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Euclid's Elements and its Prehistory

Benno Artmann

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1. Introduction

In 1904 Heiberg, the most eminent Euclid scholar of his time, wrote, '...die Elemente Euklids bieten nur spärliche Handhaben für Rückschlüsse auf die Vorarbeiten' (Heiberg (1904), 4). Mathematical and stylistic analyses of Euclid's *Elements* over the last 60 years have disproved Heiberg's claim and revealed a great deal about the prehistory of this monumental compendium of Greek mathematics. The first of these investigations was done by Becker (1934), who identified the last part of Book IX of the *Elements* as a self-contained Pythagorean theory of even and odd numbers written long before 300 B.C.E., the traditional *floruit* of Euclid. Numerous other authors followed in Becker's footsteps. Our

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1. Introduction

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present, fairly complete, picture of the different contributions of pre-Euclidean mathematicians to the *Elements* is to a large extent due to the detailed studies of Neuenschwander on the geometrical books and of Mueller on the *Elements* as a whole. Heath and the other translators of the *Elements* provide valuable commentaries, but they are primarily concerned with individual definitions, theorems, and proofs. The global picture emerges from the investigation of the logical architecture of the *Elements* in combination with the study of other ancient sources.

The investigations of logical interdependence and mathematical structure are very valuable for the history of mathematics, but they are also very dangerous if they rely too heavily on modern theories as 'isomorphic' images of earlier mathematics. Nothing, however plausible, which treats an isomorphism as an identity can be considered a certainty. One must always observe the principle that we must not ascribe to thinkers, especially those of earlier times, 'either the principles of their consequences or the consequences of their principles'.¹ My own attitude towards the *Elements* is that of a mathematician who knows how pieces of mathematics are put together, transformed, and generalized in the process of writing. Above all I think one has to keep in mind that a textbook like the *Elements* is rarely a direct reflection of the emergence of a mathematical theory. Usually the heart of the matter is the mathematical origin; the careful arrangement, preparation, and logical ordering comes afterwards. That is why I will argue, for instance, that the construction of the regular pentagon is the core of Book IV of the *Elements* and that the classification of the regular solids provided the impetus to write Book IV. Such statements are only educated guesses based on one conception of how mathematics proceeds. Other investigations will present different perspectives; and in the continuing discussion we may gradually approximate a true account of the emergence of the *Elements*.

2. The contents of the *Elements*

Traditionally the *Elements* have been divided into three main parts:

1. Plane geometry, Books I-VI;

1 Cited by Burkert (1972), 405.

2. Arithmetic, Books VII-X;

3. Solid geometry, Books XI-XIII.

It will soon be obvious that Books V and X do not really fit into this division, but it is convenient to adhere to it in the following description of the individual books.

Book I: Foundations of plane geometry

Book I starts with a set of 'definitions'. Basic concepts such as point, line, angle are described in general terms and used to define various sorts of triangles, quadrangles, etc. The very last definition describes parallel lines in the plane as lines with no common point. After the definitions we find the so-called postulates, which we might call the axioms of geometry; the last of these is the famous parallel postulate. The 'common notions' are axioms concerning magnitudes in general, e.g., 'things equal to the same thing are equal to each other.'

The theorems of Book I can be grouped into four sets, which will be discussed in greater detail in §§3 and 4.

A. (I.1-26) Fundamental theorems and basic constructions in plane geometry such as the congruence theorems for triangles or the bisection of an angle; in part A no use is made of parallel lines.

B. (I.27-32) The theory of parallel lines, including the theorem that the sum of the interior angles of a triangle is equal to two right angles (I.32).

C. (I.33-45) The theory of parallelograms; transformation and comparison of areas of parallelograms and triangles.

D. (I.46-48) The theorem of Pythagoras.

Book II: The geometry of rectangles

Compared to Book I the second book is very homogeneous. Its two definitions say (i) that a rectangle is said to be contained by its sides and (ii) what a gnomon is. (See the shaded area in figure 1. The point S must lie on the diagonal of the rectangle, or, more generally, of a parallelogram.)

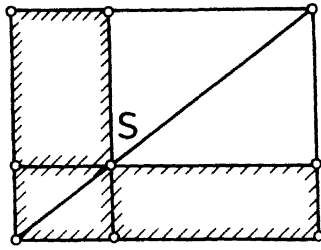


Figure 1

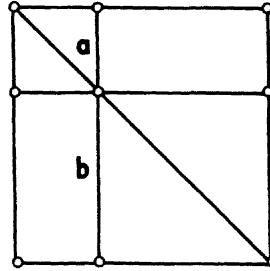


Figure 2

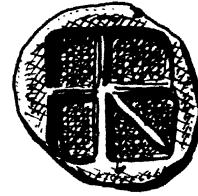


Figure 3

Most of the theorems of Book II are what we might call, in algebraic terms, variations on the theme of the binomial identity:

$$(a+b)^2 = a^2 + 2ab + b^2.$$

But the results are always expressed in the geometric language of subdivisions of rectangles and of the areas of the various parts of the subdivisions (see figure 2). A remarkable coin from Aigina (figure 3 [Kraay (1976) Nr.137; about 400 B.C.E.]) shows (except for the missing part of the diagonal) exactly the diagram of figure 2.² The theorems II.12 and 13 generalize the theorem of Pythagoras (I.47) to what we would call the law of cosines, and proposition II.14 gives the solution of the important problem of constructing a square equal to a given rectilinear figure. One might try to extract a stringent line of thought starting from parts C and D of Book I and culminating in II.14, but this would be very conjectural. Interpretation of some parts of Book II, especially II.5 and 6, has been the subject of much recent controversy centering on the concept of 'geometrical algebra', which I discuss in §4.³

Book III: The geometry of the circle

Book III has neither the obvious subdivisions of Book I nor the homogeneous structure of Book II. After some definitions it presents the basic

² For the details on coins and mathematics see Artmann (1990).

³ Fowler (1987) reads Book II as a collection of theorems auxiliary to the so-called anthyphairetic proportion theory first elaborated by Becker (1933). I discuss the anthyphairetic theory and Fowler's theses in Artmann (1988b).

geometrical facts about circles, tangents, and circles in contact. The second half of Book III is in part concerned with what one could call the basic facts about quadrangles and circles, including proposition III.21, which asserts the equality of all angles in the same segment of a circle. III.22 gives the first description of quadrilaterals in circumcircles:

III.22 The opposite angles of quadrilaterals in circles are equal to two right angles.

A second description is implicit in III.35-37, which, in modern terms, could be said to concern the power of a point with respect to a circle.

Book IV: Regular n -agons⁴ in circles

Book IV is by far the most homogeneous and tightly constructed book in the *Elements*. The following four problems are treated systematically:

- (i) to inscribe a rectilinear figure in or (ii) circumscribe it about a given circle;
- (iii) To inscribe a circle in or (iv) circumscribe it about a given rectilinear figure.

These problems are solved for

- (a) triangles in general (IV.2-5),
- (b) squares (IV.6-9),
- (c) regular pentagons (IV.10-14),
- (d) regular hexagons (IV.15),
- (e) regular 15-agons in (IV.16).

The mathematically most substantial achievement of this book is the construction of the regular pentagon in IV.10 and 11, for which much of the material of the preceding books is needed. (See §3b below.)

⁴ I use the term 'regular n -agon' for what Euclid calls (in particular cases) equilateral and equiangular n -agons.

*Book V: The general theory of magnitudes in proportion*⁵

Book V is the most 'abstract' book in the *Elements*. Whereas the other books are concerned with either geometrical objects or numbers, this one treats 'magnitudes', which include, according to Aristotle (*APo* I 5, 74a17-19), numbers, lines, solids, and times. In VI.33 angles are treated as magnitudes, and plane areas figure as magnitudes in VI.1, XII.1 and 2 (areas of circles), and in many other places.⁶ This generality makes the theory of proportions applicable throughout mathematics, so that one can see why Eratosthenes called it 'the unifying bond of the mathematical sciences'. (Proclus, *in Euc* 43.22-23; cf. Plato, *Tim* 31c and *Epin* 991ab) It has often been observed that in spite of this generality Euclid introduces in Book VII a special notion of proportionality for (natural) numbers and that he makes no effort to reconcile the two different definitions.⁷ I believe this situation reflects the fact that Euclid took his material from different sources and respected their internal logic; he was not attempting to construct a single overall coherent deductive system.⁸

Euclid's definition of proportionality (V, defs. 5 and 6) has often been compared to Dedekind cuts.⁹ Even if one overlooks the historical inaccuracy involved in rewriting Euclid's definition algebraically, there remains a fundamental difference: whereas Dedekind creates new numbers with his cuts, Euclid uses his definition only to determine the equality of given ratios. Dedekind looks at the real number system as a whole, Euclid at particular given ratios.¹⁰

Various sources indicate that Eudoxus is the creator of the theory of Book V.¹¹ The theorems in Book V, such as,

5 Beckmann (1967) gives a very thorough and detailed analysis of Book V.

6 Euclid, however, never treats numbers as magnitudes.

7 Plato suggests a similar distinction between proportions of numbers and of 'measures' at *Philebus* 25a.

8 For further discussion of this point see §3 below.

9 See, for example, the commentary on V, def. 5 in Euclid-Heath (1926), vol. 2, 124-126.

10 This point of view has been presented in more detail by Unguru and Rowe (1981), 37.

11 For references see, for instance, Euclid-Heath (1926), vol. 2, 112-113.

$$\text{V.16. } a : b = c : d \rightarrow a : c = b : d$$

were almost certainly used (with different definitions and proofs) a long time before Eudoxus. Becker (1933) tried to reconstruct a pre-Eudoxan 'anthyphairetic' theory of proportions on the basis of a remark by Aristotle (*Top* VIII 3, 158b29-35). Van der Waerden (1954), 132 discusses an even older definition of proportionality ascribed to Hippocrates of Chios (ca. 430 B.C.E.). Whether or not these specific reconstructions are correct, there can be no doubt that the theory of similar plane figures in Book VI is considerably older than Eudoxus' definition; consequently there must have been an earlier way to describe proportionality.

Book VI: The plane geometry of similar figures

In composition and general outlook Book VI is close to Book I. In fact, Books I, III, and VI represent the core of plane geometry, and their overall construction gives the impression of a standard treatment of geometry which has been reworked several times. The whole edifice of Book VI is based on theorem VI.1 and its immediate consequence VI.2, the fundamental theorem on the proportionality of lines. The proportion theory of Book V is applied immediately in the proof of:

VI.1. The areas of triangles under the same height are to one another as their bases.¹²

Note that in this theorem a proportion is established between magnitudes of different kinds, lines and areas.

