheca Mathematica*.

drawn in the manner of the « upright column » of the gnomon (77). OENOPIDES, too, knew that the perpendicular forms a right angle, but in using the « Archaic manner » he called it by the name of the instrument used for drawing right angles. The reason is evident when we will try to understand the mentality of the primitive mind and his « archaic manner » of expression. The abstract conception of a « right angle » is very plain and familiar to us, but it was not so to the primitive man. He knew very well what an « upright stick », a « pillar », a « column », a « carpenter's square » and an « upright standing man » is, but he did not know of angles and angle-geometry. When he started to observe and to study angles, he defined the right angle as an « upright » angle (78), standing between the upright column and its horizontal basis. But it would seem to him ridiculous and sophisticated to define the column as forming a right angle. The concrete upright stick or carpenter's square is the prior and the abstract notion of a right angle is deducted from it. Man first learned to know the gnomon, the mechanical angle instrument and then the right angle. Hence « right angle » is explained by « gnomon-wise » or 'ammûd (79), and the height or the upright column cannot be explained by the right angle. On the other hand the definition of EUCLID is characteristic for the mentality of the scholar who defines a concrete thing by an abstract notion.

18. — *The Right Angle as a Unit of Measurement.*

At the time of EUCLID angle-geometry and angle-mensuration reached already a high stage of development. The right angle, fractions and multiples of it, became the standard of angle-measurement and as such a very common notion, and so it is quite natural that it served as characteristic element in the definition of figures and lines. Still EUCLID's conception of the angle was confined to the idea of inclination or difference of direction. The conception of the angle as an amount of rotation was not known in Ancient

(77) Or rather : in the manner of the « carpenter's square »; see « On the Origin of the Gnomon »...

(78) γωνία ὀρθή. zawith nişşabâ.

(79) Ammûd and qeba, « column » and « basis », are in all likelihood, the names for the vertical and horizontal arm of the Hebrew gnomon. So that 'ammûd corresponds to the Greek κατὰ γνῶμονα.

Times. Hence antiquity does not recognize angles equal to or greater than two right angles, nor does it know of the zero angle or of the angle $n\pi$. Although the division of the circle into 360 degrees was familiar to the ancient nations, this division referred only to the circumference. The angle-degree and the conception of the circle as an angle of 360 degrees was strange to antiquity (80).

VI. LATERAL AND ANGULAR DEFINITION OF FIGURES.

19. — *Were the Figures named according to the number of their Sides or of their Angles?*

CANTOR (81) says : « Die Zahl der Ecken, in welchen jene Flächen, jene Linien aneinanderstossen, wird ihm (dem Menschen) der Bemerkung wert gewesen sein, wird ihn herausgefordert haben, jenen Gebilden Namen zu geben. » CANTOR, apparently, *thinks* of the *Greek* terminology, classifying the figures according to the number of their angles, as *τρίγωνον*, *τετράγωνον* and *πολύγωνον* the triangle, quadrangle and polygon (many-angle), but he *speaks* of *mankind in general*, as if this Greek terminology were the general usage common to all mankind. An examination of the facts, however, will show that mankind in general, the ancient Egyptians, Hebrews, Arabs, Hindus, Romans and probably also the Babylonians and ancient Greeks, themselves, first noticed only the dimensions of length and breadth, the sides and lines, and named the figures after their number. The observation of the corners and angles, and the classification according to their number seems to be distinctly Greek, a specific invention of Greek science, based upon the introduction of angle-geometry.

20. — *Egyptian, Hebrew and Arabic Usage.* *Not specified Terminology.*

In the Rhind Mathematical Papyrus we have the word '*ifd*' which is generally translated « square » or « rectangular ». « It

(80) See SMITH, *History*, II, p. 277; TROPKE, IV, second ed., pp. 49, 47. HEATH, *Euclid*, II, pp. 47-49, 275-76; CANTOR, I, 4th ed., pp. 47, 50, 360, 366, 416, 681.

