

A Performed Solution to the Pythagorean Problem

The Three Bodies Project

EDWARD C. WARBURTON AND GREGORY LAUGHLIN

ABSTRACT

The authors describe an art-science collaboration to devise and perform a qualitatively accurate interpretation of an elliptic-hyperbolic solution to the classical astronomical "problem of three bodies," in which three point masses execute trajectories dictated by Newton's Law of Universal Gravitation.

The Pythagorean Three-Body Problem is easy to state and hard to solve: What happens when three test masses, with $m = 3$, $m = 4$ and $m = 5$, are placed at rest at the vertices of a 3, 4, 5 right triangle and are allowed to fall in toward one another under the action of Newtonian gravity? This simple physical situation would occur, for instance, if one were to take three planets and place them at rest in inertial space with proportionally correct masses and distances. The ensuing motion of the bodies under the influence of their mutual attraction is extraordinarily complex. This problem challenged astrophysicists for many decades, from its formulation in 1893 until the problem was solved numerically by Victor Szebehely at Yale University in 1967 [1,2].

The three bodies do not simply fall into a collision. Instead, they execute an intricate series of interactions that eventually resolve with bodies 4 and 5 departing in a binary orbit and body 3 ejected in the opposite direction into empty space (Fig. 1). The solution to this extraordinary problem suggests a moving narrative replete with human emotions that can be seen to emerge from the cold columns of numbers associated with the numerical integration of the governing set of differential equations. The solution is a dance that unspools a tragedy of connections lost and found and lost again—one that is encoded deterministically, and with extraordinary compression, into the initial conditions.

The art-science project *Three Bodies* brought together an astrophysicist and dancers to interpret and perform the

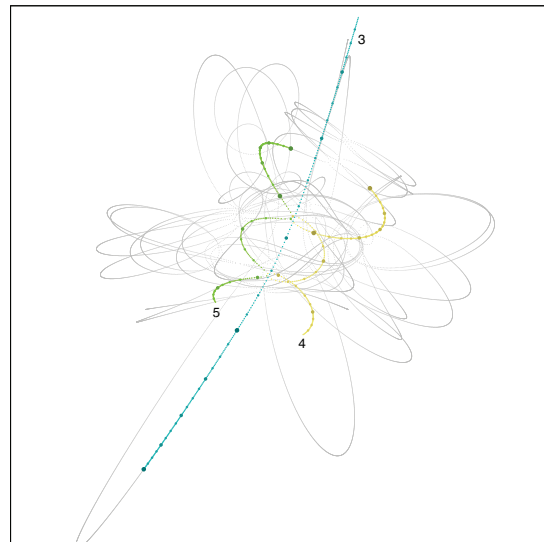


Fig. 1. Solution trajectory near conclusion of computed motion, with bodies 4 and 5 departing in a binary orbit and body 3 ejected in the opposite direction into empty space. (© Gregory Laughlin)

problem's elliptic-hyperbolic solution. Our research endeavored to adhere to the mathematical imperative and capture the emotion that we found implicit in the differential equations. We aspired to create a work that was at once correct in the purely formal sense while simultaneously revealing the aesthetics found in orbital motion and the conceptual rigor found in dance. As we grappled with an unyielding set of constraints on the choreography, we embraced a radical correspondence: The astrophysicist danced and the dancers analyzed. Our proximity, open communication and frank critiques engendered equality in all aspects of the scientific-creative process. During our collaboration, we also stumbled on, over and around key questions that underlie problem solving (and problem finding) in the arts and sciences. In the discussion section below, we consider in particular non-formal modes of concept interpretation and the role of constraints on creativity.

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THE PROJECT

Our project began as a studio-based exploration with three dancers, a choreographer and an astrophysicist. We worked with the predefined conditions—precise constraints on spatial arrangement, acceleration, timings and trajectories—to create dances that explore feelings of longing, connection and isolation as bodies are flung apart in response to the same gravitational forces that draw them together. We then invited musical and media artists to collaborate in producing a live performance and video visualization. Mirroring the iterative attempts at solving the original problem, we created several versions over the past several years, beginning with the Pythagorean problem and later with its associated variations: the periodic problem and the “butterfly” problem, which draw on the system’s extraordinarily sensitive dependence to initial conditions. Below, we describe the early phases of our research and the first edition presented at ZERO1 Bienial in 2012.

