

6626
BARKER, A.
1991

Three Approaches to Canonic Division

Andrew Barker

1. Introduction
 - 1a. Aristoxenus
2. The geometry of concords
 - 2a. The *Sectio Canonis*
 - 2b. Thrasyllus
3. Means and proportions
 - 3a. Plato
 - 3b. Archytas
4. Epimoric ratios and commensurable excesses
 - 4a. Ptolemy
5. Conclusion

1. Introduction¹

The device known as the monochord consists of a single string, stretched between two bridges mounted on a rigid base. A moveable bridge, placed on the base and making firm contact with the string

1 Works regularly cited in the notes include the *Harmonics* of Claudius Ptolemaeus (*Harm*) and the *Commentary* on this work by Porphyry (*in Harm*). The editions referred to are Düring (1930) and Düring (1932) respectively: see also Düring (1934). The texts of the Euclidean *Sectio Canonis* (*Sect Can*), Nicomachus' *Harmonicum Enchiridium* (*Harm*) and Cleonides' *Introductio Harmonica* (*Harm*) are in von Jan (1895). Aristides Quintilianus' *de Musica* (*de Mus*) has been edited by Winnington-Ingram (1963). The best edition of Aristoxenus' *Elementa Harmonica* (*Harm*) is that of Da Rios (1954). The text of Theon Smyrnaeus, incorporating the passages from Thrasyllus and Adrastus, is that of Hiller (1878). Translations of the principal texts discussed with commentary will be found in Barker (1989).

Three Approaches to Canonic Division

Andrew Barker

1. Introduction
 - 1a. Aristoxenus
2. The geometry of concords
 - 2a. The *Sectio Canonis*
 - 2b. Thrasyllus
3. Means and proportions
 - 3a. Plato
 - 3b. Archytas
4. Epimoric ratios and commensurable excesses
 - 4a. Ptolemy
5. Conclusion

1. Introduction¹

The device known as the monochord consists of a single string, stretched between two bridges mounted on a rigid base. A moveable bridge, placed on the base and making firm contact with the string

1 Works regularly cited in the notes include the *Harmonics* of Claudius Ptolemaeus (*Harm*) and the *Commentary* on this work by Porphyry (*in Harm*). The editions referred to are Düring (1930) and Düring (1932) respectively: see also Düring (1934). The texts of the Euclidean *Sectio Canonis* (*Sect Can*), Nicomachus' *Harmonicum Enchiridium* (*Harm*) and Cleonides' *Introductio Harmonica* (*Harm*) are in von Jan (1895). Aristides Quintilianus' *de Musica* (*de Mus*) has been edited by Winnington-Ingram (1963). The best edition of Aristoxenus' *Elementa Harmonica* (*Harm*) is that of Da Rios (1954). The text of Theon Smyrnaeus, incorporating the passages from Thrasyllus and Adrastus, is that of Hiller (1878). Translations of the principal texts discussed with commentary will be found in Barker (1989).

above it, can be shifted to different positions to alter the length of string, which will vibrate when it is plucked. A ruler or measuring strip lies lengthwise on the base, marked at the points to which the moveable bridge must be shifted in order to divide off, one after another, those lengths of the string which will emit the notes of a musical scale or attunement. The word *kanōn*, 'ruler', referring originally to the measuring strip, is commonly used in later sources to designate the whole apparatus.² To 'divide the *kanōn*' is to identify the points at which the strip must be marked to form a satisfactory attunement.³

The project of dividing the *kanōn* became a central preoccupation of mathematically-minded musicologists in Hellenistic and Roman times. Monochords were probably not regularly used in harmonics before about 300 B.C.E., but the problems they were designed to address had been formulated a good deal earlier, most trenchantly by Plato. In a well known passage of the *Republic* he criticizes two schools of harmonic theorists for paying too much attention to what they hear, too little to principles discoverable by reason alone. One such school in particular, labelled 'Pythagorean', is chided for seeking exclusively 'the numbers in heard concords' (*Rep* 531c1-2), that is, for making their only goal the quantification of the relations between notes in the audible attunements of contemporary musical practice. According to Plato, what they ought to do but in fact do not is to 'ascend to problems, to investigate which numbers are concordant and which are not, and in each case why.' (531c2-4) Despite some obscurities in this remark, the gist is clear. Concordance (*sumphōnia*) is to be treated primarily as a relation between numbers, not between sounds. Relations between sounds will be concordant in a derivative sense, just in so far as they instantiate concordant relations between numbers: the beauty of an audible concord is no more than an 'image' of the mathematical perfection underlying it.⁴ But the task does not end with the identification of concordant numbers. We must explain why they are so, in what their concordance consists; and

2 See, e.g., Ptolemaïs at Porphyry, in *Harm* 22.22 ff., Ptolemy, *Harm* 5.11 ff.

3 For detailed discussions of the instrument see Ptolemy, *Harm* I.8 and 11, and II.12-13. Related instruments are described in II.2 and 16, and III.1-2.

4 Thus Plato says of the concords, 'they provide pleasure to people of poor understanding, and delight to those of good understanding, because of the imitation of the divine *harmonia* that comes into being in mortal movements.' (*Tim* 80b5-8)

this explanation will fall within mathematics itself. We are to look, then, for mathematical principles such that numbers related in accordance with them are thereby concordant with one another, while those not so related are mathematically and hence harmonically uncoordinated, forming no coherent or concordant system.⁵

Not all those who set themselves to dividing the *kanōn* were Platonists — not, at least, in the sense that they treated reason as the sole arbiter of musical correctness. Some envisaged the task differently as that of finding principles whereby the perceptibly evident excellence of existing musical systems could be explained by being grounded in mathematically intelligible forms of perfection.⁶ Unlike strict Platonists, these theorists would not reject the attunements of current practice as musically improper just on the grounds that they did not conform to a set of independently excogitated rational principles. But in a weaker sense the idea expressed in the *Republic* guided their investigations too, and was current even before Plato. Simply stated, it is that genuinely musical relations, such as those between the notes sounded by the strings of a well-tuned lyre, are an expression of some corresponding form of order in the numerical domain. A well attuned system differs from an uncoordinated collection of pitches in that its elements, conceived as numbers or quantities, are integrated with one another through their joint conformity to some unifying mathematical principle. Hence the canonic divisions proposed by any theorist as ‘correct’ must be justified not only by their faithfulness to musical practice (if indeed that is reckoned a relevant consideration), but by their ‘rationality’ the harmonious coordination of the elements of the division, perceptible by the trained musician’s ear, must also be intelligible to the mind of the mathematician.⁷

5 The word *sumphōnos*, ‘concordant’, has a well defined technical sense which will be discussed below. But many writers, including Plato, sometimes use it more casually, to refer to any musically well attuned relation between pitches. It is not clear how Plato is using the word here; but in §3a we shall find indications that he believed all harmonically proper relations to be constructible through intervals that are ‘concordant’ in the more technical sense. Hence the former are in a way reducible to relations between the latter, and the vagueness of the present usage is relatively unimportant.

6 See particularly Ptolemy, *Harm* I.2 and III.3–4.

7 Ptolemy, *Harm* I.2 is again important in this connection. See also the discussions

The general character of the mathematical language, and the scope of the conceptions involved in this form of harmonic science, were determined by a set of discoveries traditionally ascribed to Pythagoras himself. However insecure this attribution may be, it was certainly among fifth-century Pythagoreans that these 'discoveries' first acquired a serious niche in musicological speculation. Their central theses were, first, that to a given musical interval between two notes or pitches there corresponds a specific ratio between two lengths of a true string;⁸ and secondly that to the three intervals standardly called *symphōniai*, concords, there corresponds an orderly set of three strikingly simple numerical ratios. The lengths of two sections of a true string which give notes at the interval of an octave are in the ratio 2:1, the ratio corresponding to the interval of a fifth is 3:2, and that of the interval of a fourth is 4:3.⁹

Two facts about musical perception and practice gave these ratios special significance. Of all the intervals within the span of the octave only these three were reckoned 'concordant'. The special feature of a concord as construed by Greek writers is that its two notes, when sounded simultaneously, present themselves to the hearing as a single blended unity; notes in other intervals form no such audible union.¹⁰ The fact that this intimate fusion of sounds in a concord seemed to rest on the simplicity of the corresponding ratios and that *harmonia*, 'attunement' in general, was regularly conceived as the coordinated unification of diverse elements gave encouragement to the quest for a mathematical interpretation of all perceptibly harmonious or 'well attuned' relations.¹¹ Sec-

of various schools of theorists by Ptolemaïs and Didymus quoted by Porphyry, in *Harm* 25.3 - 26.29, 27.17 - 28.26.