The basic similarity theorem for triangles VI.4 and 5 is essentially a reformulation of VI.2. It is worthwhile to consider this theorem more closely. Let ABC and RST be triangles with sides and angles as indicated in figure 4.

¹² Euclid's formulation is slightly different.

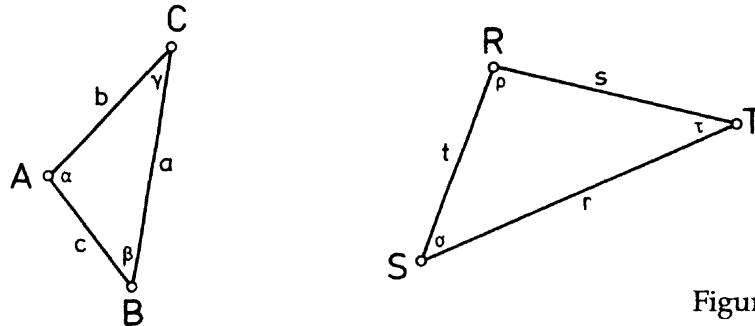


Figure 4

Proposition VI.4 and 5 state:

$$\left. \begin{array}{l} \alpha = \rho \\ \text{and} \\ \beta = \sigma \\ \text{and} \\ \gamma = \tau \end{array} \right\} \quad \text{if and only if} \quad \left\{ \begin{array}{l} b : c = s : t \\ \text{and} \\ c : a = t : r \\ \text{and} \\ a : b = r : s. \end{array} \right.$$

The essential point of this equivalence is that shape (as determined by the angles) can be expressed by means of proportions. At a time when incommensurable segments were unknown, the ratios of the segments could be expressed by numbers and consequently shapes of triangles — and more generally of figures composed from triangles — could be described by numbers. One might suspect that this theorem together with insights about harmonic intervals in music theory gave strong support to the Pythagorean doctrine that ‘everything is number’.¹³ Clearly, nothing of this kind can be found in the *Elements*, a work with no metaphysical content.

The mathematics of Book VI consists of sections on:

- A. (VI.1-8) Similarity of triangles,
- B. (VI.9-13) Lines in proportion,
- C. (VI.14-17, 32) Lines and areas in proportion,

¹³ The Pythagorean doctrine of numbers is discussed in detail by von Fritz (1971), 47-62 and Burkert (1972), Ch. I, §2.

D. (VI.18-23) Similarity for more general plane figures,

E. (VI.24-31) The application of areas.

The application of areas is only loosely connected with A-D and will be discussed in §4 of this paper.

Book VII: Basic arithmetic

The definitions at the beginning of Book VII are intended to serve for all of VII-IX. First come descriptions of the basic concepts of unit, of number, and of 'measuring' numbers; these definitions are analogous to the descriptions of point, line, etc. in Book I.¹⁴ There follows a group of definitions concerning even and odd numbers (defs. 6-10) which clearly belong to the theory of the even and odd in the second half of Book IX. Primes, composite numbers, and multiplication are defined in 11-15. Geometric shapes of numbers (e.g., 'squares' and 'cubes') are the subject of 16-19 and 21. Proportionality of numbers is defined in 20, and, finally, 22 says that a perfect number is the sum of its factors other than itself; this definition is invoked only at the end of Book IX.

Euclidean arithmetic is founded on the Euclidean algorithm for determining if two numbers are prime to one another (VII.1-4). In fact, the Euclidean algorithm gives the greatest common divisor (gcd) of any two numbers a, b . The next part of Book VII uses definition 20 to establish the fundamental properties of proportions for numbers, to a great extent proving again for numbers more general theorems established in Book V.¹⁵ The mathematical core of Book VII is the theory of the gcd (VII.20-32), which has its counterpart in the theory of the least common multiple (lcm; VII.33-39).

Book VIII: Numbers in continued proportion

Whereas Book VII, like Book I, has a rather clear internal structure, Book VIII and the first part of IX are entangled like Book III. Numbers in continued proportion are the main focus of Book VIII.1-10. The second

¹⁴ It should be kept in mind that the unit is not a number for Euclid, i.e., that the first number is 2.

¹⁵ Compare for example V.16 and VII.13.

part (VIII.11-27) is more concerned with special types of numbers ‘in geometrical shape’, such as squares and cubes. One important question in this context is how to characterize numbers a, b , for which there exists a mean proportional x , i.e., an x such that

$$a : x = x : b.$$

If a, b are squares r^2, s^2 , the answer is easy because

$$r^2 : r \cdot s = r \cdot s : s^2 \quad (\text{VII.11})$$

The general answer is given in VIII.18-20 using the concept of similar plane numbers, which is defined in VII, def. 21. The analogous problem for cube numbers and two mean proportionals is solved in VIII.19-21.

Book IX: Numbers in continued proportion; the theory of the even and the odd

There is no break in subject matter between Books VIII and IX; IX.1 picks up where Book VIII ends. Euclid’s other books have well defined subjects, but in this case the book division is artificial. It is even more curious that after IX.20 Euclid turns to a completely new subject, the theory of even and odd numbers (‘the even and the odd’), which has no connection with what precedes, but rests only on definitions 6-10 of Book VII. The theory culminates with the construction of even perfect numbers (IX.36).¹⁶

Prime Numbers in Euclid’s arithmetic. Prime numbers are the multiplicative building blocks of the natural numbers (positive integers). The starting point of modern number theory is the fundamental theorem of arithmetic, which says that every natural number has a unique factorization into primes. Euclid’s definition of prime numbers is:

VII, def. 11. A prime number is that which is measured by a unit alone.

¹⁶ The origin of the concept of a perfect number is discussed by Taisbak (1976), who puts it, contrary to Becker (1934), in the context of Book VII. For a general discussion of the prehistory of Greek arithmetic see Waschkies (1989).

In other words, p is prime if p has no divisors other than 1.¹⁷ One of Euclid's most famous theorems asserts that there are infinitely many primes:

IX.20. Prime numbers are more than any assigned multitude of prime numbers.

In modern notation Euclid's proof is as follows:

Let A, B, C be the assigned prime numbers; I say that there are more prime numbers than A, B, C . For consider the number $D = A \cdot B \cdot C + 1$. It is either prime or has a prime factor (by VII.32). If D is prime, then, since it is different from A, B, C , we have another prime number. If D is not prime, then it will have a prime factor G . Since G divides D , it cannot be equal to one of A, B, C , each of which divides $A \cdot B \cdot C$ and therefore does not divide $A \cdot B \cdot C + 1$; hence again we have found a new prime number.

IX.20 is situated between two sections of Book IX which have quite different subjects. The theory of continued proportions comes to an end with IX.19. The new theory of the even and the odd starts at IX.21. The proof of IX.20 makes some use of theorems from Book VII, but it has no connection with the material in Books VIII and IX. It seems then to be an isolated fact.

In proving IX.20 Euclid uses the following theorem:

VII.32. Any number is either prime or measured by some prime number.

In modern theories, this statement is used to establish the existence of a factorization of any number into prime factors. For the uniqueness of this factorization another lemma is needed. Euclid formulates it as:

VII.30. If two numbers by multiplying one another make some number, and any prime number measure the product, it will also measure one of the original numbers,

i.e., in modern terms:

¹⁷ p itself is not considered to be a divisor of p by Euclid. The unit is not a prime number in Greek mathematics, since it is not a number at all.

If the prime number p divides the product $a \cdot b$, it will divide one of the factors a, b .

The closest Euclid comes to actually proving the fundamental theorem is in IX.12-14.

IX.12. If $1 : a_1 = a_1 : a_2 = \dots = a_{n-1} : a_n$ and the prime p divides a_n , then it must divide a_1 .

If, for instance, a_1 is a prime p , no other prime than a_1 can divide $a_n = p^n$ (IX.13).

IX.14. If $p_1, \dots, p_k$ are prime numbers and $a = \text{lcm}(p_1, \dots, p_k)$, then no prime different from $p_1, \dots, p_k$ will divide a .

This gives us the uniqueness of the prime factorization of $p_1 \cdot \dots \cdot p_k$ (if $p_1, \dots, p_k$ are all different), and, in fact, the proof of IX.14 uses VII.30 just as a modern proof of the fundamental theorem would. Thus Euclid comes rather close to the fundamental theorem of arithmetic, but he never takes the last step of asserting it. The situation here is typical of the history of mathematics: all the tools needed to formulate an important insight are present, but the insight is not formulated.

Book X: Incommensurable line segments

Book X is the most voluminous book of the *Elements*, occupying about one quarter of the whole work. In it the Euclidean algorithm of Book VII is adapted to magnitudes in general in order to get the criterion for commensurability:

X.5. Commensurable magnitudes have to one another the ratio which a number has to a number.

X.6. If two magnitudes have to one another the ratio which a number has to a number, the magnitudes will be commensurable.

In X.9 Euclid states as an immediate consequence of this that the side of a square of area n is incommensurable with the side of a square of area 1 when n is not a square number.¹⁸ The bulk of the material of Book

18 There is a serious gap in his argument; see Euclid-Heath (1926), vol. 3, 31.

X, up to proposition 115, consists of a careful study of various types of incommensurable lines which cannot possibly be described here. The reader interested in more detailed information about Book X should consult Mueller (1981), chapter 7.2, Taisbak (1982) and Knorr (1985).

Book XI: Foundations of solid geometry

Book XI begins with a long set of definitions for Books XI-XIII. There are no postulates of the kind we find in Book I, so that there is no foundation for Euclid's deductions at the beginning of Book XI.¹⁹ The general composition of Book XI is very parallel to that of Book I. It has the following sections:

- A. (XI.1-19) Fundamentals of solid geometry (lines, planes, parallelism and orthogonality).
- B. (XI.20-23) Solid angles, their properties and construction.
- C. (XI.24-37) Parallelepipedal solids.

I discuss the similarities and differences between Books I and XI in §3c.

Book XII: Areas and volumes; Eudoxus' method of exhaustion

Some infinitesimal method is needed to determine the area of a circle in relation to a square, or the volume of a pyramid. The method of exhaustion, which Euclid employs, is said to have been first applied rigorously by Eudoxus, to whom most of the contents of Book XII are attributed.²⁰ The method of proof is quite different from — and much more intricate than — anything in the preceding geometrical books. In my discussion I shall focus on the treatment of the circle and the sphere. Euclid first proves:

- XII.1. Similar polygons inscribed in circles are to one another as the squares on the diameters.