Because the three-body problem proscribes the pathways and timings through general space, it acts as a road map (and speed limit) for the choreography. Hence, our initial research focused on devising a movement vocabulary that could travel along these precise trajectories. We employed William Forsythe’s technique of Improvisation Technologies because of its spatial emphasis: Dancers improvise by designing lines in and through space [3]. Forsythe’s fundamental insight is that movement can start at any point in the body, travel to another point in the body and be perceived visually as a line drawn in space. Thus, this has been called the “point-point-line” method, describing movement in geometric terms: i.e. movement through space from a point to another point, from point to line, line to plane, and so on.

The improvisation of physical tracings was not the final objective, however. Dynamic quality is a key aspect of expressive movement. After we devised a series of inscribing motifs, we made decisions about how those movements would be rendered with specific qualities using Laban Movement Analysis (LMA). Based on the theoretical study of motion by Rudolf Laban (1879–1958), LMA addresses the more ephemeral and qualitative aspects of dynamics, providing a common vocabulary for how energy could be used to convey the feeling, tone or texture of movement [4]. Effort change is generally associated with change of mood or emotion and, hence, is an inroad to expressivity. In the context of dance, the quality of a body (or planet) “gliding” around another body can be defined as having the features of light weight, sustained timing and direct spatial intention.

Almost immediately, questions arose about the literal nexus of artistic intention and mathematical certainty. Exactly where would dancers travel and when should they experience the onsets (or offsets) of mounting gravitational attractions? To achieve this layer of precision, we used the Bulirsch-Stoer algorithm, which numerically integrates the equations of

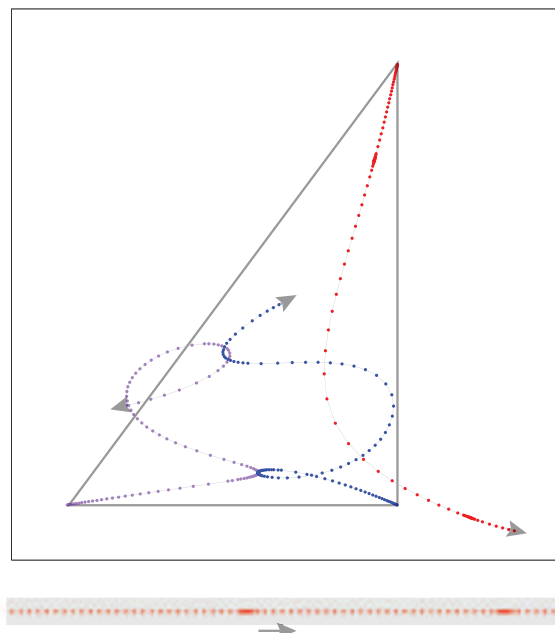


Fig. 2. Initial condition and beginning trajectory in equal units of computational time. (© Gregory Laughlin)

motion to extremely high accuracy (better than one part in 10 trillion per step), to explore the very beginning of the evolution. The diagram in Fig. 2 plots the positions of the bodies in equal units of “computational” rather than physical time, underscoring the rigors of the close approaches.

The diagramming of computational time raised the problem of how time should flow in the choreography. Ultimately, we opted for the performance to unfold in real time. The next step was to construct timing grids that were imported into Apple’s Logic software as rhythmic skeletons. Our timing interpretation envisioned a 137.5-measure framework in 4:4 musical time at 120 bpm to define a performance lasting 4 minutes 35 seconds. Figure 3 shows the first 24 measures. Each frame plots 8 measures. Points are resolved as 16th notes, full measures and four-measure blocks. Accumulated motion is plotted as faint gray points spaced at 32nd-note cadence.

To develop a correct animation that could guide performers’ movements, we required precise numerical integration. Even tiny numerical errors can accumulate to corrupt the simulation, which required single-step convergence of one part in 10 trillion, a severe demand. Using Processing, a flexible software framework for coding visual materials, the completed animation of the entire sequence contained 8800 (1024 × 1536) .png frames. Each frame has a duration of a 1/64th note, and the current measure, the current beat and the current 16th note within the beat were marked on each frame. Because the animation was projected from the ceiling to the floor of the dance studio, this level of detail was imperative for rehearsals; the choreographic process demanded accurate coordination of movement phrases with spatial patterns, timings and relationships.