- 8 A 'true' string is one that is consistent throughout its length in tension, thickness, and material constitution. For tests designed to check that a string has this property see Ptolemy, *Harm* 18.9-21, and compare 26.15 - 28.12.
- 9 For some of the stories associating the discovery of these ratios with early Pythagoreans, see, e.g., the scholion to Plato, *Phaedo* 108d4, quoted at Diels (1956), 18.12, Theon, 59.4-21 (= Diels (1956), 18.13), Nicomachus, *Harm* ch. 6, Aristides Quintilianus, *de Mus* III.1.
- 10 See e.g. Plato, *Tim* 80b4-5, Aristotle, *Sens* 7.448a9-11, Euclid, *Sect Can* 149.17 ff., Porphyry, in *Harm* 35.26 ff., Nicomachus, *Harm* 262.1 ff., Cleonides, *Harm* 187.19 ff.
- 11 For the interconnection between unity and *harmonia* see especially the passages of Philolaus printed by Diels (1956) as fragment 6.

ondly, the concords form the framework, the basic structure, of all the varieties of attunement studied by the theorists. In the kinds most frequently considered, an octave is divided into two subsections each spanning the interval of a fourth and separated by an interval known as the tone (*tonos* or *toniaion diastēma*). Since an octave is the sum of a fifth and a fourth, the tone is the difference between them; and the structure shared by all these systems can be defined by reference to the octave, the fifth and the fourth. In figure 1 the letters refer to notes on a modern keyboard, which can be used to illustrate the relations involved.

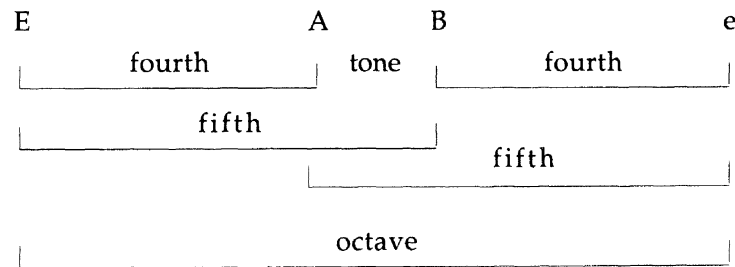


Figure 1

Thus the essential framework of any octave attunement can be straightforwardly defined by reference to an orderly group of three very simple ratios. (Given these ratios, that of the tone is readily computed as what we would call the quotient of $3/2$ and $4/3$; the value $9:8$ is assigned to it in all relevant sources from the late fifth century onwards.) But the structure so far formed is incomplete. The challenge facing dividers of the *kanōn* was to find ratios representing the intervals between the notes that have not yet been located in the system. A standard attunement over the range of an octave consisted of eight notes, like a modern octave scale. So far we have four, and their positions remain fixed. The others are located within the boundaries of each of the fourths, two in each; and the eight-note system is thus resolved into two subsystems of four notes each, two tetrachords, separated from one another by a tone. In any one form of attunement (at least among those we shall consider), the relations between notes in one tetrachord are identical with those in the other. Thus the location of the remaining notes in the octave is dictated by the manner in which any one tetrachord is divided up. As we have already seen, it will not be enough merely to quantify the intervals and express them as ratios

on the basis of empirical trial and error: it must be shown that the system thereby formed displays some sort of mathematical coherence.

The situation is complicated by the currency among the Greeks of several different systems of attunement, differing in the internal structure of their tetrachords. Theorists disagreed both about the precise form proper to each attunement and about the number of forms that should be recognized as genuinely distinct. It will be useful to offer some preliminary indication of their structures. But this should not be treated as a set of neutral data which it was the mathematical theorist's task to accept and analyze. On the contrary, each different mathematical account is claiming, in effect, the right to redefine the real nature of the data themselves, either on the grounds that its quantifications and no others, are correct representations of attunements that the ear *does* accept or because they are those that an ideal hearer (though perhaps no real one) *would* accept, since they are the expressions of perfect mathematical relations.

1a. Aristoxenus

To provide our preliminary indications I shall set out the principal forms of attunement recognized by Aristoxenus, towards the end of the fourth century. He at least is a theorist who claims explicitly to be analyzing real musical practices; and since he wholly rejects the procedure whereby intervals are expressed as ratios, drawing rather on the familiar language of listeners and musicians, his formulations, at an impressionistic level, are immediately comprehensible. There are, on his account, three 'genera' of attunement, diatonic, chromatic and enharmonic. There are several different species (or 'shades', *chroai*) of diatonic and of chromatic, and beyond these there exists an indefinite number of possible variants in all three genera; those that he describes are only the most familiar.¹² This last complication need not detain us. The 'familiar' forms of attunement, according to Aristoxenus, employ tetrachords of the kinds indicated in figure 2.¹³ It will be seen that the differences between the attunements are created by expansions or contractions of the two lowest intervals of the tetrachord, or, equiva-

¹² See, for instance, Aristoxenus, *Harm* 48.10 ff., 50.20 ff.

¹³ See especially *Harm* 50.20-52.9.

lently, by relocations of the two 'moveable' notes lying between its boundaries. (Aristoxenus conceives intervals as quasi-linear distances between points of pitch. These distances are expressed as multiples or fractions of the tone, which is treated as an empirically recognizable unit of distance in this dimension. His approach contrasts sharply with that of the Pythagoreans for whom pitches differ in quantity not by their positions in an acoustic 'space', and for whom the relation between two different pitches is not a distance, but the ratio between the quantities characterizing them.)

		Lowest note		Highest note	
Enharmonic		$\frac{1}{4}$	$\frac{1}{4}$	2	
Chromatic					
(i)	Soft chromatic	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{11}{6}$	
(ii)	Hemiolic chromatic	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{7}{4}$	
(iii)	Tonic chromatic	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	
Diatonic					
(i)	Soft diatonic	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{5}{4}$	
(ii)	Tense diatonic	$\frac{1}{2}$	1	1	

Figure 2

Few mathematical theorists address themselves to as many attunements as these. Ptolemy is the exception, offering eight divisions in all. More commonly we find just three, one for each genus; and there is an important group of writers who admit one form of attunement only as mathematically perfect, one that corresponds roughly to Aristoxenus' tense diatonic. Despite his enormous influence, theorists rarely construed their task as that of directly translating Aristoxenus' systems

into the language of ratios.¹⁴ Exact translations into ratios of integers are in fact impossible since if the tone is in the ratio 9:8, no ratio of integers can express any exact fraction of it. A theorem with this consequence was known from the early fourth century.¹⁵ The table in figure 2 of Aristoxenian tetrachords will serve us, then, only as a guideline, to give some suggestion of what the theorists have in mind when they classify systems as 'enharmonic', 'chromatic', and 'diatonic'.

By what procedure is a string to be divided to yield a genuine form of attunement, rather than a mere collection of pitches? The procedures must be mathematically intelligible, and they must also give results that connect in some way with known musical practices. Some theorists, admittedly, were more concerned to uncover metaphysically ideal relations than to analyze attunements in actual contemporary use. But even they recognized the perceptible beauty of human music as a sign of the approximation of its attunements to the theoretically perfect, and could therefore not represent as ideal any structure entirely alien to the musical ear. Their constructions might be conceived as rational corrections of those involved in the perceptible music of their day, rather than as direct representations of them; but the rational and the perceptible could not be entirely unrelated. In the following sections we shall consider three distinct but interconnected approaches to the problems that the theorists confronted.