(81) I, 4th ed., p. 15.

is clearly a derivative of *fdw*, the numeral 4, and its original meaning must therefore have been four-sided, perhaps with the tacit addition of rectangular » (82). Every one will agree with PEET that the word '*ifd*' = « four-fold » refers only to the four sides and not to the four angles. As we have seen, the Egyptians are primarily concerned with the mensuration of areas and volumes by their sides and not by their angles. The angle can hardly play any part in their mathematics since it is not mentioned at all. Hence it would be improbable to assume that with '*ifd*' they meant to say the four-angle. The very fact that they simply say « four-fold » proves that they thought only of one thing, the side, to be taken four times. Would they give much thought to the other characteristic, the angle, which also occurs four-times they certainly would have specified their expression as to four sides or four angles.

The same thing is the case with the Hebrew word *rabu'a*, *merubba'*, commonly used in Bible and Mishnah. It is a derivative of *arba'*, « four », and means literally « four-fold », not specifying sides or angles; but there is no doubt about it that it refers only to the four sides of the figure. It is so much so that we even have the word, *reba'*, meaning « the fourth part », « the side, » of the square (83). In the same way the biblical word *shalish* (I Sam. XVIII, 6), meaning « threefold », is conceived by many commentators to mean, like the Greek *trigonon*, a musical instrument in the shape of a triangle; cf. GESENIUS, *Hebrew Dictionary*. In the *Mishnath ha-Middoth* the words *merubba'ath* and *meshullesheth* and in Arabic the corresponding forms *murabba'ah* and *muthallathah* are the usual terms for trilaterals and quadrilaterals (84). When the *Mishna* and other *Tannaitic* (85) texts speak of triangular or quadrangular houses, then the Greek words *trigon*, *tetragon* are used (86) not *meshullash*

(82) PEET, p. 85. On the tacit addition of rectangular see hereafter.

(83) See *Ezekiel*, I, 8, 17; X, 11; XLIII, 16, 17.

(84) The Hebrew-Arabic *murabba'ah*, *merubba'* like the Egyptian '*ifd*', the Greek *tetragonon* and the Latin *quadratum*, are commonly used to mean the square also, the square quadrilateral being mostly used as the standard measure for all the areas and not differentiated from the quadrilateral with oblique angles.

(85) Hebrew sources contemporary to the Mishnah.

(86) Cf. *Mishnah*, Ne ga'im, XII, 1; *Thosephtha*, ib., VI, 3 and *Nazir*, I, 2; *Yer.*, *Nazir*, I, 2; *Sifra*, Mešora' VI, 2, (ed. WEISS, p. 73a); *Babli*, *Nazir*, 8 b; *Baba Bathra*, 164 b. See also LEVY's Talmudic dictionary, I, pp. 392, 441 b;

or *rabbū'a*. It is interesting to note that they spoke of a circular, biangular, triangular, quadrangular and pentangular house (87). The biangular house was apparently a house in the shape of a semicircle, (88) and the word = *digōn* = *δίγωνον* is a new formation not known from Greek sources (89). So the Euclidean series of figures according to the number of sides (90) finds here its parallel in a series of figures according to the number of angles; the circular house being a house with one angle, and the biangular house having the shape of a semicircle.

21. — *The Greek Usage. Specified Terminology.*

The next stage in the development of mathematical thought is represented by Greek terminology. The Greeks were the first to abandon the primitive unclear usage of « three-fold », and « four-fold » figures. They considered not only the sides but also the angles of the figures and were confronted by the problem as to which characteristic is more important, the side or the angle. Hence the indistinct word « threefold, fourfold » could no more be satisfactory to them, and so they created the double terminology, classifying the figures according to the number of the sides and of the angles. They speak of the trilaterals, quadrilaterals and multilaterals, (91) and on the other hand also of the triangular, quadrangular and multiangular figures (92) these latter terms being the usual ones.

22. — *The Problem and its Solution.*

The question now arises as to which terminology is the older one. The general opinion is that the nomenclature according to the number of angles is the older one, and that the lateral nomenclature was introduced by EUCLID. HEIBERG (92) even

KRAUSS, *Qadmoniyyoth ha Talmud*, I, 2, p. 288, and *Lehnwörter*, II, pp. 197, 260, 272, 429. The term *trigon* occurs also in *Baraita di Shemuel ha-Qatan*, VI, end.

(87) Babli, *Nazîr*, 8 b, *Baba Bathra*, 164 b.

(88) See commentary of *R. Sh. b. M.* to *Baba Buthra* 164 b.