The floor animation clarified for dancers the moments of attraction (and reaction) to gravitational forces. Yet, predict-



Fig. 3. Timing interpretation in first 24 measures of the solution. (© Gregory Laughlin)

ably, it failed to inspire creative expression. We recognized that the biggest performance payoff would come from an embodied memory that obviated the animation. We invested great effort in grasping the overall pattern, choreographing from our initial improvisations, building longer movement phrases and repeating time and again. Once the patterns of physical interactions and sequences emerged, confusions about artistic intent arose. The dancers became immersed in the idea of intention. They posed hard questions about

creative interpretation, e.g. “Why am I gliding around this body in this way?” They required direction: a feeling or narrative that could motivate interaction or suggest an initiation point for the movement. At a loss for explanation, the choreographer (Warburton) turned to the astrophysicist (Laughlin), who located concrete answers to questions of artistic intent in the differential equations themselves. In a series of written notes and studio sessions, Laughlin described and demonstrated narrative interpretations that encapsulated a creative nonfiction retelling of the three-body story.

At the halfway mark of our performed solution to the mathematical problem, we realized there is a near, *but not exact*, return to the 3-4-5 starting configuration. We saw that the immediate outcome of the near-return was to illustrate how the subtlest perturbation can give rise to a profoundly different outcome. As shown in opening measures 1–5 (Fig. 3, top-left panel), bodies 4 and 5 move immediately toward one another, suggesting strong physical attraction. They interact, as body 3 veers off with barely a glance. In contrast, midway through the

dance, at measures 65–68 (Fig. 4), bodies 3 and 5 make first contact well before body 4 reacts. We realized that body 4 is somehow lazy at this point, or late to grasp the import of the situation, and is marginalized as a result. This is the first real opportunity for bodies 3 and 4 to express emotion—shock for body 4, joy for body 3. In the following measures (Fig. 5), we find a surprising episode: body 4 is marginalized, sulky, scheming, whereas body 3 is doing its best to impress body 5. In the set of looping, private engagements we find “A reverie!”

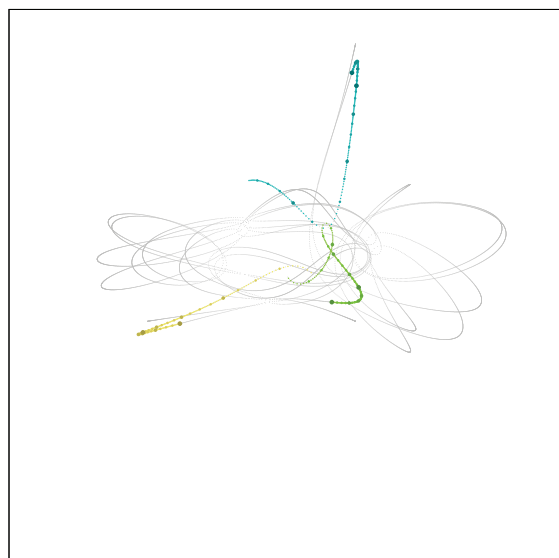


Fig. 4. Interpretation of “midway doldrums” at measures 65–72. (© Gregory Laughlin)

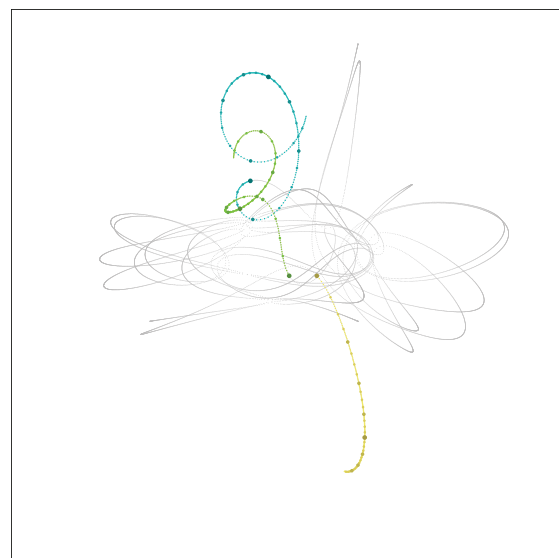


Fig. 5. Interpretation of “A reverie” at measures 73–80. (© Gregory Laughlin)