2. The geometry of concords

We shall look first at the method of division adopted in the Euclidean *Sectio Canonis*, together with a variant attributed to Thrasyllus by Theon of Smyrna. We begin here not because their strategy is the earliest on record — for it is not — but because it is in several ways the simplest. The ascription of the *Sectio* to Euclid has been questioned, and its date

14 The divisions of Thrasyllus, considered in §2b, constitute a gesture in this direction, and there were others like them. More sophisticated attempts at 'Pythagorizing' Aristoxenian systems were made by Eratosthenes and perhaps by Didymus. See the divisions attributed to them in Ptolemy, *Harm* II.14, with the footnotes relating to them on pp. 346-9 of Barker (1989).

15 See Boethius, *de Institutio Musica* III.11 (relating to Archytas) and Euclid, *Sect Can* propositions 3 and 16, with the discussion of Knorr (1975), ch. 7.

is uncertain.¹⁶ I am inclined, however, with some hesitation, to accept the traditional dating around 300 B.C.E., whether its author is Euclid or perhaps some associate or pupil of his. Thrasyllus belongs to the early first century of the common era.

2a. *The Sectio Canonis*

The core of the *Sectio Canonis* is a set of theorems proving a sequence of propositions about ratios (propositions 1-9), and deriving with their help the ratios of the musical concords and the tone (propositions 10-13). A group of subordinate propositions follows, concerned especially to show that none of the concords is equal to any number of exact tones or half-tones, and that neither the tone nor certain other small intervals can be divided into equal parts, that is, into subintervals corresponding to equal ratios of integers (propositions 14-18).¹⁷ The treatise ends (propositions 19-20) with a division of the *kanōn* which draws on the results of propositions 10-13 to form an attunement over the range of two octaves.

In order to understand the construction we need first to set out schematically the framework of fixed notes that forms the basic structure of the system assumed by this author, and by most others whose analysis extends to the double octave. (The octave previously analyzed in figure 1, to which we shall later return, corresponds in structure to the one in the center of this system, between *hypatē mesōn* and *nētē diezeugmenōn*.) Figure 3 includes the names of the fixed notes and the tetrachords whose boundaries they are, since we shall need to refer to them in the subsequent exposition.

16 For a recent discussion of these matters see Barbera (1984).

17 These propositions depend on the theorem referred to in note 15 above.

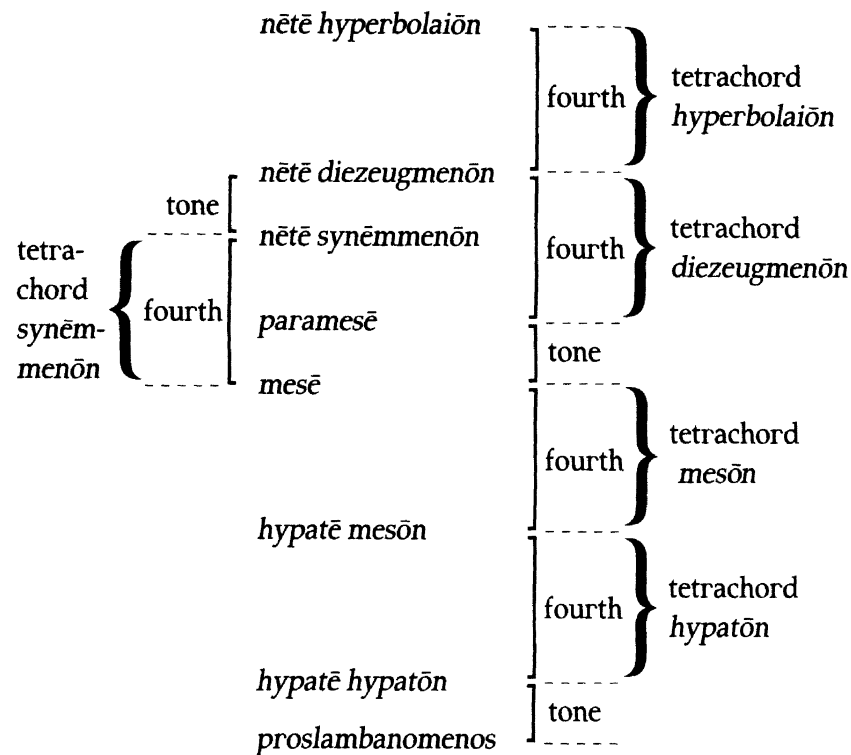


Figure 3

The tetrachords *hypatōn*, *mesōn*, *diezeugmenōn* and *hyperbolaiōn* together with *proslambanomenos* comprise what came to be called the Greater Perfect System (GPS). The tetrachord *synēmmenōn*, together with those below *mesē*, was usually treated as a separate structure, the Lesser Perfect System (LPS).¹⁸ The fact that *nētē synēmmenōn* is not a note, let alone a fixed note in the GPS will be of some significance later.

18 A different analysis is offered by Ptolemy, *Harm* II.6.

Proposition 19 of the *Sectio Canonis* begins with the words: 'Let there be a length of the *kanōn* which is also the length of the string.' Its task is then to determine the points at which the monochord's measuring strip should be marked to indicate the proper locations of the moveable bridge: it is not manipulating numbers in abstraction from any application. The author first divides his length, AB, into four equal parts, by marking points C, D, E (figure 4).

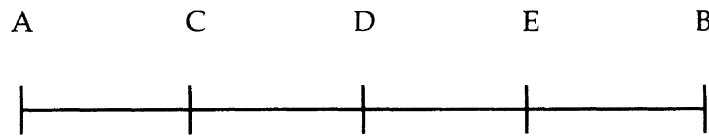


Figure 4

Length AB of the string gives the lowest note of the double octave, *proslambanomenos*. DB is an octave higher, giving the note *mesē*. EB, a quarter of the whole length, is two octaves above the original pitch, giving the upper boundary of the system, *nētē hyperbolaiōn*. CB, as three quarters of the original, sounds a fourth (4:3) above it, giving the note *diatonos hypatōn*. The role of this note in the system raises problems to which we shall return. Next, CB is divided in half, at F, and DB is divided at G, which is found by reducing DB by one third (figure 5).

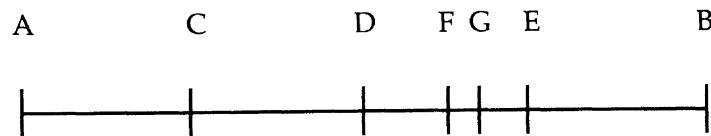


Figure 5

Since FB is half CB, it sounds an octave above *diatonos hypatōn*. Half of three quarters of the string is equivalent to three quarters of a half; hence FB is a fourth higher than DB (*mesē*), and is *nētē synēmmenōn*. GB, being two thirds of DB, sounds a fifth higher than *mesē*, giving *nētē diezeugmenōn*.

In the last phase of proposition 19, three further points are marked (figure 6). H is found by making GH equal to GB, so that HB is an octave below GB, and is *hypatē mesōn*. KB, which is two thirds of HB, is found by subtracting one third of HB. Since HB:KB is 3:2, KB is a fifth higher than *hypatē mesōn*, and is *paramesē*. Finally, LK is a length equal to KB, so that LB is double KB and sounds an octave lower, *hypatē hypatōn*.