(89) Cf. however, *δύοκλις* « with two knots », or « joints ».

(90) See hereafter, p. 477.

(91) *Tripleuron*, *tetrapleuron* and *polypleuron*.

(92) *Trigonon*, *tetragonon* and *polygonon*.

(92) *Mathematisches zu Aristoteles*, pp. 15, 31, 32.

goes so far as to doubt the Aristotelian origin of the *Mechanics*, mainly from the fact that the word *tetrapleuron* occurs there for the quadrangle. HEATH (94), following HEIBERG, states that EUCLID first introduced the lateral terminology, and that the reason for it was that he wanted to put an end to the ambiguity in the use by the older mathematicians of the word *tetragonon*, to mean «square» and any «quadrilateral». EUCLID therefore created the term *tetrapleuron* for the quadrilateral and retained *tetragon* for the square.

The writer does not agree with this theory. 1) It is not proved that *tetrapleuron* was first introduced by EUCLID. The fact is that it does occur in one passage of the Aristotelian books, the *Mechanics* and the *Problems*, respectively (95). HEIBERG, himself, admits that its introduction by EUCLID is only a conjecture, and CANTOR seems to be very sceptical in regard to this conjecture (96). 2) Anyhow the reason given by HEATH does not account for the introduction of the terms *τρίπλευρον* and *πολύπλευρον* (def. 19). As a rule EUCLID in his *Elements* uses only *τρίγωνον* and *πολύγωνον* (97). Apart from that HEATH, himself, concedes (98) that «though EUCLID enabled ambiguity to be avoided, there seem to be traces of the older vague use of *τετράγωνον* in much later writers (e.g. in HERO and PROCLUS).»

The writer therefore thinks that EUCLID in using the terms «three-four-and manysided» in *def. 19* had an entirely different aim before his mind, and this aim was to stress the point of the series of figures increasing in the number of their boundary-lines. The order of the figures spoken of in *definitions 15-20* of *Book I* is (1) the circle, 2) the semicircle, 3) the trilateral, 4) the quadrilateral and 5) the multilateral. PROCLUS and SIMPLICIUS in their

(94) *Euclid*, I, pp. 187-188; II, p. 239. Cf. also TROPFKE, IV, second ed., p. 92.

(95) HEATH, *Euclid*, I, p. 187.

(96) CANTOR, I, 4th ed., p. 254, note 3, says: «HEIBERG, Seite 31 flg., bezweifelt die Echtheit des Aristotelischen Ursprunges (der *Mechanik*) aus sprachlichen Gründen, namentlich wegen des Vorkommens des Wortes *τετράπλευρον* für Viereck, welches erst EUCLID eingeführt habe. Er gibt aber, Seite 32, selbst zu, dass diese, Seite 15, behauptete Einführung durch EUCLID 'nur eine Vermutung, wenn auch eine sehr wahrscheinliche' sei.»

(97) See I, *Def.* 20, 21, *Propositions*, 1, 4, 5, 6 etc. for the use of *trigonon*, and VI, 20, XII, 1 seqq. for the use of *polygonon*.

(98) *Ib.*, p. 188.

commentaries already pointed out that, in the order adopted by EUCLID for these definitions of figures, the first figure taken is that bounded by *one* line (circle), then follows that bounded by two lines (semicircle), then the trilateral bounded by three lines etc. (99). This explanation of the commentaries is well founded in the words of EUCLID, I, Deff. 15, 18, 19. EUCLID, himself, expressly mentions and emphasizes the point that the circle is bounded by one, the semicircle by two, the trilateral by three, the quadrilateral by four and the multilateral by more than four lines (100). In fact EUCLID in his *Elements* makes little use of the semicircle. Apparently the only reason for introducing the semicircle and its definition (def. 18) is to have a figure bounded by two lines, and to complete the series of figures bounded by an increasing number of lines, most likely taken from an earlier textbook (101). The terms *tripleuron*, *tetrapleuron* and *polypleuron*, however, were probably not created by EUCLID, as HEIBERG and HEATH believe. They were, in the writer's opinion, rather very old archaic terms. In ancient times, as already pointed out above, people were concerned only with mensuration by means of lines, but they little cared for the angles of the figures, since their value for mensuration was not yet discovered (102). Hence the figures were named and classified according to the number of sides and not of the angles. With the beginning of angle-geometry in Greece in the times of THALES and his school (c. 600-500 B. C.) the terms *trigonon*, *tetragonon* and *polygonon* were invented and obtained so much weight that the older names *tripleuron* etc. were almost entirely discarded and became obsolete (103). As a remnant of an ancient time the term *tetrapleuron*

(99) See HEATH, *Euclid*, I, p. 186.