19 The logical difficulties in the proofs of the first theorems of Book XI are pointed out in Heath's commentary in Euclid-Heath (1926), vol. 3. See also the remarks on solid geometry and on arithmetical elements below.

20 For the sources see e.g., Euclid-Heath (1926), vol. 3, 365-368 or Neuenschwander (1975).

He then approximates the area of a circle by inscribing in it successively larger polygons, and proves:

XII.2. Circles are to one another as the squares on their diameters.

That is, if two circles have areas A_1, A_2 and diameters d_1, d_2 or (to use the modern convention) radii r_1, r_2 , then:

$$A_1 : A_2 = d_1^2 : d_2^2 = r_1^2 : r_2^2.$$

Consequently,

$$A_1 : r_1^2 = A_2 : r_2^2,$$

so that the ratio of the area A of a circle to the square r^2 on its radius is constant. This constant is now called π , and we write:

$$A : r^2 = \pi, \text{ or } A = \pi r^2.$$

The second (and secondary) question is to determine the constant π as precisely as possible. In his *Measurement of a Circle* Archimedes established:

$$3 \frac{10}{71} < \pi < 3 \frac{10}{70}.$$

Propositions XII.3-15 are concerned with pyramids, cones, and cylinders. The final section of Book XII deals with spheres and establishes:

XII.18. Spheres are to one another in the triplicate ratio of their respective diameters.

We may express this result by saying that if V_1, V_2 are the volumes of two spheres with diameters d_1, d_2 and radii r_1, r_2 , then:

$$V_1 : V_2 = d_1^3 : d_2^3 = r_1^3 : r_2^3,$$

or, as before, for some constant k ,

$$V : r^3 = k$$

One of the major achievements of Archimedes (*On the Sphere and the Cylinder*, prop. 34) establishes the following relation between this constant k and π :

$$k = \frac{4}{3} \pi.$$

Book XIII: The Platonic Solids

The first part of Book XIII, that is, XIII.1-12, consists of various planimetric propositions. Some of them are evidently lemmas for the subsequent theory of the regular polyhedra, and some others are concerned more generally with the division of a straight line in extreme and mean ratio (the golden section).²¹ Because division in extreme and mean ratio is indispensable for the constructions of the icosahedron and the dodecahedron, we may regard the whole first part of Book XIII as preparatory for the second.

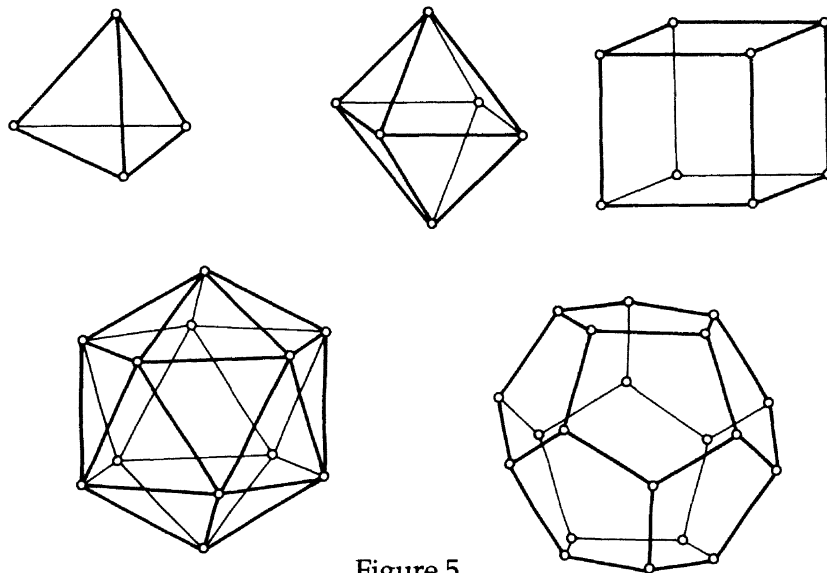


Figure 5

Each of the regular solids is treated in a separate theorem with two parts:

- (i) To construct the solid and to comprehend it in a given sphere.
- (ii) To compare the diameter of the sphere with the side of the polyhedron.

²¹ For the definition see §3b below.

Elaborate constructions are required for the first parts, especially in the case of the icosahedron and the dodecahedron. For the second parts, the classification of irrational lines from Book X is used to describe the (lengths of) the lines (edges) of the icosahedron and the dodecahedron; the other three cases admit simpler descriptions.

Euclid's treatment of the regular polyhedra is especially important for the history of mathematics because it contains the first example of a major classification theorem. Such theorems start with some definition (or axiomatic description) and end up with an explicit list of objects satisfying the description. There is no official definition of a regular solid in the list of definitions at the beginning of Book XI, but only definitions of cube, octahedron, icosahedron, dodecahedron (XI, defs. 25-28) and of a general type of pyramid or tetrahedron (XI, def. 13). There is, however, a definition of regularity in the last theorem (XIII.18a) of the *Elements*, which asserts the completeness of the preceding list of polyhedra:

XIII.18a. No other figure, besides the said five figures, can be constructed which is contained by equilateral and equiangular figures equal to each other.

Euclid apparently regards a solid as regular if its faces are congruent regular polygons. This is essentially the modern definition except for two omissions: the requirement that the polyhedron be convex, and the specification that the solid angles (vertex figures) of the polyhedron be congruent. Euclid probably took the first of these conditions for granted, but overlooked the existence of the kinds of convex polyhedron illustrated in figure 6, which falsify proposition 18a.²²

22 As was pointed out by Freudenthal and van der Waerden (1946). One might add the condition that the solid have a circumsphere instead of insisting that the vertex figures be equal. This condition is implicit in Euclid's procedure in XIII.13-17. Plato seems to have such a description of a regular solid in mind when he speaks (*Tim* 55a) of a solid 'that gives rise to a subdivision of the (circum-)sphere into equal and similar parts'. Similarly Proclus (*in Euc* 202.15-18) speaks of the problem of inscribing in a sphere 'polyhedra with equal sides and angles and composed of similar faces', which again makes the existence of a circumsphere part of the definition. Adding this condition to 18a would make the theorem true but complicate its proof.

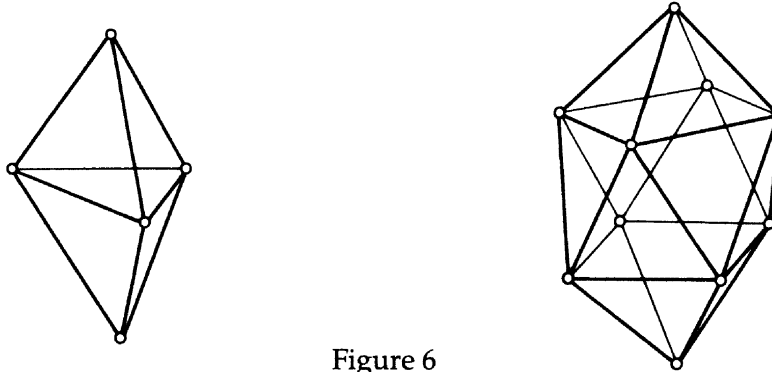


Figure 6

Proclus (*in Euc* 71.22-24) states that the aim of the *Elements* is 'both to furnish the learner with an introduction to the science as a whole and to present the construction of the several cosmic figures [the regular solids].' This characterization is too humble. The *Elements* contains much more than a first introduction to mathematics, and, as we have seen, the regular solids are one among several highlights. I shall discuss the origins of this great book in the next section.²³

3. On the prehistory of Euclid's *Elements*

What historical developments lie behind the *Elements*? To begin to answer the question, let us first consider people who wrote on mathematics before Euclid. One source of information here is Diogenes Laertius, who gives lists of writings for the philosophers he discusses. These lists include some mathematical works:

Simon the Shoemaker (*ca.* 440): *Mathēmata*, about number;

Democritus (*ca.* 420): several books on geometry and arithmetic;

²³ I do not discuss the development of the theory of the regular polyhedrons, for which see Waterhouse (1972), Sachs (1917) and Artmann (1984). Roughly it seems that Theaetetus, starting from one particularly interesting example, the dodecahedron (of Hippasus the Pythagorean), discovered the general concept of a regular solid, and, adding octahedron and icosahedron to the well known tetrahedron (pyramid), cube, and dodecahedron, succeeded in giving the complete classification of all possible examples which we find in *Elements* XIII.

Crito, the friend of Socrates (ca. 430): 'Elements' (of what?);

Simmias of Thebes (ca. 400): 'Elements';

Speusippus (ca. 350): *Mathēmatikos*;

Heracleides Ponticus (ca. 340): books about geometry;

Xenocrates (ca. 330): 5 books on geometry, a book about numbers, one on the theory of numbers, and two more books on geometry;

Theophrastus (ca. 330): books on geometrical researches and on arithmetical researches.

These are books of philosophers or mathematically inclined philosophers rather than of mathematicians proper, with whom Diogenes Laertius is not concerned.²⁴ We get information about mathematicians in the strict sense from Proclus. In his famous quotation from Eudemus' history of mathematics (*in Euc* 65-68) Proclus identifies Hippocrates of Chios as the first person recorded to have written elements. After him Leon, a member of Plato's Academy 'was able to compile a book of elements more carefully designed to take account of the number of propositions and of their utility.'... 'Eudoxus of Cnidus, a little later than Leon and a member of Plato's group, was the first to increase the number of the so-called general theorems.' Amyclas of Heracleia, Menaechmus, and Dinostratus, the brother of Menaechmus, 'made the whole of geometry still more perfect.' 'Theudius of Magnesia... produced an admirable arrangement of the elements and made many partial theorems more general.' 'Hermotimus of Colophon... discovered many propositions in the elements'... 'Not long after these men came Euclid, who brought together the elements...' This short passage gives us quite a bit of information on the predecessors of Euclid, and we will see that in some instances Eudemus' remarks fit nicely with what we can gather from the analysis of the *Elements* itself.

After describing Euclid's works, Proclus goes on to talk about the possible meaning of the word 'element' in the sense of 'elementary treatise' and continues:

24 We get a sense of the separation of mathematicians from philosophers in general in the *Theaetetus*. There (165a) Theodorus of Cyrene, friend of Socrates and mathematical teacher of Plato, declines to defend his teacher Protagoras, and declares that he himself left philosophy at an early age to become a mathematician.