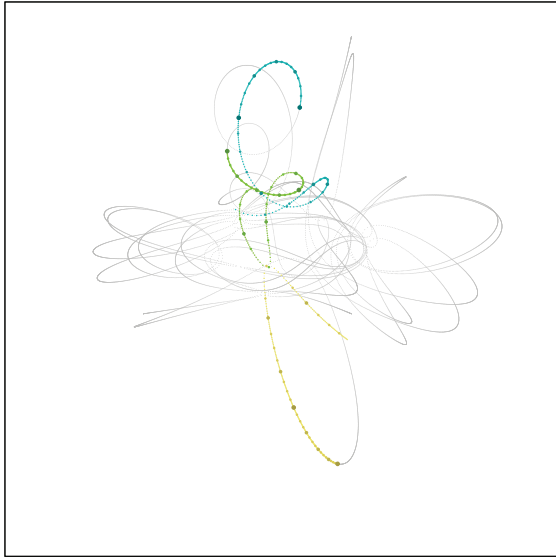


Fig. 6. Interpretation of “Last chance” at measures 81–88.
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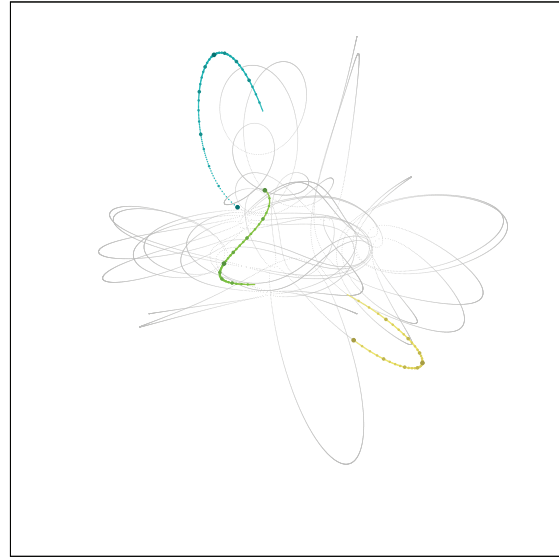


Fig. 7. Interpretation of “Unfulfillment” at measures 89–96.
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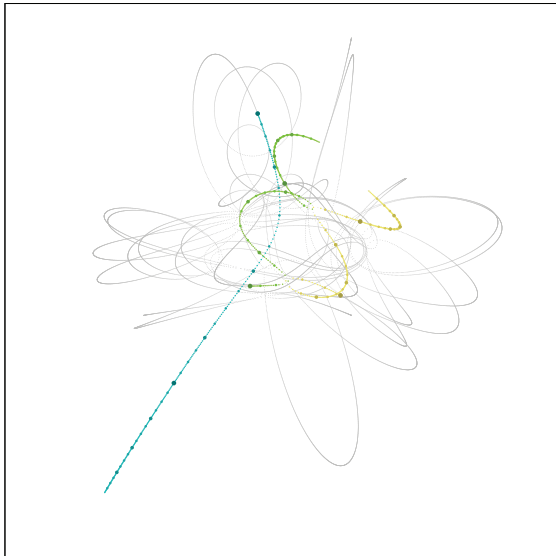


Fig. 8. Interpretation of “Run out” at measures 97–104.
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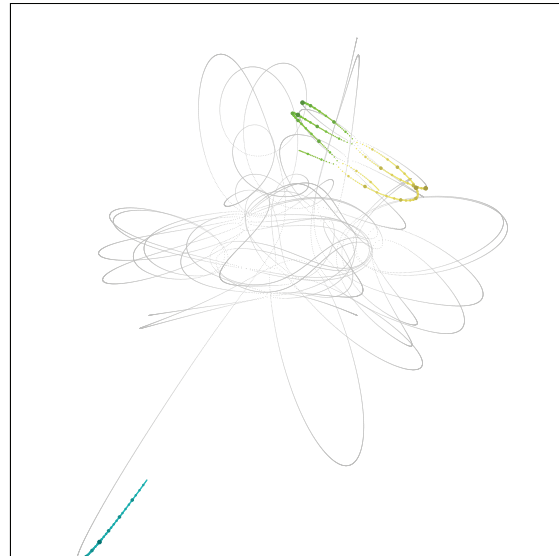