(100) It is, however, very interesting to note that the Talmudic sources, quoted above § 20, speak of the circular, biangular, triangular, quadrangular and pentangular houses. Thus they seem to think of a series of figures increasing in the number of their angles.

(101) The same reason, namely that the classification was taken from an earlier textbook, is also given by HEATH, (*Euclid*, I, p. 189) for the inclusion of definitions of the oblong, the rhombus and the rhomboid in *def. 22* dealing with the quadrilaterals, although the oblong, rhombus and rhomboid are not further discussed in the *Elements*.

(102) See above, end of chapter II.

(103) The rare terms *tripleuron*, etc. are, like the phrase *κατὰ γνώμονα* used by OENOPIDES for the perpendicular (see above, chapter V, 17), and the phrase of the *similar angles* at the base of the isosceles triangle used by THALES (see HEATH,

is still preserved in some of the Aristotelian writings mentioned above. EUCLID reintroduced these ancient terms for a special purpose, but he retained only *tetrapleuron* (104). Later on, when MENELAUS wrote on spheric trigonometry (c. 100 C. E.) and the need arose for the distinction between two kinds of the triangle, the word *tripleuron* was again revived and introduced by MENELAUS as a term for the spheric triangle in contrast with the plane triangle for which he used *trigonon* (105).

22. — *A curious Fact.*

In conclusion the writer wishes to note a rather curious fact. In the Palestinian Talmud the Greek word *γωνία*, *gōnia*, « angle » is supposed to be used in the meaning of « side », *latus* (106). The Greek dictionaries cite also the meaning « ein eckiger Pfeiler » « an edged pillar ». In the Talmudic sources, quoted above (107), speaking of the house as being circular, *digon*, *trigon*, *tetragon* or *pentagon*, it certainly would be more fitting to understand these words as meaning having two sides (or pillars), three sides (or pillars) etc. (108). If the meaning of « side », « pillar » for *γωνία* should find more support in Greek literature we might perhaps venture the conjecture that originally *trigonon* and *tripleuron* etc. had the same meaning of « trilaterals » etc.

23. — *The Roman and Hindu Usage.*

In Roman literature too we find the double terminology characterising the figures according to the number of the sides and

Greek Mathematics, I, pp. 130-131), rather vestigial remnants of an ancient time. It would be very desirable to have a monography written on such archaic terms in Greek mathematics, which were later crowded out by abstract and technical terms invented in the scientific schools from the times of the Sophist still EUCLID. These archaisms form the connecting link between Greek and ancient Oriental mathematics.

(104) *Book IV*, 20, porism; ed. HEIBERG, II, p. 138.

(105) See CANTOR, I, 4th, ed., p. 413 for the occurrence of the word *tripleuron*.

(106) This is the opinion of LEVY in his Talmudic dictionary, I, p. 348. Cf. also KRAUSS, *Lehnwörter II*, pp. 168-169. JASTROW, I, p. 256; *Arukh Completum*, II, pp. 320-21; HILDESHEIMER, *Beiträge zur Geographie Palestina's*, pp. 73-74 explains *ginyâ di-ashqalôn* as « die Ausbuchtung », the bay, of Askalon.

(107) See § 20.

(108) The Bible verses, *Leviticus*, XIV, 37-39, to which these Talmudic passages refer, speak of the walls of the house and not of the angles.

of the angles : *triangulus*, *trigonum*, *trigonium* and *trilatera*, *quadrilatera*, *multilatera* or *plurilatera*. However, the latter terms, stressing the number of the sides, are preferred by the Roman agrimensores, e. g. by BALBUS (1st century C. E.) Apparently because they are the original ancient Latin terms. (109)

The Hindus also name the figures both according to the number of angles and sides. They have *tryasra*, *tricóna* = « triangle », and *tribhuja* = « trilateral »; *chaturasra*, *chaturcóna* = « quadrangle », and *chaturbhuja* = « quadrilateral » (110). It seems, however (111), that the classification according to sides is more frequent and usual. The terms *tricóna* and *chaturcóna* seem to show Greek influence, while *tribhuja* commonly used for the oblique trilateral, and *sama-chaturbhuja* the « equilateral quadrilateral », commonly used for the square and rhombus seem to be the native, ancient Hindu terms.