This is also the meaning the word has in *numerous compositions in arithmetic and astronomy* called elementary treatises. It is a difficult task in any science to select and arrange properly the elements out of which all other matters are produced and into which they can be resolved. Of those who have attempted it [for geometry] some have brought together more theorems, some less; *some have used rather short demonstrations, others have extended their treatment to great lengths; some have avoided the reduction to impossibility, others proportion*; some have devised defenses in advance against attacks upon the starting-points; and in general many ways of constructing elementary expositions have been individually invented. (*in Euc* 73.12-25. Morrow translation, my italics)

Proclus mentions three specific ways of writing elements of which we might try to find traces in Euclid's work:

- (i) shorter or longer proofs;
- (ii) no proofs by contradiction;
- (iii) proofs without using proportion.

There may be traces of variations in length in the stereometric books XI and XII.²⁵ The Arabic tradition seems to have generally preserved shorter proofs. From Proclus' remark it seems probable that he knew versions of elements which survive only in Arabic.²⁶ With respect to (ii), it is very hard to say if a proof avoiding contradiction would be more 'natural' than one using it. I.48, the converse of the theorem of Pythagoras, is a case in which Euclid gives a direct proof instead of a longer, indirect one. On the other hand, proofs by contradiction are quite common in the *Elements*. They are given for I.27, III.16, V.10, VI.7, VII.20, IX.30, to mention just a few examples. So one cannot say Euclid provides any striking evidence concerning (ii). This leaves us with (iii), proofs without proportion, and here we find some distinct traces in Euclid's work which merit more extended treatment.

25 See Euclid-Thaer (1965). Neuenschwander (1975) discusses this question in detail.

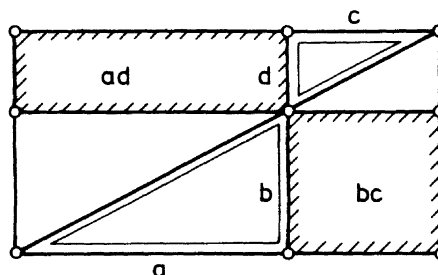
26 We still do not have critical editions of some important Arabic texts.

3a. 'Elements' without proportions

Because Euclid defines proportions and studies their properties for the first time in Book V, he cannot use proportions in the first four books. However, these books include some theorems with curiously involved proofs, which could be proved much more simply using proportion. Closer inspection reveals a standard way of transforming statements about proportions into statements about areas using VI.16, which I state in abbreviated form as:

$$a : b = c : d \leftrightarrow a \cdot d = b \cdot c$$

Figure 7



This equivalence makes it possible for Euclid to avoid using proportionality, for example in the construction of the mean proportional x for two straight lines a, c . Euclid carries out this construction using proportion in VI.13 by means of a right triangle in a semicircle (see figure 8), arguing that since BEH and HEG are similar triangles:

$$a : x = x : c.$$

In II.14, however, Euclid proves $a \cdot c = x^2$ using essentially the same figure (see figure 9) but a quite different argument:

Let $a = BE$ and $c = EF$. Then by II.5 one has $BE \cdot EF + EG^2 = GF^2$.

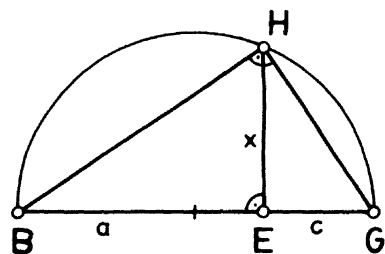


Figure 8

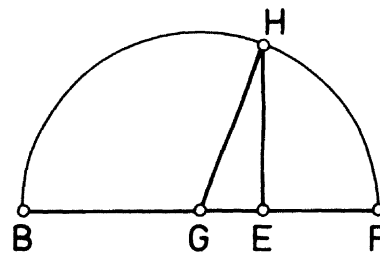


Figure 9

By construction $GF = GH$ and by the Theorem of Pythagoras

$$GF^2 = GH^2 = GE^2 + EH^2;$$

hence

$$BE \cdot EF + EG^2 = EH^2 + EG^2,$$

and,

$$BE \cdot EF = EH^2.$$

Several other proofs follow the same pattern: a clever combination of II.5 or 6 with the theorem of Pythagoras gives the desired result without invoking proportionality.

In Artmann (1985) I present a detailed study of the Books I-IV, based on the investigations of Neuenschwander (1972) and Mueller (1981). Logical and stylistic observations confirm the conclusion that, essentially, Euclid's Books I - IV are the 'Proportion Free Elements' mentioned by Proclus. However, these *Elements* were probably altered in at least two ways before being incorporated in Euclid's work. The first is the addition of the theory of parallels and the consequent modifications of the beginning of Book I. The second is the treatment of parallelograms, also in Book I. In Artmann (1985) I tentatively identified Theudius as the author of the 'Proportion Free Elements', but the arguments for this identification are not conclusive. Whoever the author of the 'Proportion Free Elements' was, his aim was to go as far as possible in geometry without proportions. There is no reason to think that he added new theorems; he merely gave new proofs for known results. That this was no mean achievement can be seen, for instance, in III.35 or in the construction of the regular pentagon in IV.10 and 11.²⁷

3b. Book IV and the regular pentagon

I have already described the rigorous and systematic plan of Book IV. One can infer from this tight plan that the book is a monograph written by a single author. Moreover, Book IV is a sort of end-point in the

²⁷ It is worth pointing out that Hilbert (1899) succeeded in developing the whole of similarity geometry from the axioms of congruence, bringing to completion the ideas of 'Theudius'.

composition of the *Elements*. It uses much of the material of the preceding books, but its contents are not really used subsequently. It is especially noteworthy that in Book XIII Euclid implicitly constructs the pentagon a second time in connection with the regular dodecahedron. On the surface there are no obvious seams in Book IV, but Euclid does appear to have deviated from the original monograph in two ways. First, he has reworked the construction of the regular pentagon to eliminate the use of proportions. And second he does not use the first two definitions of the book, which he presumably preserves from the original monograph.²⁸

In what follows I will first describe the status of the main problem of Book IV in modern terms, then try to sketch some stages in the historical development of the book, and finally comment on the problem of inscribing a circle in a regular n -agon.

Regular n -agons. The basic problem of Book IV is the construction of the regular n -agon in a given circle by ruler and compass. We find an implicit construction for $n = 3$ (in IV.2 combined with I.1) and explicit ones for $n = 4, 5, 6$ and 15 (in IV.6, 11, 15, and 16).

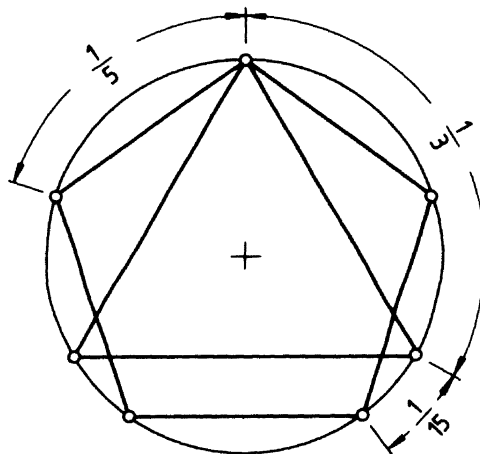


Figure 10

28 Cf. Heath's commentary on definition 22 of Book I (Euclid-Heath (1926), vol. 1, 189).

Implicit in the solution for $n = 15$ is a general principle: if we are able to construct the regular r - and s -agon, and, moreover, we know integers x, y such that $x \cdot r + y \cdot s = 1$, then we can construct the $r \cdot s$ -agon as well. Euclid does this for $r = 3, s = 5, x = 2$, and $y = -1$. Since, for any r, s with $\gcd(r, s) = 1$, we can use the Euclidean Algorithm to find x, y with the desired properties, we are easily led to item (iii) listed below. This general formulation is, of course, modern. The author of Book IV has no reason to formulate the general principle (iii) because he needs only the specific case he deals with. The same is true for the trivial observations (i) and (iv). Bearing this point in mind, we can say that the author of Book IV 'knew' the following facts about regular n -agons:

- (i) For all $n > 1$, the 2^n -agon is constructible;
- (ii) The 3- and the 5-agon are constructible;
- (iii) If the r - and s -agon are constructible and $\gcd(r, s) = 1$, then the $r \cdot s$ -agon is constructible;
- (iv) If the n -agon is constructible and k divides n , then the k -agon is constructible.

Given (i) - (iv), the problem for general n is reduced to prime powers p^i for odd primes p . Gauss (1777-1855) decided to become a mathematician instead of a classical philologist at the age of 17 when he succeeded in constructing the regular 17-agon. Two years later he proved

- (v) The p^i -agon is constructible if and only if $i = 1$ and p is a prime number of the form $p = 2^{2^k} + 1$, i.e., if p is a so called Fermat-prime.

This is where the problem stands today. Except for the first few values of k , it is not known if $2^{2^k} + 1$ is prime or not.

The Pentagon and the Prehistory of Book IV. Two scholia to Book IV²⁹ say that its theorems are the discovery of the Pythagoreans. It is not known whether the author of the present form of Book IV was a Pythagorean. In any case, the theorems are rather easy except for the construction of the pentagon, which gives the book its *raison d'être*.

29 2 and 4. These are discussed by Neuenschwander (1972), 372.

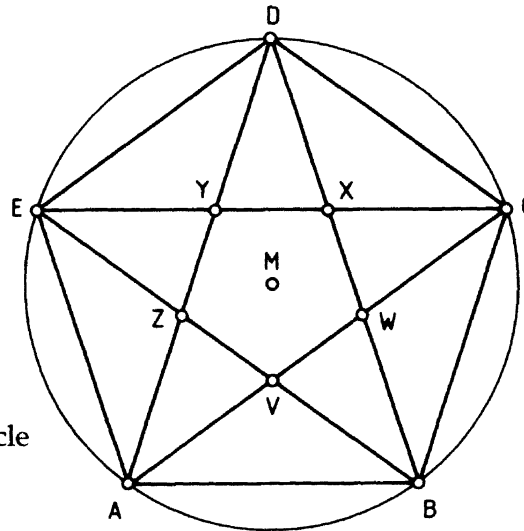


Figure 11. Regular Pentagon with circumcircle and diagonals

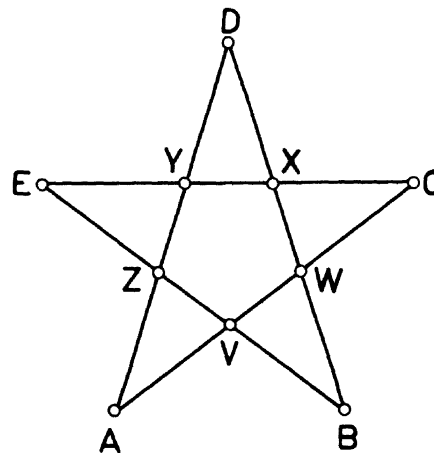


Figure 12. Pentagram

The pentagram/pentagon is commonly assumed to have been a Pythagorean symbol.³⁰ Hofmann (1926) describes a relatively simple construction of the regular pentagon using verging lines (*neusis*). In

30 For details see Burkert (1972), 176 or von Fritz (1945). Compare also the coin from Metapontum in figure 16.

figure 13 the point Z lies on the circle k , and D lies on the perpendicular bisector m of the segment AB ; furthermore the distance ZD is equal to AB . Starting from AB one constructs m and k and slides the line l into position so that D is on m , Z on k and l passes through A . The completion of the pentagon is then easy.