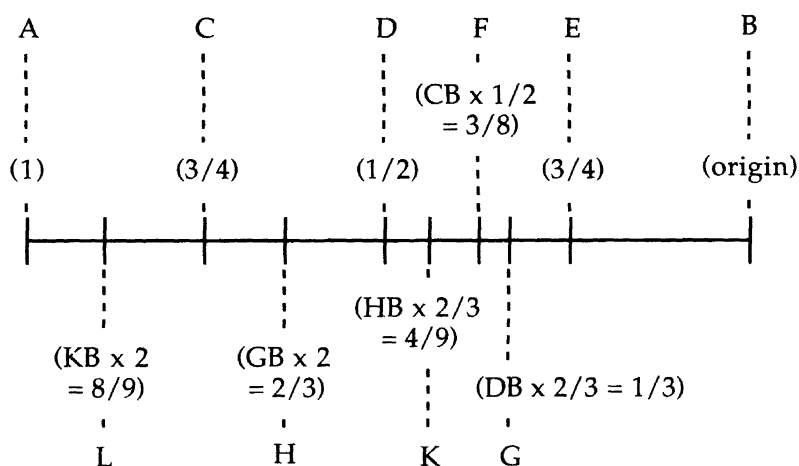


Figure 6

This completes proposition 19. It is evidently designed to show how the basic framework of the two octave system emerges from a very simple sequence of operations on a geometrical length. It will be the task of proposition 20, whose procedure is rather different, to locate the subsidiary 'moveable' notes inside the boundaries of each tetrachord. But it should be noticed that the correspondence between the first phase of the construction and the fundamental framing notes of the system is not exact. Figure 7 should be compared with figure 3 above.

Two points deserve some comment. First, one of the 'fixed notes' identified here is *nētē synēmmenōn*, which, as I have explained, is a fixed note only in the context of the LPS. If the author had intended to include the LPS in his account, it would have been necessary to complete the tetrachord *synēmmenōn* by locating its moveable notes in proposition 20. This, however, is not done. The system completed in proposition 20 does require a note at the pitch of *nētē synēmmenōn*, but it is a moveable note, *paranētē diezeugmenōn*, which has this locus only in the diatonic genus. The designation of the holder of this pitch by the name *nētē*

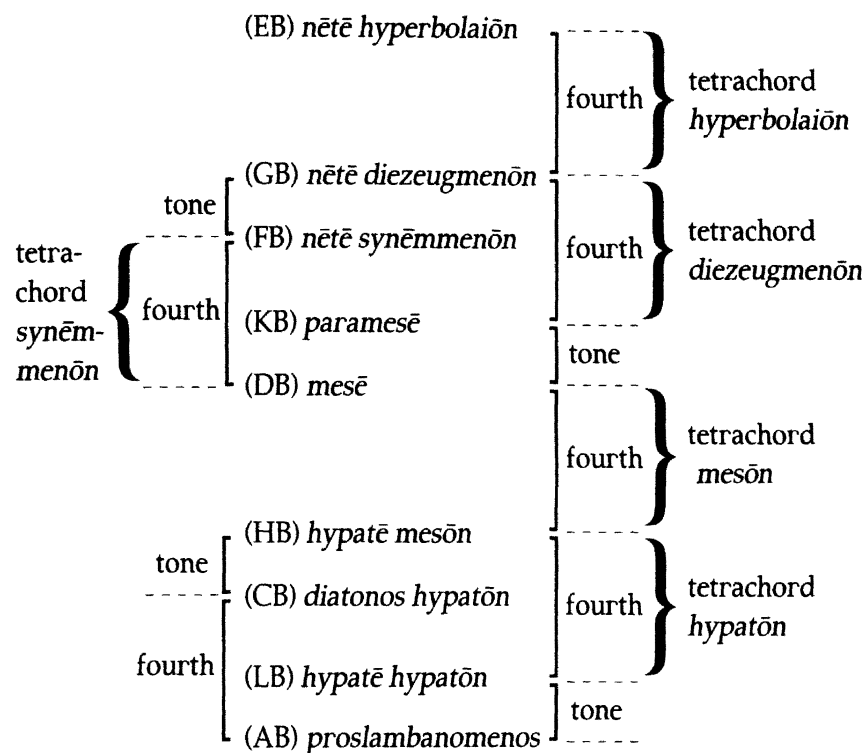


Figure 7

synēmmenōn implicitly attributes to it a status in the system which the note actually involved does not possess. Secondly, one note appears here which is not a fixed note at all: *diatonos hypatōn*, a tone below *hypatē mesōn*. As its name indicates, it can occur in this position only in diatonic systems; it is a moveable note, one lying inside the boundaries of a tetrachord, and the corresponding notes in chromatic and in enharmonic are differently placed, both lying more than a tone below *hypatē mesōn*. The insertion of the two problematic notes creates a neat-looking pair of variations in the overall structure, the lower fifth of each octave being divided either as a fourth above a tone or as a tone above a fourth. Musicologically, however, this symmetrical pattern is quite spurious; it corresponds to nothing that is of any structural significance, and it will not, of course, survive a shift from the diatonic to any other genus. It seems clear, then, that the author's sense of orderliness in his pattern of geometrical divisions has taken precedence over musical considera-

tions, those that determine the musical sense of the structure he is analyzing.¹⁹

Proposition 20 turns to the construction of the moveable notes in each tetrachord. It proceeds in two stages. In the first those of the tetrachord *hyperbolaiōn*, the highest tetrachord, are independently established. The length corresponding to the highest note, EB, is divided into eight equal parts, and is then extended to M by a length equal to one such part. Then $MB:EB = 9:8$, and MB sounds a tone below EB. The procedure is then repeated using MB as its starting point, so that the new length, NB, stands to MB in the ratio 9:8. Thus the fourth (ratio 4:3) embraced by the highest tetrachord has been divided, from the top down, into two tones each of ratio 9:8, and a small residue which the present author does not quantify. (It is in fact the so-called *leimma* or 'remainder', in the ratio 256:243, since $4:3 = 9:8 \times 9:8 \times 256:243$. We shall meet it again later.) The second phase of construction in proposition 20 turns on the fact that the division of each tetrachord in the GPS will be the same, in any one genus, and that corresponding notes in adjacent tetrachords are always either a fifth or a fourth apart. Hence the relevant lengths will be related in the ratios 3:2 or 4:3; and these facts provide the author with a simple strategy for locating the remaining points to be marked on the *kanōn*.

If we look back over the procedures involved in the two propositions, it seems obvious that they have been determined, in part, by considerations of simplicity and symmetry. Proposition 19 requires only the construction of quarters, thirds, and halves of given lengths, together with the operation of doubling, conceived as adding a length equal to the original. It proceeds in a tolerably systematic order. First, the string is quartered. Then one of the resulting lengths is halved, to find its counterpart in the higher octave, and the reduction of this length by one third locates the next note. Next this length is doubled to return us to the lower octave, and this new length is in turn reduced

19 It has been argued that there was a structural role for a note a tone below *hypatē mesōn* in some early musical systems; see Winnington-Ingram (1932), 205-206. But the evidence is, I think, unconvincing; and the use of the term *hyperhypatē* for this note in a few later sources (including Thrasyllus), which apparently invests it with a role independent of genus, is probably only a reflection of the way it emerges in constructions like the present one. The word is plainly a theorist's coinage, not part of the jargon of practising musicians.

by a third. Finally the last length is again doubled to give the corresponding note an octave below. In proposition 20, however, the author does not proceed by the simplest steps available. He places two tones in the highest tetrachord by adding one eighth to each of two successive lengths. But since the tone is the difference between a fifth and a fourth (assumed in proposition 13), the note a tone below *nētē hyperbolaion* could have been found by ascending through a fourth from the note a fifth below it, which has already been found in its guise as *nētē synēmmenōn*: this will be done by constructing three quarters of the length FB. If we then extend the resulting length by half, descending through a fifth, we shall find the second lowest note in the tetrachord *diezeugmenōn*, a tone below *nētē synēmmenōn*; and ascending once again through a fourth we find its counterpart in the highest tetrachord. This manner of constructing tones by movements of a fifth down and a fourth up, or the converse, was known to the writer of this work since he draws on it in propositions 17 and 18. It is therefore curious that he does not deploy it in this way in proposition 20. In §3a we shall find a similar puzzle in the account given by Plato; but the explanations I shall offer there do not seem so plausible in the present case. Perhaps the reason is only that the Euclidean writer, having established the ratio of the tone in proposition 13, wishes to put it to some direct use in his division. Similarly, the other moves in his procedure are not merely simple from a geometrical point of view. They are just those that correspond to the ratios of the concords which he has previously derived.