After these lines were written, my esteemed friend, Professor B. DATTA, was kind enough to send in the following information for which the writer wishes to express his best thanks :

« The oldest Hindu terms for rectilinear geometrical figures are found to be compounds of number names, such as *tri* (= 3), *catur* (= 4), *sat* (= 6), etc., with the words *ásra* (*asra* or *asri*) and *srakti*. Such names occur commonly in the Vedic literatures, as also in later works. The word *srakti* means « angle ». But there is difference of opinion about the real import of the word *ásra* or *asri*. Later lexicographers, such as AMARAKOṢA (c. 350 A. D.) interpret it as meaning « angle ». Still later scholiasts are usually loose in their interpretation. They sometimes put *ásra* = « angle » and at other times, = « side » or « edge ». ĀRYABHATA (499 A. D.) calls a « triangular pyramid » as *ṣaḍ-ásrī* and a cube as *dvādaś-āsṛī*. So in these two instances, *asri* certainly means « edge » or « side ». Similar instances occur in the *Arthaśāstra* (c. 350 B. C.) and in the *Suryaprajñapti* (500 B. C.) in which *ásra* and *kona* (= angle) must be interpreted differently, the former meaning « side ». Hence it will have to be concluded

(109) See TROPFKE, IV, second ed., p. 61, quoting *Schriften der römischen Feldmesser*, I, pp. 105-106, and *Pseudo-Boethius*, p. 375, lines 15, 16, 20.

(110) See COLEBROOKE, p. 58, note, and BURGESS, E., *Sūrya-Siddhānta*, New Haven, 1860, p. 95, last line. The book of BURGESS appeared also in the *Journal of the American Oriental Society*, 1860.

(111) From the further information given by COLEBROOKE, *ib.*

that the ancient Hindus named the geometrical figures according to the number of angles as well as of sides. In any case your inference that the « angle-geometry » is due to the Greeks cannot be supported. For the names with *srakti* (angle) occurs much earlier, long before the birth of Greek geometry. Can you account why the Greeks who learnt Geometry from the Egyptians adopted an altogether different nomenclature? At least their earlier literatures ought to reveal names according to the number of sides. I hope you will look into this side of the problem also ».

VII. BRIEF SUMMARY AND CONCLUSION.

24. — It may be proper to conclude this discussion with a short recapitulation. At the outset a definition was given of the terms geometry of lines and geometry of angles, angle-geometry and trigonometry. The attempt was then made to demonstrate that these two branches, geometry of lines and geometry of angles, represent two stages in the evolution of geometry. The old Egyptian and Babylonian geometry was characterised as a geometry of lines and the invention of the geometry of angles was credited to the Greeks. Greek tradition was vindicated against the theories of some modern historians of mathematics, and in connection with that instinct and science, the prehistoric phases in mathematics were discussed. Hebrew and Arabic geometry were then investigated and shown to constitute a phase of transition from a geometry of lines to a geometry of angles. From this point of view an inquiry was made into the history of the Pythagorean theorem, and it was shown how the conception of the height and of the right angle developed. Finally the question was discussed as to whether the sides or angles formed the characteristic in the classification of the figures.

It is hoped that a little contribution was made towards the explanation and elucidation of these questions. Yet, the examination of these problems requires a thorough knowledge and familiarity in both ancient philology and history of mathematics, and the writer is fully aware of his shortcomings in both these fields. Should this discussion prove of some use to students of ancient philology and history of mathematics, and be it only by arousing their interest in these problems and by stimulating

them to correct and modify some of the findings of this paper,
the writer shall deem himself amply rewarded for his labours. (*)

*The Rabbi Isaac Elchanan Theological Seminary
and Yeshiva College New York.*

SOLOMON GANDZ.

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