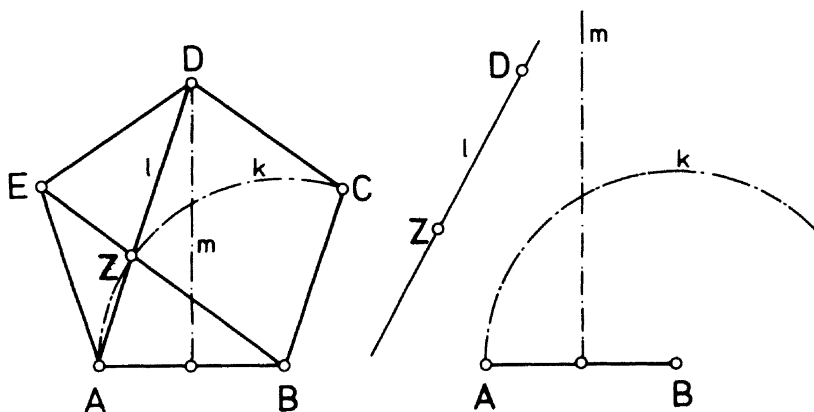


Figure 13

This construction has as a by-product the division of a line in extreme and mean ratio (golden section), which is intimately related to the pentagon and the dodecahedron.³¹ Euclid describes the golden section in:

Theorem XIII.8 (abbreviated). Two diagonals in a regular pentagon cut each other in extreme and mean ratio. (See figure 14.)

³¹ Cf. Artmann (1984).

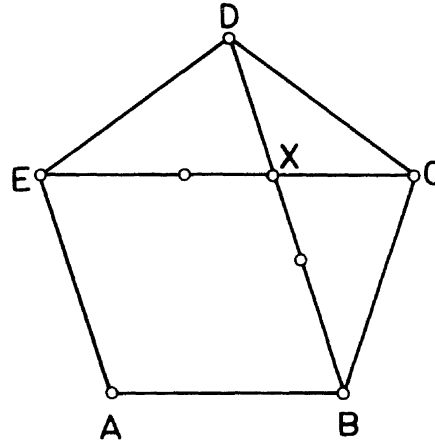


Figure 14

From the similarity of the triangles BCD and CXD Euclid concludes via the basic similarity theorem for triangles (VI.4 and 5):

$$DX:DC = DC:DB$$

Because $XB = DC$,

$$DX:XB = XB:DB.$$

That is, the smaller segment DX is to the larger segment XB as the larger segment is to the whole line DB, which is to say that DB is divided in extreme and mean ratio (VI, def. 3).³² I suggest that the discovery of the relation of the pentagon (or better of the dodecahedron) to the division in extreme and mean ratio and some construction like the one described above were the first steps towards the contents of Book IV. Since we know that the Pythagorean Hippias was interested in the dodecahedron³³, this step could well have been taken by one of the early followers of Pythagoras.³⁴

It seems that at some time and for some unknown reason *neusis* constructions were decided to be unacceptable. The construction of the pentagon becomes difficult if only ruler and compass are permitted for

³² Herz-Fischler (1987) gives a very detailed history of division in extreme and mean ratio.

³³ See von Fritz (1945) or Artmann (1984).

³⁴ See also the early fifth century coin from Chalcis in figure 15.

geometrical constructions.³⁵ XIII.8 reduces the construction of the pentagon to the problem of cutting a line in extreme and mean ratio. Within the framework of Greek mathematics, this division of the line has to be done by application of areas.³⁶ According to Eudemus (Proclus, *in Euc* 419.15-18), the application of areas is an ancient discovery 'of the Pythagorean muse'. Hence there is good reason to believe that the construction of the pentagon (using proportions) with ruler and compass is due to Pythagorean mathematicians.

For the third stage in the history of Book IV I assume that somebody wrote a systematic monograph, still using some type of proportion theory, but essentially giving us the text we find in the *Elements*. Such a treatise was probably written on the basis of some 'Elements' rather than being incorporated into such a text. I speculate that the writing of a systematic text on regular n -agons could have been motivated by Theaetetus' theory of the regular solids, so that the third stage could be located in the context of the Platonic Academy. The fourth stage of our story would be the writing of the new proof of IV.10 not using proportionality, and the incorporation of the book into the 'Elements Without Proportions', from which it passed into Euclid's *Elements*.

Some Greek coins are of interest for the history of the pentagon.³⁷



Figure 15

Franke and Hirmer (1964)

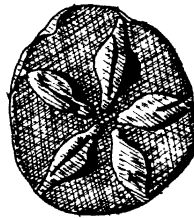


Figure 16

Kraay (1976), Nr.S95



Figure 17

Kraay (1964), Nr. 3

35 Steele (1936) discusses the restriction to ruler and compass constructions but offers no convincing explanation for it.

36 I discuss application of areas in detail in §4. Euclid uses the method to divide a line in extreme and mean ratio in VI.30. He makes the same division in II.11 without using proportion, but, as I will show, that construction can only be understood by reference to VI.30.

37 A systematic description of coins with mathematical motifs is given in Artmann (1990).

Figure 15 shows a five-spoked wheel on a coin from Chalcis of about 480 B.C.E. Five spokes would make a very awkward design for a real wheel, which usually had 4, 6, or 8 spokes. Figure 16 shows a coin of about 440 B.C.E. from Metapontum, known to have been a Pythagorean stronghold until about 440. The coin provides strong numismatic confirmation of the Pythagorean use of the pentagon/pentagram. Finally in figure 17 we have a pentagram on a coin of ca. 420 B.C.E. from Melos. The coins from Chalcis and Melos suggest that non-Pythagoreans were also interested in the pentagon, but we know that not all Greek mathematicians were Pythagoreans. In any case the dates of the coins testify to an interest in the pentagon long before Euclid.³⁸

The construction of the incircle. In contrast to the problem of inscribing a regular n -agon in a given circle, which has to be solved for each n individually, the construction of an incircle in a regular n -agon has a general solution, as does the construction of a circumcircle. The procedure of IV.13, where Euclid inscribes a circle in a regular pentagon, is generalizable since the proof of IV.13 does not make any real use of the fact that the regular n -agon being dealt with is a pentagon. Euclid knows this, because he says in VI.15 for the hexagon and in IV.16 for the 15-agon that the construction can be carried out as in the case of the pentagon.³⁹

3c. 'Elements of solid geometry'

Neuenschwander (1975) noted close affinities between Books I, VI, XI and XII. Analogies in the structures of the Books I and XI were pointed out by Mueller (1981), 207. I summarize here the results of a systematic inspection of the relevant books carried out under the guidance of these ideas (Artmann (1988a)). I assume, as in the discussion of Book IV, the existence of a standard treatment of plane geometry including a theory of proportions. The 'new' definition of proportion in V and the theory

38 Schouten (1968) studies the pentagram as a medical symbol. According to Käppel (1989) the design of the theater at Epidauros is based on the pentagon.

39 There is an analogous situation in the proof of IX.20 described in §2, where Euclid takes three prime numbers instead of an arbitrary finite collection. In both cases he takes a definite example but carries out the proof in such a way that the general procedure becomes obvious. (Similarly in II.1, V.12, VI.20, VII.33, VIII.6, IX.8, and elsewhere.)

of volumes in XII are generally assigned to Eudoxus.⁴⁰ Probably a treatise on solid geometry was worked out by his disciples. It would have contained the following parts of Euclid's *Elements* :

1. The theory of proportions as in Euclid's Book V.
2. The adaptations of similarity geometry made necessary by the new theory of proportions; for this a new proof of the fundamental theorem VI.1 would suffice.
3. Some transformations of the older theory of rectangles into a theory of parallelograms. (Book I, part C and some theorems in Book VI.)
4. The foundations of solid geometry (XI.1-23).
5. The theory of parallelepipedal solids of Book XI, which is very closely related to the respective parts of the theory of parallelograms.⁴¹
6. Book XII.

Euclid could have taken this treatise and incorporated it into the *Elements* with only very small modifications. He would have had to move the first steps in the theory of parallelograms into Book I. In fact, the generalization from rectangles to parallelograms is superfluous for mathematical purposes prior to Book XI, and disturbs rather than clarifies the line of thought. Parallelograms are forced on Euclid by mathematics itself only in XII.3, when he wants to determine the volume of a pyramid.

The first part of Book XI is needed not only for Book XII, but for Book XIII as well. Neuenschwander (1975), 118-120 points out that most of XI.1-23 must have been known to Theaetetus. But although the systematic treatment of these propositions may be new, some essential theorems of the second part of XI must be much older than the theorems of Eudoxus in Book XII. The most prominent example is:

40 For detailed discussions of the sources see Beckmann (1967) and Neuenschwander (1975).

41 It is worth pointing out that Euclid introduces both parallelograms and parallelepipedal solids without defining them.

XI.23. Similar parallelepipedal solids are to one another in the triplicate ratio of their corresponding sides.

If we here replace the 'later' term 'parallelepipedal solids' with the 'earlier' one 'cubes', XI.23 is reduced to a theorem ascribed to Hippocrates of Chios:⁴²

Cubes are to one another in the triplicate ratios of their sides.

The proofs of these two propositions are almost identical and have an identical counterpart in the proof of the corresponding theorem for similar solid numbers (VIII.19).