Fig. 9. Interpretation of “Spellbound” at measures 113–120.
(© Gregory Laughlin)

wherein the successive body 3/body 5 interchanges highlight the difference in masses between bodies 3 and 5. Body 3 is light footed, fleet, and body 5 glides smoothly, deliberately (but not dully) as an anchor. To recover, body 4 must come back from its run-out with a renewed sense of determination and purpose. The ensuing encounter between bodies 4 and 5 must somehow telegraph body 4’s charms and strengths. In a very real sense, this encounter is the tipping point that determines the outcome of the relationship for all time (Fig. 6).

The next sequence shows the consequence of body 4’s efforts (Fig. 7). There is a sense of unfulfillment in the body 3/body 5 encounter—a longing gaze that fails to connect—as body 4 pulls body 5 away. Figure 8 shows how this missed opportunity sets up body 3 to dive frantically through the others

on its way to the penultimate run-out. This section requires that body 3 must somehow telegraph the failure to live up to the expectations that it so brightly promised. The outcome is now determined, and the bodies know it, although the audience does not. While body 3 is suspended at the arc of its final run-out for almost 16 measures or 64 counts, body 4 weaves a spell on body 5, cementing the outcome ever more decisively (Fig. 9). Indeed, body 5 is only briefly engaged as body 3 makes its final dramatic run through the orbit of bodies 4 and 5. As displayed in the introduction to the Pythagorean three-body problem (Fig. 1), the final encounters of giving, receiving and returning between bodies 4 and 5 grow ever more identical, signaling the finality of the celestial dance.

In 2012, an evening of 4-min 35-sec performances of *Three Bodies* premiered at ZERO1 Biennial, the international digital art and new media festival in San Jose, California. With our media and musical artists, we envisioned a sound and data environment that could amplify and guide the dancers as they moved through the performative space. To give the audience a bird's-eye perspective of the dancers' movements as they "solved" the problem, we decided on a real-time visualization projected on a screen behind the stage. The use of interactive media allowed us to deploy a sophisticated structuring system to engineer the integration and layering of multiple strands of the dance, music and visual media. Using a custom light-tracing application to detect the light emitted from LEDs on their soft circuitry costumes, we tracked dancers' movements to a visual model of their trajectories across the 2D plane of the stage. The three dancers in illuminated costumes thus simultaneously embodied and traced a real-time video visualization of the elliptic-hyperbolic solution (Color Plate A). Live accompaniment from three jazz musicians intensified the performance as each body was interpreted in sound.

In retrospect, we felt this first edition of the project evolved into more of a "performative installation" of the multimodal interaction among performers than an embodiment of the three-body problem. Subsequent editions refocused attention on the solution itself [5]. We reduced the complexity of the digital media, amplified the dancers' physical intensities during the onsets and offsets of gravitational attraction and increased the number of variations in the solution set. These variations created new opportunities for interpretation. In one case, the *periodic solution*, the three bodies return precisely to the start and begin again. We learned that audience members appreciated this event repetition as a second go-round for perceiving the larger pattern. In the *butterfly variation*, we nudged one body ever so slightly at the start, thereby altering all of the resulting motions and generated entirely different patterns of behavior. Each variation required a creative narrative to allow the performers to discover patterns of longing, connection and isolation as the solutions unspooled. In performance, we stage these variations in sequence, allowing the audience time to immerse in the magic of nonlinearity.

DISCUSSION

In our collaboration, artists saw how scientific and mathematical truths differ from malleable everyday concepts, and they faced the concurrent challenges of learning and interpretation. Scientists learned how the creation of an original artistic outcome involves significant complexities transcending those of everyday thinking. Our work revealed key questions centered on problem solving (and problem finding) in the arts and sciences. We were particularly struck by the use of nonformal modes of concept interpretation and the role of constraints on creativity.