Considered musically, the system constructed has important properties, as the remarks above will have suggested. It can be built up from a given starting point by moves through concordant intervals only, octaves, fifths, and fourths. The practice of attuning lesser intervals through concords alone was well known in antiquity. It seems to have been a technique used among performing musicians, not just by theorists, and was grounded not merely in an abstract sense of the priority and the unifying power of the concords, but in the well-founded belief that pure concords are easier to attune accurately than are the smaller, discordant scalar intervals.²⁰ If the procedure is used, by itself, to

20 See especially Aristoxenus, *Harm* 55.3 ff., Euclid, *Sect Can* proposition 17, Ptolemy, *Harm* 40.8-17.

generate a complete attunement over a set of tetrachords, it is inevitable that the tetrachords will be divided, as here, into tones and *leimmata*. This division of the tetrachord has, therefore, a significant appeal to performer and theorist alike. There is evidence, nevertheless, that performers found this kind of attunement less than satisfactory, and that the preferred forms of diatonic attunement were normally constructed, in practice, either by another route altogether, or through additional adjustments to the results that the 'method of concordance' could yield.²¹ To those theorists, however, for whom mathematical simplicity took precedence over faithfulness to the empirical phenomena, this system remained the paradigm of rational perfection. We shall find another, very influential approach to it in Plato, and the special attributes of this 'Pythagorean diatonic', as it was later known, made it a prime focus of speculation and controversy in the Renaissance.²²

2b. Thrasyllus

Let us now consider, very briefly, the approach to this division taken by Thrasyllus.²³ Its results are the same, and the procedures are similar, but not identical. Here the counterpart of the Euclidean proposition 19 is an even simpler and more orderly series of steps. The whole string provides the lowest note; and it is then divided first into two, next into three, and finally into four equal parts. Division in two gives *mesē*. Division into three gives *hypatē mesōn* (two thirds of the length, a fifth above *proslambanomenos*) and *nētē diezeugmenōn* (one third of the length, an octave above *hypatē mesōn*). Division into four, the initial step in the *Sectio*, gives *diatonos hypatōn* (three quarters of the length, a fourth above *proslambanomenos*) and *nētē hyperbolaiōn* (one quarter of the length, two

21 Note the comments of ps.-Plutarch, *de Musica* 1145b-c, a source derived from Aristoxenus. For Ptolemy's discussion of what he calls the 'ditonic' diatonic see *Harm* 39.12 - 40.20. The role of the 'concordant' system in musical practices is discussed in Winnington-Ingram (1932) and Barker (1989), 49-52; cf. Burkert (1972), 389. On the difficulty of constructing chromatic and enharmonic divisions on the basis of this system, see the comments on Thrasyllus in the next section.

22 See, for instance, Walker (1978), 9-10, 41-42.

23 A detailed account, involving elaborations ignored here, is given by Theon of Smyrna, 87.4 - 93.9.

octaves above *proslambanomenos*), in addition, of course, to *mesē*, which has already been found.

This procedure reproduces the oddity found in the *Sectio* in that it constructs the moveable note *diatonos hypatōn* in the same phase of the operation as the fixed notes. It adds peculiarities of its own. In the first place, *mesē* is, by implication, constructed twice, by halving and by quartering. Secondly, two regular fixed notes are missing, *hypatē hypatōn* and *paramesē*. They are located later, but through procedures designed primarily to capture the moveable notes, not to establish the fixed framework. Thirdly, there is as yet no mention of *nētē synēmmenōn*, whose appearance at this stage of the *Sectio*'s division seemed anomalous: here the boot is on the other foot, since Thrasyllus, unlike the Euclidean writer, does construct the tetrachord *synēmmenōn* in the later phase of his division, and it would have been musicologically appropriate to locate its fixed terminus in the earlier, foundational phase.

Thrasyllus sets out on the second stage of his division in the same way as the *Sectio Canonis*. He too constructs the two tones in the highest tetrachord by extending the relevant lengths by one eighth. But he does not go on, as the *Sectio* does, to locate the corresponding notes in other tetrachords by movements through concordant intervals, preferring to repeat, for each tetrachord, the independent construction of each tone. He also adopts one further manoeuvre, that of constructing a tone upwards by reducing a length by one ninth. This is used for two purposes. In the first place it serves to locate the missing fixed notes of the GPS, *paramesē* and *hypatē hypatōn*, each of which stands a tone higher than an existing fixed note. Secondly, whereas the system constructed in the *Sectio*, and so far by Thrasyllus, represents an attunement in the diatonic genus, Thrasyllus now uses this same manoeuvre to locate one additional note in each tetrachord, a tone above its lower boundary, which will permit the representation of a chromatic division. The intervals of a Thrasyllan diatonic tetrachord, taken from the bottom upwards, are *leimma*, tone, tone. (See figure 8.) In the chromatic tetrachord, the first interval is again the *leimma*, while the upper boundary of the second is no longer a note a tone below the top of the tetrachord (this note does not appear in the chromatic division), but a note located a tone above the bottom. The highest interval is therefore the residue of a fourth after a tone, that is, a tone plus a *leimma*.

Just as this version of the diatonic corresponds roughly to Aristoxenus' tense diatonic, so this chromatic division approximates to his tonic chromatic. (See figure 2.) But Aristoxenus' account involves reference to half-tones, and the assumption that the fourth spans exactly

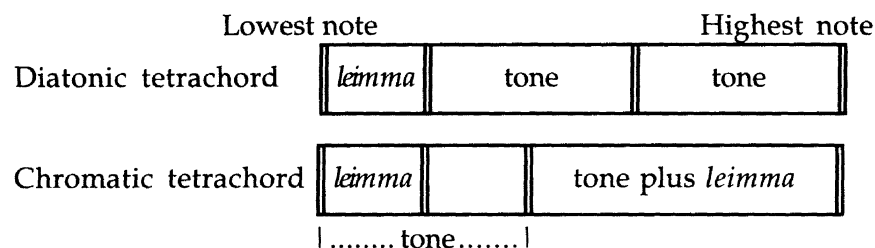


Figure 8

two tones and a half.²⁴ Now if the ratio of the tone is 9:8, an exact half-tone, as we have seen, cannot be represented as a ratio of integers, and if the fourth is 4:3, it is less than two and a half tones (that is, $4:3 = 9:8 \times 9:8 \times 256:243$, and $256:243$ is less than 9:8). None of Thrasyllus' chromatic intervals, then, is exactly equivalent to its Aristoxenian counterpart. Further, the two lowest intervals in the tetrachord will not be equal. The first is 256:243, the second $9:8 \times 243:256$, or 2187:2048; and the latter is slightly greater. The highest interval will be $9:8 \times 256:243$, or 32:27. Thrasyllus goes on to gesture in the direction of an enharmonic division. In Aristoxenus its tetrachord appears as quarter tone, quarter tone, ditone. Thrasyllus clearly has this in mind when he remarks that an enharmonic division requires the removal from the diatonic tetrachord of its second-highest note. But he makes no attempt to suggest a way of constructing subdivisions of the *leimma* to represent 'quarter tones', nor does he identify the ratios which such subdivisions might be assigned.