3d. The background of plane geometry

In discussing both the treatise on solid geometry and the monograph on regular n -agons I assumed a text on plane geometry as a sort of standard reference for basic theorems and constructions. What can we say about the contents of such a text? Most of part A of Book I, that is I.1-26, is indispensable. Van der Waerden (1978) thinks that most of this material was already present in the very first 'Elements' written by Hippocrates of Chios. Eudemus (Proclus, *in Euc* 379.2-18) reports on a Pythagorean proof of I.32. Neuenschwander (1973) shows how the 'squaring' of a general rectilinear plane figure can be done within the framework of a geometry of triangles and rectangles, so this part of Book I (together with II.14) is probably rather old, too. On the other hand the theory of parallels and parallelograms were not much anterior to Euclid.⁴³

The geometry of the circle is needed for many applications. Except for the proofs of III.35 and 36 the whole of Book III seems to be old.⁴⁴ Neuenschwander (1972), 369 also conjectures that the geometry of similarity stood right at the beginning of a pre-Euclidean Elements. I agree with this and assume that these Elements must have contained a substantial part of Book VI. An indication that the concept of similarity was historically prior to that of congruence is Euclid's phrase 'equal

⁴² See van der Waerden (1954), 136.

⁴³ For a general discussion of the problem of parallels see Toth (1967).

⁴⁴ For details see Neuenschwander (1972), 374-378.

and similar' (XI, def. 10) for congruent solids. Whereas he defines similarity in VI, def. 1, he has no definition for congruence of plane figures. The gap in the proof of VI.22, which is corrected by a lemma interpolated later, may also be due to an older version using 'equal and similar'.⁴⁵ Plato uses (*Tim* 55a) 'equal and similar' for the congruent faces of the regular polyhedron (or for the corresponding parts of the circumsphere). We may conclude, then, that the early foundation for plane geometry consisted of most of the contents of the Books I, III and VI. We have, however, no way of being more precise about the contents of the 'Elements' of Leon in Plato's Academy or of the first 'Elements of Geometry' written by Hippocrates of Chios.⁴⁶

3e. Elements of arithmetic

The arithmetical books of the *Elements* have been thoroughly analyzed by van der Waerden (1947), Becker (1934) and Taisbak (1971). Becker isolated the theory of the even and the odd in Book IX.21-36 and modified some of its proofs so that it became entirely independent of the preceding material. He was the first person to identify a substantial part of the *Elements* as a self-contained work of an earlier author. His approach to the text was very influential in subsequent treatments of the *Elements*.

Van der Waerden (1947) postulated three different authors for Book VII, Book VIII, and the first part of Book IX. Comparison of Book VIII with theorems of the Pythagorean Archytas of Tarentum known from other sources led van der Waerden to the highly probable conclusion that Archytas was the author of Book VIII. Books VIII and IX A (1-20) are written in very different styles. IX A contains advanced propositions and, according to van der Waerden, was written in Plato's Academy. Since much of the mathematics in Book VII is needed in VIII, van der Waerden inferred that VII must have been known to Archytas. Archytas,

⁴⁵ Cf. Euclid-Heath (1926), vol. 2, 242-247.

⁴⁶ I have not discussed a pre-Euclidean *theory* of proportions, but simply assumed the laws of proportion to be known. A strong case has been made by Becker (1933) and many subsequent authors for the existence of an 'anthyphairetic' theory of proportions using the Euclidean algorithm to determine proportionality. Aristotle provides evidence for the existence of such a theory or definition, but the reconstruction of the theory can only be conjectural. For discussion see the references in note 3.

however, could not be its author because VII and VIII differ so much in their structure and logic. Thus, according to van der Waerden, Pythagoreans before Archytas are the most probable authors of Book VII.

Van der Waerden considered Book VII to be a homogeneous, perfect whole with an impeccable logical architecture. His judgment concerned the mathematical content of Book VII. Malmendier (1975) and Mueller (1981), chapter 2 directed their attention to the definitions and fundamental concepts of Euclid's arithmetic, where there are several important anomalies. Hankel (1874) and Zeuthen (1896) pointed out two different descriptions of the multiplication of integers. Definition 15 describes $a \cdot b$ asymmetrically as the a -fold repeated addition of b ; accordingly the commutativity of multiplication is proved in VII.16. On the other hand, definition 16 describes a plane number $a \cdot b$ as the (symmetric) product of its 'sides' a and b .⁴⁷

As in the case of Euclid's solid geometry, one looks in vain for substantial arithmetical axioms in Book VII. The starting point of both theories seems to be a kind of Platonic or Socratic thinking for which the foundation of reasoning is correct definitions of the object under discussion. Even if one makes generous allowance for tacit assumptions about the properties of the integers, most of VII.5-10 are best regarded as axioms rather than as propositions, according to Malmendier and Mueller. These propositions, which use the terminology of 'part' and 'parts', are rephrased in the immediately following VII.11-14 using the word 'proportion'. The rephrasing is illustrated by the following two propositions:

VII.10. If a number be parts of a number, and another be the same parts of another, alternately also, whatever parts or part the first is of the third, the second also be the same parts or the same part of the fourth.

VII.13. If four numbers be proportional, they will also be proportional alternately.

In modern terms, both propositions mean:

If $a : b = c : d$, then $a : c = b : d$.

⁴⁷ Multiplication of numbers is already contained implicitly in the definition of a part of a number (VII, def. 3): a is part of c , if it 'measures' c (i.e. if $c = n \cdot a$).

As Taisbak (1971), 33 has pointed out VII.5-10 look like an attempt to provide a foundation, using VII, def. 20, for the previously known laws VII.11-14. It is, however, not clear what, if any, previous foundation these laws were given.⁴⁸

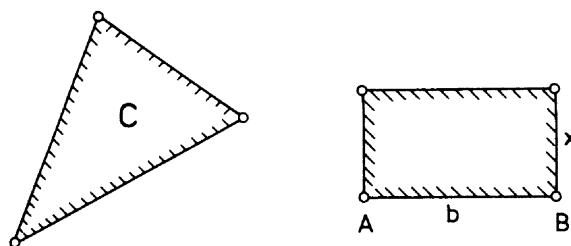
4. The application of areas

Translated into modern mathematics, the Greek procedure of application of areas amounts to the solution of quadratic equations (or problems), where the coefficients and unknowns are represented by line segments. Recently, there has been a heated debate about the right interpretation of this method, which traditionally has been called geometrical algebra. Before discussing the debate itself in §4e, I want to present the theorems of the *Elements* (VI.28 and 29) which are central to it.

4a. What is the method of application of areas?

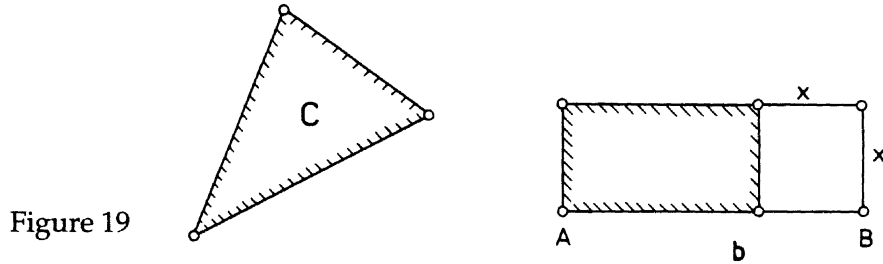
When Euclid speaks of applying a given plane area C to a line b he refers to constructing a rectangle with side b and area C . In modern terms, the problem of applying C to b is the problem of solving the equation ' $C = b \cdot x$ ', where C is an area and b and x are straight lines.

Figure 18

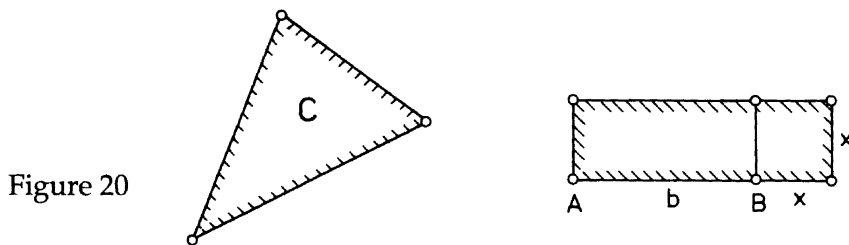


48 Taisbak (1971) offers an answer based on an interpretation of a reckoner's multiplication table. Itard (1961), 78-79 proposes a geometrical definition of proportionality based on the representation of numbers on a checker board. (Cf. Euclid's definition of similar numbers (VII, def. 21).)

To apply a given area C with square defect to a line b means to construct a rectangle of area C with one side shorter than b , say $b-x$, and the other side x .



Again in modern terms we have to solve (geometrically) $C = (b - x) \cdot x = b \cdot x - x^2$, for the unknown x . The same problem with square excess means to find x such that $(b+x) \cdot x = C$.



Plato refers to application of areas at 527a of the *Republic*. Proclus ascribes the discovery of the technique to the Pythagoreans:

Eudemus and his school tell us that these things - the application of areas, their exceeding and their falling short - are ancient discoveries of the Pythagorean muse. ... Those godlike men of old saw the significance of these terms in the describing of plane areas along a finite straight line. (*in Euc* 419.15-24)

Euclid presents the application of areas in a more general way than I have indicated. Instead of rectangles he uses parallelograms and instead of a square defect/excess he asks for a defect/excess similar to a given parallelogram. However he only applies the procedure in terms

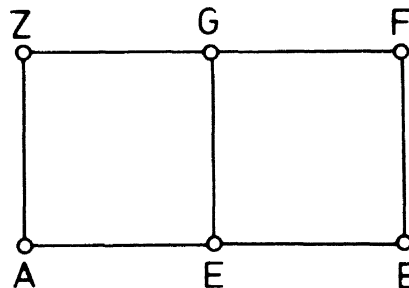
of rectangles with square defect/excess or without defect/excess.⁴⁹ I shall, therefore, confine myself in the next section to the more restricted case, consideration of which makes the essential geometrical issues much clearer, and then discuss Euclid's more general procedure in §4c. Moreover, I will concentrate on VI.28 (application with defect) before considering the very similar VI.29 (application with excess) in §4d.

4b. The interpretation of Elements VI.28

VI.28 (specialized to square defect). To a given straight line to apply a rectangle equal to a given rectilinear figure and deficient by a square: thus the given rectilinear figure must not be greater than the square described on the half of the straight line.

Let AB be the given straight line and C the given rectilinear figure.
Construction. Let AB be bisected at the point E, and let EBFG be a square; let the square AG be completed (figure 21).

Figure 21



Case 1. If the square AG is equal to C, we are done, because GB is a square defect.

Commentary. There are two ways to determine whether AG is equal to C:

- (a) Apply the area C to the line AE, using I.44, and compare the resulting rectangle with the square AG. Since the rectangle and the square have one common side, one has to compare the other sides and see if $AX < AZ$ (figure 22a).
- (b) Transform C into a square c^2 by VI.13 or II.14 (using I.44 as an essential prerequisite), and compare the sides c and AE (figure 22b).