Scientific concepts must be consistent and unambiguous to allow explanation and prediction. It is generally assumed

that formal methods, such as declarative specifications, provide the most explicit and reliable ways of interpreting scientific concepts. However, novice scientists and nonscientists alike are often intimidated by mathematics and, although formal methods are sufficient to interpret these concepts, their systematic implementation may be slow and cumbersome, inadvertently building barriers to engagement and comprehension. This raises a question about modes of interpretation that are optimally suited for understanding and using scientific ideas. Just as dancers have devised informal methods for learning and recalling every detail of hours-long choreographies, we wondered what role more informal methods could play in interpreting mathematical abstractions.

Concepts are often grasped in an effortless, intuitive manner, albeit without maximal precision and generality. So-called nonformal modes of interpretation have the virtue of being cognitively efficient if one has a repertoire of compiled knowledge encompassing both the standard and various special cases of a concept [6]. If knowledge in this repertoire is sufficiently familiar, it can be recognized and retrieved automatically without conscious processing. How does one build this repertoire? In astrophysics, one can draw on the use of mental and physical imagery that "reduce" and "re-present" information. One example is the use of gesturing for scientific sense-making [7]. The resulting repertoire can be highly useful despite its intrinsic limitations on precision, accuracy and coherent generality.

In dance, the equivalent nonformal modality is "marking": a movement-reduction strategy that involves enacting the sequence of movements with curtailed size and energy by diminishing the size of steps, height of jumps and leaps, and extension of limbs. Marking in dance is most often used during rehearsal. In contrast to dancing "full out," dance marking has been shown to confer processing benefits by relieving cognitive load and supporting more efficient encoding and consolidation, resulting in better subsequent performance [8]. Our experience in *Three Bodies* suggests that it is possible that dance itself can "mark" important aspects of mathematical and scientific concepts by "loosely going through the motions" of those solutions.

Where dance may inform astrophysics via nonformal conceptualization, science can inform art by placing precise constraints on the production problem space. In *Three Bodies*, the mathematical solution prescribes the motion. What is the nature of choreographic cognition when creativity is applied within strict bounds? In cognitive psychology, constraints are succinctly defined as tools that come in pairs that are hierarchically organized and specific to domains [9]. The domain provides both the elements that one can work with and the elements one can work against. One of the most important constraints in dance, located at the top of the hierarchy, is goal constraints, which specify artistic style, e.g. ballet. Subject and task constraints follow. Subject constraints speak to contents of movement in space and time, e.g. the five basic positions in ballet. Task constraints involve materials: how to use breath or weight to move through the five positions. Choreographers create new work when they extend currently

accepted goal criteria and stylistic standards. This can involve goal-directed specification of paired constraints that structure the problem space and produce a solution path that simultaneously defines and satisfies a novel criterion.

Consider Balthasar Beaujoyeulx, the great “geometer” of Renaissance “celestial dances.” Beaujoyeulx designed rapidly shifting geometric patterns and substituted the traditional, floor-level perspective for an elevated seating and bird’s-eye view [10]. His specification of paired subject constraints—spatial arrangement and rhythmic timing—turned an initially ill-defined problem (i.e. how do I make a new dance) into a well-defined one. In doing so, Beaujoyeulx changed the course of dance history: His shift in perspective and rhythm evoked the written page and punctuation. Spectators could now “read” dance, engendering a nonintuitive framing of dance as a rhetorical modality on par with epic poems. In *Three Bodies*, by placing computed restrictions on our use of time and space, we learned anew how constraints can help

experts find, specify and solve new problems, effectively clarifying and directing the creative process

Our goal has been to create original, less abstract solutions to the Pythagorean Three-Body Problem. We are intrigued by the possibility that “three point particles attract each other according to Newton’s Universal Law of Gravitation” can become a familiar situation to novice scientists and artists alike, which may also expand their repertoire of nonformal knowledge and constraint-based problems in the arts and sciences. Certainly, we expect that doubtful conclusions based on snap intuitions will be checked against more formal knowledge. But it is also possible that nonformal knowledge acquired by “dancing the three bodies” may be useful in providing intuitive checkpoints for more abstract arguments. It is also possible that the creative cognition acquired by dancing the three-body problem may provide a case study for dispelling the rigor mortis that sometimes seizes art-science connections.

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