Thrasyllus has taken the method of division pioneered in the *Sectio* about as far as it can go. His presentation is notably orderly, especially in its first phase, where, as Theon suggests, his exploitation of the possibilities of dividing a length into halves, thirds, and quarters indicates the allure of the numbers 1, 2, 3, 4, the constituents of the Pythagorean *tetraktys*. But, as in the *Sectio*, the considerations determining the selection of steps through which divisions may proceed seem ultimately to be musical rather than numerological. It is because 2:1, 3:2, 4:3 are the ratios of the primary concords, and because 9:8 constitutes

²⁴ See, among other passages, Aristoxenus, *Harm* 46.1-2 and 56.14 - 58.5.

the difference between the two latter, that these steps and no others are used. By contrast the order imposed on them in the opening phase has no real musical credentials, and generates anomalies of the kinds to which I have drawn attention. Musical and mathematical procedures may produce the same final results, but the mathematical construction does not adequately reflect the relations between fundamental structure and subordinate detail which are essential to this mode of musical organization. Again, the determination shown by these authors to use only those ratios that correspond to concords or to relations between concords ensures that no scalar steps can be constructed other than tones and *leimmata*, and, in Thrasyllus' procedure, the interval by which the tone exceeds the *leimma*. In the context of Greek harmonics, this is a serious obstacle to the construction of divisions corresponding to the attunements of musical practice; a process of division restricted to moves through concords and tones cannot hope to capture attunements in all three genera, let alone the variants recorded by Aristoxenus and Ptolemy.

The method also has disadvantages from a theoretical point of view. Though the steps of the division proceed through simple ratios, the intervals constructed inevitably include some whose ratios are unsatisfactorily cumbersome, specifically the *leimma* of 256:243, and the interval by which it differs from the tone, sometimes called *apotomē*, whose ratio is 2187:2048.²⁵ For some theorists, as we shall see, the awkwardness of these ratios is sufficient to ensure that they cannot correspond to any genuinely melodic intervals. Finally, although we have seen the theoretical as well as the practical attractions of the 'method of concordance', there is nothing in the way in which Thrasyllus and the *Sectio* set to work that would meet the challenge set by Plato, the demand that we explain mathematically why certain relations between numbers count as 'concordant' while others do not.

25 See Boethius, *de Institutio Musica* II.5 (Diels (1956), 44.A26); cf. Burkert (1972), 395.

3. Means and proportions

We must now return to the earlier fourth century, to the work of Archytas and of Plato himself. Archytas of Tarentum, the distinguished Pythagorean statesman, mathematician and philosopher, introduces a quite different and more sophisticated line of thought with his classification of three kinds of mathematical mean (*mesotēs*), which define three kinds of proportion (*analogia*). These, he says, are used in music:

There are three means in music. One is arithmetic, the second geometric, the third subcontrary, which they call 'harmonic'. There is an arithmetic mean when there are three terms, proportional in that they exceed one another in the following way: the second exceeds the third by the same amount as that by which the first exceeds the second... There is a geometric mean when they are such that as the first is to the second, so is the second to the third... There is a subcontrary mean, which we call 'harmonic', when they are such that the part of the third by which the middle term exceeds the third is the same as the part of the first by which the first exceeds the second... (Porphyry, in *Harm* 93.6-17 [= Diels (1956), 47.B2])

Archytas' own harmonic divisions are recorded by Ptolemy, and we shall return to them shortly. Ptolemy's account, however, makes no allusion to the theory of proportions, and gives no indication of how it might have been applied. It will be helpful to look first at another division in which the theory's role is clear and explicit.

3a. Plato

The division in question is the best known and most influential of them all, the one set out in Plato's *Timaeus*. The context is not a musical one: it concerns the structure of the soul of the perceptible universe. Nevertheless, all subsequent commentators recognize its affinities with musical divisions, and aspects of Plato's language,²⁶ as well as his results, point unambiguously in that direction. Besides, as we saw at the outset,

26 Note in particular his uses of the word *diastēma*, 'interval', at 36a1, a3, b1; and see my comments below on his handling of the 'epogdoic' ratio.

according to Plato the principles underlying musical divisions are not specific to music. They belong to the wider domain of number theory in general, and their relevance to music is only one instance of the various subordinate domains in which they might be applied.

Plato's divine craftsman begins the division of his strip of meta-physical stuff by cutting off segments corresponding in length to the terms of two geometrical progressions, 1, 2, 4, 8 and 1, 3, 9, 27. Each intermediate term in each sequence, at this preliminary stage, is therefore the geometrical mean between two other terms. The craftsman's next step is to introduce two further means between each term and its successor in each progression; these means are arithmetic and harmonic and are defined in a manner very close to that of Archytas. Let us consider just one representative pair of these means, those between 1 and 2.²⁷ The arithmetic mean is $3/2$, and the harmonic is $4/3$. If we then set the four terms out in order, 1, $4/3$, $3/2$, 2, it is obvious that the ratio of the second to the first is $4:3$, that of the third to the first is $3:2$, and that of the fourth to the third is $4:3$. The ratio of the fourth to the first, $2:1$, is that of the octave. Hence this phase of the proportional division has generated precisely the structure set out in figure 1, the basic framework for any attunement, namely an octave constituted by two intervals each spanning a fourth ($4:3$) and separated by a tone ($9:8$, the ratio of $3/2$ to $4/3$).

The craftsman now proceeds to locate further terms within the fourths or 'epitritics intervals', as Plato calls them.²⁸ From a musical point of view, this will correspond to the insertion of the intermediate, 'moveable' notes in each tetrachord. The way in which the craftsman does this is by now familiar: 'he filled up all the epitritics with the epogdoic kind of interval, leaving a part of each of them, where the interval of the remaining part had as its boundaries, number to number, 256 to 243.' (*Tim* 36b1-5). Since 'epogdoic' is the adjective referring to the ratio $9:8$, Plato's tetrachords are divided as $9:8 \times 9:8 \times 256:243$, two

27 The reasons for Plato's introduction of the series of triples and its combination with the other series are cosmological rather than musicological, and will not concern me here. Nor shall I consider the whole span of the system thereby constructed, which again has no specifically musical significance. I shall focus on only one of its constituent octaves.

28 An epitritic interval is one whose ratio is the *epitritos logos*, $4:3$.

tones and a *leimma*, and are identical with those of the diatonic system set out in the *Sectio Canonis* and by Thrasyllus.²⁹

Plato's procedure here shares also the minor awkwardness found in those later accounts, in that all three of them undertake the formation of tones by independent constructions of the ratio 9:8. Thrasyllus and the *Sectio* abandon their sequences of moves through pure concords, and in Plato's version the principle of proportional division drops out of sight. But it can readily be reinstated. Since a tone can be constructed as the difference between a fourth and a fifth, and since these can be formed by the insertion of arithmetic and harmonic means in the octave, it is clear that Plato's results can be generated straightforwardly by his proportional procedure. It is not difficult to guess why he chooses a different mode of expression. In the first place, the reference to 'epogdoic' intervals constitutes a direct hint (of which there are several others in the passage) at the musical connections of the analysis, which Plato seems concerned to bring out, but could scarcely give explicit attention to in the context. The ratio 9:8 has recognizable significance only in the musicological domain. Secondly and more obviously, a complete exposition in terms of means and proportions would have been disagreeably cumbersome, sufficient reason by itself for Plato to avoid it.

But the shift in his method of procedure should not disguise the significance, as he understood it, attaching to the system's complete definability in terms of concordances, or, more fundamentally still, through his application of the theory of proportion. We have already noticed the close affinity between conceptions of concordance and of coherence or unity, and the emphasis laid by Plato in the *Republic* on the need to identify 'concordant numbers'. In the *Republic* he also demands an account of the reasons why some numbers and not others are concordant, and the explanation, it would now appear, is to be provided by the theory of proportions. Numbers are concordant with one another if they are so organized as to interweave the three kinds of proportion in a maximally economical way, through the insertion of arithmetic and harmonic means between terms in a geometrical progression of the simplest sort (that is, the series of doubles). That the role

29 This is only to be expected, at least in the case of Thrasyllus, since it is highly probable that his account of the division occurred in a commentary on the *Timaeus*.

of proportional relations is to bring diverse entities into a coherent unity is a thesis outlined a little earlier in the *Timaeus*:

It is not possible for two things by themselves to be well put together without a third, for there must be a bond between them that will unite the two of them. The finest bond would be that which makes itself and the things bonded one, to the greatest possible extent, and it is proportion (*analogia*) that can achieve this in the finest way. For whenever, of three numbers, the middle term between any cubes or squares³⁰ is such that, as the first is to it, so is it to the last, and again, as is the last to the mean, so is the mean to the first, then the mean becomes first and last, and the last and the first conversely both become means, and all of them thus necessarily come to be the same, and in becoming the same as one another they will all be one. (*Tim* 31b8-c7)

The intriguing details of this passage and its sequel need not concern us here. What matters is the general sense — to which the closing statements give heavy rhetorical, if not perfectly intelligible, emphasis — that the role of *analogia* is to bring unity to separated elements, in a closely coordinated system of relations. That, certainly, is the main motive for Plato's use of Archytan proportion theory in his quasi-musical account of the world soul's structure. It would give an equally comprehensible rationale for attempts to define explicitly musical structures through patterns of proportional relations on the grounds that a well-ordered attunement is, precisely, one whose various elements are welded into a coherent unity.