⁴⁹ VI.30 with square excess, X.20, 22, 60-65 and 97-102 for the ordinary application, X.17, 18, 33, 34, 91-96 with square defect.

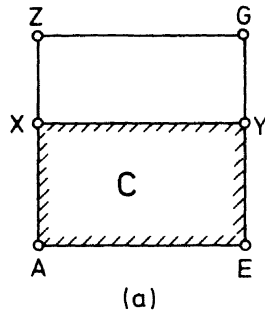


Figure 22a

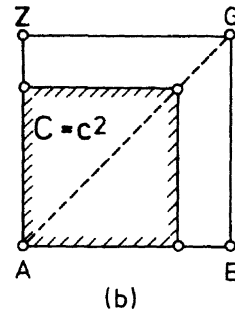


Figure 22b

Case 2. The square AG is greater than C.

Construction. Let the square KLMN be constructed equal to the excess by which GB (= AG) is greater than C.

Commentary. There are two ways we can construct KLMN:

(a) Transform the rectangle XYZG into a square by VI.13 (or II.14).

(b) If C is already transformed into a square, we can use the theorem of Pythagoras (I.47) with $C = c^2$ and $GB = t^2$ in order to find a square equal to $t^2 - c^2$, which is easily done (figure 23): Draw the half circle about EB, make $BS = c$ and let $ES = s$. By III.31 EBS is a right triangle, so that $s^2 + c^2 = t^2$.⁵⁰

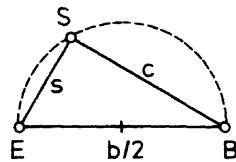


Figure 23

Observation. Because GB is equal to KM plus C, the line GE is greater than KL.

⁵⁰ Note that in (b) we have used the theorem of Pythagoras to transform a gnomon (difference of two squares) into an area of prescribed form (a square). We did essentially the same thing in (a), where we also arrived at a square. In both cases the 'squaring' made possible by VI.13 or II.14 was an essential step. This procedure is generalized from constructing a square equal to a given area to constructing a figure equal to the area and similar to a given figure in VI.25, which I discuss in the next section.

Construction. Let GO be equal to KL and GP equal to LM ; complete $OGPQ$.

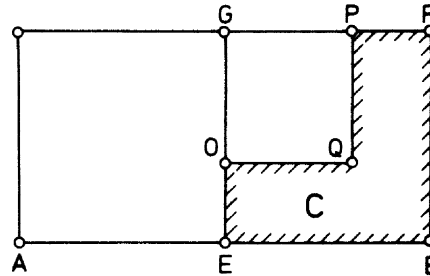


Figure 24

Observation. Q is on the line GB (VI.26).

Construction. Complete the figure as indicated in figure 25.

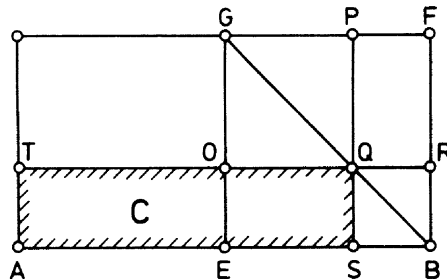


Figure 25

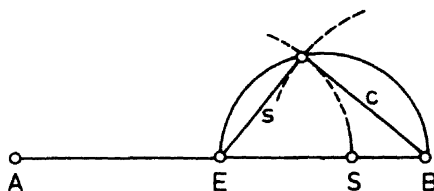
Observation. (i) BQ is a square (VI.24).

(ii) By construction, the gnomon around Q is equal to C . But rectangle SF = rectangle ET ; hence rectangle AQ = (gnomon around Q) = C , and the square BQ is missing.

Commentary. It is clear from the start that the main point in the whole problem is to find the point S on AB (or, equivalently, P or Q). What we have done gives a simple construction for S which can be used routinely in later applications with square defect: Let $AB = b$ and $C = c^2$ be given. Let E be the midpoint of AB , and carry out the construction indicated in figure 26.⁵¹

51 Similar constructions are given in Euclid-Heath (1926), vol. 1, 383-388.

Figure 26



I will now translate the above construction into algebraic formulae, where $a, b, \dots$ stand for (the length of) lines, and products like $a \cdot b$ or c^2 for areas.⁵² Let $AB = b$ and $C = c^2$. Then the original problem was to find x such that

$$(b - x) \cdot x = c^2,$$

or

$$-x^2 + b \cdot x = c^2.$$

In the construction we made $EB = b/2$. In case 1,

$$(b/2)^2 = c^2,$$

and we need only set $x = b/2$. In case 2, $c^2 < (b/2)^2$, so that $(b/2)^2 - c^2 > 0$, and we can construct a square with side s satisfying

$$s = \sqrt{(b/2)^2 - c^2}.$$

Our final solution is then

$$x = b/2 - s = b/2 - \sqrt{(b/2)^2 - c^2}.$$

It is to be noticed that we started from the equation

$$-x^2 + b \cdot x = c^2$$

or

$$x^2 - b \cdot x + c^2 = 0,$$

and the formula for x obtained by the construction is exactly the same as by the ordinary method of 'completing the square' in elementary algebra:

$$(x - b/2)^2 = (b/2)^2 - c^2.$$

⁵² Euclid does not measure lengths and areas, but deals with specific lines and figures.

A modern reader familiar with quadratic equations may find the algebraic manipulations easier than Euclid's construction. One should, however, not be deceived by habits. The arithmetic procedure of taking square roots (as limits!) is much more subtle than the construction of $\sqrt{pq}$ by way of VI.13.

4c. Euclid's generalizations

Apollonius (*Conica* I.12 and 13) makes use of the application of areas with rectangular defect or excess. This forces some modifications on the proof I have just described, most notably the introduction of Theorem VI.25. I will, therefore, first state and comment upon VI.25, restricted to rectangles (instead of Euclid's more general parallelograms):

VI.25 To construct one and the same figure similar to a given rectilinear figure and equal to another given rectilinear figure.

In the proof of this proposition Euclid takes the first given figure to be a triangle ABC. I take it to be a rectangle ABCD.⁵³ The construction for VI.25 proceeds as follows. Apply the area Q to the line BC (I.44), and construct the mean proportional BH to the lines AB and BF (VI.13).⁵⁴

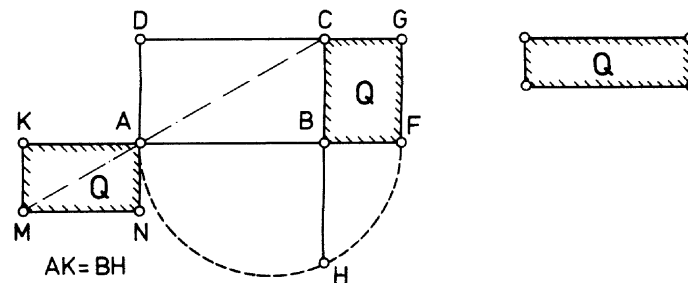


Figure 27

⁵³ In neither case is there any loss of generality.

⁵⁴ The essential application of VI.13 (or II.14) in VI.25 has been overlooked by Unguru and Rowe (1982), 27-28, who claim that 'of course, nothing, even mildly resembling II.14 is ever used' [in the proof of VI.29]. The construction of the mean proportional BH amounts algebraically to the extraction of a square root because $BH = \sqrt{AB \cdot BF}$. Consequently the finding of a square root is hidden in the prefabricated VI.25 in Euclid's solution of the 'quadratic problems' VI.28 and 29.

Then the rectangle AKMN on AK (= BH) which is 'similar and similarly situated' to ABCD will have area Q.⁵⁵

In the proof of VI.28 and 29 with rectangular defect/excess VI.25 replaces the theorem of Pythagoras, which is used to transform the difference (VI.28) or sum (VI.29) of two squares into a square. As we have seen, VI.25 in turn is based on the construction of the mean proportional by way of a right triangle in VI.13. Thus it is mathematically quite plausible to regard VI.25 as a generalization of the theorem of Pythagoras, and to associate VI.25 with Pythagoras,⁵⁶ as Plutarch does:

Among the most geometrical theorems, or rather problems, is the following: given two figures, to apply a third equal to the one and similar to the other, on the strength of which discovery they say moreover that Pythagoras sacrificed. This is indeed unquestionably more subtle and more scientific than the theorem which demonstrated that the square on the hypotenuse is equal to the squares on the sides about the right angle. (*Quaest Conviv* 720A, cited from Euclid-Heath (1926), vol. 1, 343-344).

We are now in a position to consider:

VI.28 (specialized to rectangular defect). To a given straight line to apply a rectangle equal to a given rectilinear figure and deficient by a rectangular figure similar to a given one: thus the given rectilinear figure must not be greater than the rectangle described on the half of the straight line and similar to the defect.

Let AB be the given line, C the figure with given area and D the rectangle given in shape.

Construction. As before we bisect AB at E, and then construct on EB a rectangle EBFG 'similar and similarly situated' to D.

⁵⁵ For the proof of this equality see VI.25. We can characterize VI.25 in modern terms as a similarity transformation for the first given figure. A similarity factor x for lines transforms areas by the factor x^2 . Euclid expresses this geometrically in a series of theorems starting with VI.19. In VI.25 he has λ as a ratio of areas but needs $\sqrt{\lambda}$ as similarity factor; he constructs $\sqrt{\lambda}$ using VI.13 or II.14.

⁵⁶ Unguru and Rowe (1982), 3-5 doubt the Pythagorean origin of VI.25.

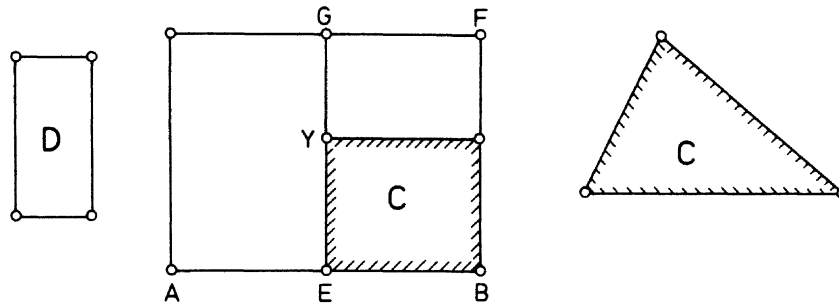


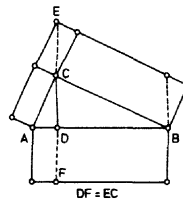
Figure 28

Commentary. Again we have to compare the areas of C and AG. For that purpose we could, as before, use I.44 to apply C to AE (without defect or excess) and compare the resulting rectangle YA with AG. If $C < AG$ the (now rectangular) difference has to be brought into the desired shape by VI.25; the rest of the proof is then quite similar to the one for squares. There is, however, another attractive way to get the solution.⁵⁷ If we *first* transform C into the shape of D and then compare the areas of C and AG, we could, for the construction in the case $C < AG$, use the generalized theorem of Pythagoras for similar rectangles (as in the figure accompanying VI.31), and proceed exactly as for squares. This procedure seems more plausible historically because it involves VI.31,⁵⁸ although Euclid himself provides no indication of what procedure he has in mind. In any case we have a substantial generalization from one special similarity class of rectangles, namely the squares, to any similarity class. The proof

⁵⁷ First proposed by Zeuthen (1896), 30.

⁵⁸ Note that VI.31 for rectangles has a proof similar to that of I.47, the theorem of Pythagoras (figure 29).

Fig. 29



requires new tools (i.e., VI.25), and the result has important new applications (conics).⁵⁹

Mathematically, the generalization of geometrical theorems like I.44 or VI.28 from rectangles to parallelograms is very simple. If we shear a rectangle to a parallelogram on the same base and under the same height, areas and ratios are preserved. Hence any statement concerning

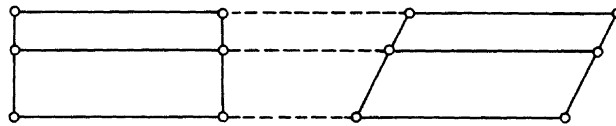


Figure 30

areas and proportions remains true when generalized from rectangles to 'parallelograms with a given angle'. The relation of triangles and parallelograms is in certain cases (e.g., I.41) more straightforward than that of triangles and rectangles, but for most purposes, including the application of areas, the generalization to parallelograms is pointless.⁶⁰ Euclid appears to be following the well-known postulate of the modern mathematician: 'Be wise, generalize'.

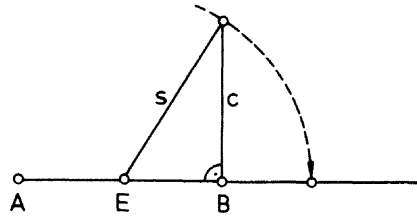
4d. Application with excess in VI.30

We have analyzed VI.28 in detail. Its counterpart VI.29 for squares leads to a construction similar to the one in figure 26, except that we have to construct the sum (instead of the difference) of the two squares GB and c^2 . If, as before, we let $AB = b$, we obtain $s^2 = (b/2)^2 + c^2$ and $x = -b/2 + \sqrt{(b/2)^2 + c^2}$.

⁵⁹ Euclid studies similarity classes of 'arithmetical' rectangles in Book VIII.11, 18, and 20 in terms of the concept of 'similar plane numbers', which he uses to establish that there exists a mean proportional number c to two numbers a and b if and only if a and b are similar plane numbers.

⁶⁰ Cf. Zeuthen (1896), 29 and Mueller (1981), 168. For a more detailed discussion of the problem of the generalization from rectangles to parallelograms see Artmann (1988a), 129-131.

Figure 31



VI.29 is used immediately for the construction of the golden section in VI.30, where we are given a line AB and asked for the point H on AB such that

$$AB:AH = AH:HB,$$

or equivalently (VI.16)

$$AH \cdot AH = AB \cdot HB.$$

To solve this problem Euclid first applies the square b^2 on AB to the line AB' ($= AB$) with square excess, determining the point D as in figure 32.

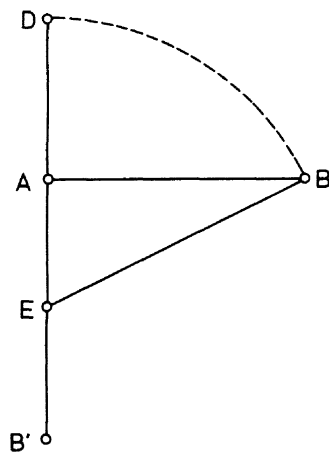


Figure 32

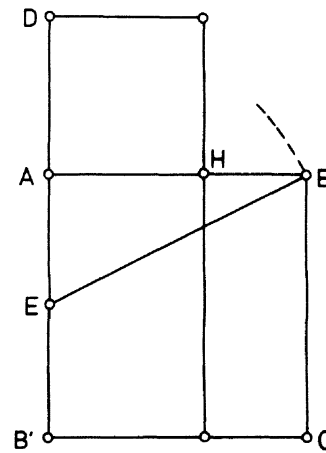


Figure 33

In the second step, he makes AH equal to AD, completes figure 33, and observes that:

$$AH \cdot AH = BH \cdot BC = AB \cdot BH.$$

Thus the application of b^2 to AB with square excess in VI.30 gives us exactly the figure of Euclid's proof in II.11, for which Euclid carries out the same construction with no explanation, using II.6, a similarly reduced form of the last step in the proof of VI.29.⁶¹ The mysteriousness of Euclid's procedure is cleared up by reference to the pre-Euclidean 'Proportion Free Elements'. The author of these 'Elements' succeeded in proving important theorems such as II.11 or constructing the pentagon with the help of the condensed parts II.5 and 6 of VI.28 and 29.

If we accept II.11 as originating from VI.30, the historical development involves the following sequence, which is quite different from the one we find in the *Elements*:

- (1) application of areas without and with *square* defect/excess (using proportion theory);
- (2) construction of the golden section as in VI.30;
- (3) elimination of proportion theory, as in II.5, 6, and 11.

This sequence would presuppose knowledge of:

- (i) the theorem of Pythagoras (I.47);
- (ii) the technique of 'squaring' a rectilinear figure (II.14, probably in the form of finding a mean proportional between straight lines (VI.13));
- (iii) the theorem that the angle in a semicircle is right (III.31).

This sequence is quite different from the sequence hypothesized by Unguru and Rowe (1982), 5. They see II.5 and 6 as fundamental presuppositions of VI.28 and 29. They are correct in the logical sense, but, as I have indicated, VI.28 and 29 (in some form) have historical priority

⁶¹ Cf. Mueller (1981), 169-70. Unguru and Rowe (1982), 18 ff. do not refer to VI.30 in their discussion of II.11.

because they involve the construction of solutions whereas II.5 and 6 can only be used to confirm known solutions.⁶²

We have seen that there is good reason to associate the Pythagoreans with the discovery of the application of areas for rectangles without or with square defect/excess.⁶³ The coins in figures 15-17 show that the regular pentagon was already familiar in the early fifth century B.C.E., so that the middle of that century is a plausible date for the introduction of the application of areas into Greek geometry.⁶⁴ It is hard to say when the generalization from square to rectangular defect/excess took place, but as we have seen, it presupposes the knowledge of VI.25 for rectangles. Perhaps it should be associated with the first investigations of conics and Menaechmus, the associate of Plato.⁶⁵ The generalization from rectangles to parallelograms, was presumably carried out a short time before Euclid, most likely in the school of Eudoxus.

4e. The debate about 'geometrical algebra'

Traditionally VI.28 and 29 have been considered under the rubric 'geometrical algebra', a concept introduced by Zeuthen (1896), 7, following Tannery (1882). Subsequently Neugebauer (1936), van der Waerden (1954), Freudenthal (1977) and Weil (1978) adapted and extended Tannery's and Zeuthen's position. Heath followed Tannery in his comments on II.5 and 6, which he interpreted as solutions to quadratic equations. This traditional position was attacked by Szabó

62 The method of isolating part of a more complicated proof as a lemma or partial theorem is quite common in mathematics. For an example see Artmann (1986), 85, Bemerkung 1. Pickert (1984) reflects on the method of 'abstraction by analysis of proofs'.

63 Burkert (1972), 450-452 accepts the attribution of the discovery of application of areas to them, although he is generally skeptical about early Pythagorean mathematics.

64 As noted above, major theorems in Book II are posterior to the mathematics of step (1). It is an open question whether these theorems were first established in the context of the 'Proportion Free Elements' or earlier. I am inclined to think that II.2-10 were first developed independently. Høyrrup (1987a) and (1989) points out the similarity of the diagrams in Book II with Old Babylonian methods.

65 See Proclus, in *Euc* 67.9-10 and 111.20-25.

(1969), Unguru (1975) and Unguru and Rowe (1981), (1982).⁶⁶ Van der Waerden (1954), 118-126 gives a clear statement of the position of the proponents of 'geometrical algebra'. His main claims are:

- (i) The real content of VI.28 and 29 is algebraic (as solutions of quadratic equations); geometry is only a mode of expression.
- (ii) Geometrical algebra originated with the Pythagoreans, who took it (somehow) from the Babylonians.
- (iii) The Greeks had to use a geometrical formulation of the theory of quadratic equations because they had no other way to deal with incommensurable magnitudes.

The issue of Babylonian origins has to be considered moot at this time.⁶⁷ Similarly extant texts provide us with no good evidence concerning (iii), which remains a matter of speculation. As far as point (i) is concerned, no one doubts the possibility of translating propositions such as VI.28 and 29 into algebraic formulations. The question is what purpose such a translation serves. On this question Unguru writes:

Does this tell us anything about the Greeks in general, and about II.5 in particular? *Nothing. There is not the slightest shred of genuine historical evidence that Euclid (or the other great Hellenistic mathematicians, let alone the Pythagoreans) ever used equations in their geometrical works. The sources do not contain equations.* (Unguru (1975), 91, his italics)

What Unguru says is certainly correct. However, algebraic translation does make some of Euclid's geometrical statements more transparent to the modern reader. I see nothing wrong in describing Euclid's applications of VI.28 and 29 as 'quadratic problems' or 'problems leading to quadratic equations in modern reformulations'. In my opinion the two positions in the debate about geometrical algebra are a reflection of two quite different points of view, which might be called mathematical and philological. Mathematicians tend to stress isomorphisms; they like to

⁶⁶ Berggren (1984) reviews the debate.

⁶⁷ For a discussion of the debate on this question see Unguru and Rowe (1981) and (1982). Recent investigations by Høyrup (1987b) and (1989) show the likelihood of some connection between Greek and Babylonian mathematics, but the evidence is not unambiguous.

see the same structure in different guises; by contrast a philologist puts great value on expression and literary form. Mathematicians are accustomed to separating the content of a proposition from its form of expression, whereas philologists are likely to stress the particularity of different forms of expression. Since the history of Greek mathematics lives at the crossroads of these 'two cultures', debates such as the one about geometrical algebra are unavoidable. Indeed, they are essential to progress in our understanding of ancient science.