3b. Archytas

We may now return to Archytas himself. Plato's procedures, when the aspects relevant to our field of interest have been isolated, will generate

30 The opening of this sentence raises problems to which I have no solution. For discussion see Cornford (1937), 44-50. The present translation gives a sense very close to his. I am not convinced that Cornford explains satisfactorily why, on this interpretation, Plato requires the extreme terms in these relations to be either cubes or squares. Translations which avoid this restriction on the terms are certainly possible but pose problems of their own which are at least equally serious.

the attunement over the range of an octave indicated in figure 9, in which higher notes correspond to higher positions.

Notes	Ratios		Intervals		
1		upper tetrachord	tone	fourth	fifth
2	9.8		tone		
3	9.8		<i>leimma</i>		
4	256:243				
5	9.8	disjunction	tone	fourth	fifth
6	9.8	lower tetrachord	tone		
7	9.8		tone		
8	256:243		<i>leimma</i>		
octave					

Figure 9

None of the attunements attributed to Archytas is divided in this way. Ptolemy records three Archytan divisions, one for each of the genera mentioned in section 1, enharmonic, chromatic, and diatonic.³¹ They are described in figure 10.

31 See Ptolemy, *Harm* I.13-14 (quoted in part at Diels (1956), 47.A16) and II. 14.

Notes	Enharmonic	Chromatic	Diatonic
1	5:4	32:37	9:8
2	36:35	243:224	8:7
3	28:27	28:27	28:27
4	9:8	9:8	9:8
5	5:4	32:27	9:8
6	36:35	243:224	8:7
7	28:27	28:27	28:27
8			

Figure 10

It is obvious that Plato's simple application of the three kinds of proportion cannot generate these results. But a different and only slightly more complex manipulation of them will serve the purpose. The construction has two stages. First, although the ratios of some of these scalar steps are distinctly inelegant, especially in the chromatic division, each note in all three divisions can nevertheless be constructed, from a given starting point, by movements through intervals whose ratios are simple and form a simple set. No ratio is required beyond the first eight 'epimorics',³² those whose terms lie within the decad, i.e., between 1 and 9. Thus notes 1, 4, 5 and 8, in all the genera, are 'fixed' notes, related through the ratios of the concords, 2:1, 3:2, 4:3. Note 3 is also the same in all three divisions. It is most simply located by the ratio in which it stands to note 5, which is 7:6 (= 28:27 × 9:8). Note 3 is straightforwardly related to its counterpart in the lower tetrachord, note 7, standing to it in the ratio 3:2. Only notes 2 and 6 in each division

32 An epimoric ratio, now usually called 'superparticular,' can be informally described as one whose form is $n+1 : n$. For further discussion see the next section.

remain, and once one of them is found the other is automatically located, since these two also stand in the ratio 3:2, that of the fifth. In enharmonic notes 1 and 2 are in the ratio 5:4, and note 2 is also in the ratio 6:5 to note 5. In diatonic note 2 is related to its neighbors by the ratios 9:8 and 8:7. The case of the chromatic looks more complicated, but in fact it is not. As Ptolemy points out — attributing the explanation to Archytas himself — note 2 in chromatic is lower by a *leimma* than its counterpart in diatonic (that is, $9:8 \times 256:243 = 32:27$).³³ Since the interval between notes 1 and 4, whose ratio is 4:3, amounts to $9:8 \times 9:8 \times 256:243$, it follows that the ratio between notes 2 and 4 in chromatic is 9:8 ($= 243:224 \times 28:27$). Hence every note in the three systems can be constructed by operations involving only the ratios 2:1, 3:2, 4:3, 5:4, 6:5, 7:6, 8:7, 9:8.

It is then easy to express the divisions in terms of Archytas' scheme of means and proportions. Plato's system required the insertion of arithmetic and harmonic means only in the octave relation, between terms in the ratio 2:1, where the ratios between means and extremes are 3:2 and 4:3. If, in addition, such means are located between terms in the ratio of the fifth (3:2), the ratios between means and extremes are 5:4 and 6:5; and if they are inserted between terms in the ratio of the fourth (4:3), the corresponding ratios are 7:6 and 8:7. The ratio 9:8 has of course emerged already as that between the two means in the octave.

This exposition has been designed to show, in as simple a way as possible, both how the principle of proportional division can be seen at work in Archytas' divisions, and how it might be used to support the credentials of various other forms of division; for it is plain that the set of ratios we have employed could be rearranged to form a number of other tetrachordal patterns.³⁴ Nevertheless, the principle, for all its power and versatility, is mentioned rather rarely in harmonic treatises, and its direct use in the derivation of harmonic divisions is rarer still. Archytas' procedures do seem to have been influential, but the direc-

33 On the significance of this point, see the discussions of Burkert and Barker cited in note 21 above.

34 For example, the diatonic and chromatic divisions of Didymus, Ptolemy's soft chromatic, and all but one of his five diatonics: see the tables in Ptolemy, *Harm* II.14.

tion which they prompted later theorists to take did not involve, at least explicitly, any further developments in proportion theory.

4. Epimoric ratios and commensurable excesses

A thorough study of Archytas' divisions would take us too far afield. It seems clear that they were destined (as Plato's was not) as direct representations of attunements in practical use; and the differences between them and other such representations opens up a rich selection of musicological issues.³⁵ These are not our concern; but one further mathematical feature of the divisions requires attention.

We have already had occasion to refer to the class of 'epimoric' ratios. A cursory glance at the ratios uncontroversially involved at the simplest level of harmonic analysis, 2:1, 3:2, 4:3, 9:8, will suggest that ratios of the form $n+1:n$ have some special role. It became a commonplace of mathematical harmonics (although one challenged by Ptolemy³⁶) that all genuinely concordant intervals, at any rate, must have ratios that are either epimoric or multiple.³⁷ The *Sectio Canonis*, for example, enunciates this thesis about concords and makes substantial use of it.³⁸

In introducing Archytas' divisions, Ptolemy remarks that he 'tried to preserve what follows the principles of reason not only in the concords but also in the divisions of the tetrachords, believing that a

35 Some of them are sketched in the appendix to chapter 1 of Barker (1989).

36 See Ptolemy, *Harm* 1.5-6. The most obvious difficulty is that posed by the interval of an octave plus a fourth, which by ordinary aesthetic criteria and some theoretical ones was a genuine concord, but whose ratio is 8:3.

37 Multiple ratios have the form $m:n:n$. The octave ratio, 2:1, is properly classified as multiple rather than epimoric because an epimoric ratio (*epimorios logos*) is one in which the larger term is constituted by the lesser and one part (*morion*) in addition (*epi*). In the ratio of the octave the amount added to the lesser to form the greater is equivalent to the whole, not a part, of the lesser.

38 The principle is stated at the end of the introduction to the *Sectio*, and is drawn on especially in proposition 11, where, however, the reasoning is faulty and cannot be repaired; see Barker (1981) and (1989), 200, note 28. The problem of the octave plus fourth is evaded in this treatise: the ratios of the octave plus fifth and the double octave (3:1, 4:1) are derived in prop 12, but the interval that creates the difficulty is not even mentioned.

commensurable relation between the excesses is a characteristic of the nature of melodic intervals.' (*Harm* 30.10-13) Here 'melodic' intervals are the individual scalar steps. We shall return shortly to the notion of a 'commensurable relation'. For the present, anticipating some results, we need only note Ptolemy's intended implication that Archytas' pursuit of the 'principles of reason' required him to attach an epimoric ratio to each scalar step as well as to each concord.

If that was Archytas' aim, he failed in the case of the chromatic division, as Ptolemy is quick to point out. The analysis offered above suggests that Ptolemy may have misconstrued Archytas' intentions: not every scalar step is epimoric, but each system is completely determined by the interweaving of epimoric ratios, and ones whose terms are small numbers. There is no indication in our sources, however, that any fourth-century theorist found a satisfactory way of explaining what is so special about epimoric ratios and why they should be privileged above others in the context of harmonic analysis. The author of the *Sectio Canonis* does indeed offer a justification for his claim that the ratios of concords must be either epimoric or multiple. But the argument is impressionistic and weak;³⁹ and even if accepted, its application could not extend to intervals other than the concords. Archytas himself might have pointed to the way in which new epimorics are formed by the insertion of arithmetic and harmonic means between terms in some given epimoric ratio and thereby have grounded the status of epimorics in his theory of proportions. But, although this strategy would be sufficient to generate epimorics suitable for harmonic division (granted that the ratio from which division begins is that of the octave, 2:1), it is still theoretically inadequate to sustain the attribution of special mathematical status to epimorics in general. The primacy of the ratio 2:1 (which is not properly epimoric at all) remains grounded in nothing but intuition combined with empirical observation. And the ratios formed between means and extremes when the means are inserted between terms in non-epimoric ratios will themselves, of course, not be epimoric.

Nevertheless there can be no doubt of the importance attached to ratios of this sort. All later Greek divisions are expressed either exclusively or almost exclusively as concatenations of epimorics.⁴⁰ We know

³⁹ See Barker (1981), 2-3.

⁴⁰ See particularly the tables in Ptolemy, *Harm* II.14.

that Archytas propounded a theorem concerning them, to the effect that between terms in such a ratio there is no (geometrical) mean proportional or proportionals. A version of the proof reappears in the *Sectio Canonis*.⁴¹ As we have seen, the consequence of this theorem is that no interval in epimoric ratio can be subdivided into two or more equal intervals whose values can be expressed as ratios of integers. Since the basic ratios to be subdivided in harmonics were uncontroversially epimoric in form (especially that of the fourth, 4:3), or else shared with the epimorics the feature ensuring that there is no mean proportional between the terms (this is the case with the octave, 2:1), they could not be divided either equally or in any way that implied the possibility of equal division into integral ratios. One of the major challenges that Archytas faced, given his proof, was to find some other governing principle that could rival that of equal division in simplicity and rational appeal. His proportional approach offered a persuasive solution to the problem.

But the question how different sorts of ratio could properly be invested with different levels of significance remained so far without a satisfactory answer. Theon of Smyrna records an ingenious procedure for the systematic generation of ratios, which may possibly have been devised with these issues in mind.⁴² He attributes the form in which he presents it to Adrastus (first century C.E.). However, he remarks that it was originally due to Eratosthenes (third century B.C.E.), who, he says, gave a less clear account. The procedure takes as its starting point the relation of equality. It then applies to the terms of the equality a simple operation, which is then repeated for the terms that result. Further repetitions follow, with modifications in the order in which the terms are taken. The procedure generates classes of ratio out of the initial equality in a determinate order of precedence: first multiples, then epimorics, then ratios having neither of these forms ('epimerics'), among which several subspecies are distinguished. The ratios within each class also emerge in a systematic order, those with smaller terms having priority.⁴³ It seems clear that the possibility of this orderly kind

41 See note 15 above.

42 The discussion runs from 106.12 to 111.9; detailed exposition begins at 107.23.

43 The procedure begins from three equal terms, 1, 1, 1. We then take 'one term equal to the first, one compounded from the first and the second, one from the first and

of derivation might have given grounds for assigning different status to different classes of ratio in a hierarchy whose pinnacle is the relation of equality; and it would have been possible to make use of this ordering in theoretical harmonics. There is no direct evidence, however, that it was so used. The approach taken to harmonic divisions by Adrastus shows no trace of it;⁴⁴ and Eratosthenes' divisions offer little encouragement to the hypothesis that he drew on it in this connection.⁴⁵

4a. *Ptolemy*

A thoroughgoing justification of the principle that the ratios of melodic intervals must be epimoric is found in no source before Ptolemy. His discussion, and his use of the principle to generate his harmonic divisions, are too complex and subtle to be treated adequately here.⁴⁶ I shall attempt only a sketch of some central points. His argument turns on two claims. The first is that the excellence of any ratio must be assessed by its 'closeness to equality' in a sense to be defined. (It is not the same as the relation involved in the Eratosthenes-Adrastus derivations.) Secondly, all simple melodic intervals, that is, all scalar steps, must be

two of the second and the third,' that is, 1, (1+1), (1+2+1), or 1, 2, 4. Repeating the operation on 1, 2, 4 gives 1, 3, 9, and further repetitions give 1, 4, 16, etc. If we then start again from the series of doubles but take the terms in reverse order, 4, 2, 1, the same operation gives 4, 6, 9, a series in which each term stands to its predecessor in the first epimoric ratio, 3:2. Operating in the same way on 9, 3, 1 we get 9, 12, 16, where the terms are related in the second epimoric ratio, 4:3; and further repetitions will give successive epimorics, 5:4, 6:5, etc. Operations on these epimoric ratios, taking the terms first in one order, then in the other, will give the primary classes of epimerics, and these in turn can be treated in the same way to generate progressively less simple and fundamental forms of ratio.

44 Theon does not attribute to Adrastus a formal division of the *kanōn*. But the material he derives from this source throughout the passage from 49.6 to 72.20 makes it clear that Adrastus' interest was largely or exclusively in the Platonic scheme of division, to which the hierarchy of ratios is not directly relevant. There is no sign of any attempt to put it to use here.

45 See Ptolemy, *Harm* II.14, with notes 117, 122, 125 in Barker (1989), 346-9.

46 See especially Ptolemy, *Harm* I.7 and 15, and, for some further discussion, Barker (1990).

such that their 'excesses are commensurable' (*Harm* 16.12-17). We have already met the notion of commensurable excesses in Ptolemy's comments on Archytas. It emerges that the sense of this expression is that the excess of one term in the ratio over the other (that is, the difference between the terms) must be a 'simple part' (*haploun meros*), a factor, of the smaller term (*Harm* 16.19).⁴⁷ In that case it follows immediately that the difference will be a simple part also of the greater term, since if, in the ratio $A:B$, $B = n \cdot (A-B)$, it is evident that $A = (n+1) \cdot (A-B)$. And in that case the ratio will be epimoric, since $(n+1) \cdot (A-B) : n \cdot (A-B) = (n+1):n$.

Returning to the first thesis, that the more excellent ratios are those closer to equality, we find that this does not refer to those ratios whose terms are more nearly equal. The equality holds rather between the smaller term and the difference between the terms (*Harm* 15.24-5, 16.17-21). Hence the finest of the ratios is 2:1, in which the smaller term is precisely equal to the difference; and since, after this beginning, only epimoric ratios are to be considered, they will plainly fall into an orderly sequence from the more excellent to the less, the more excellent being those whose terms are smaller.

Ptolemy's account makes sense. But why does he suppose that it provides cogent reasons for privileging and ordering the epimoric ratios in the context of harmonics? The fact that epimoric ratios have interesting features is obviously not enough by itself to show that all melodic intervals must take this form or that those whose ratios have smaller terms are better, 'more melodic' (*emmelesteroi*, e.g., *Harm* 16.18). The matter is complex, but two points are of special significance. First, we should notice that in treating the special feature of epimorics as one that is bound up with the relation between the smaller term and the difference between terms, Ptolemy is drawing quite naturally on the manner in which Greek mathematicians referred to and expressed these ratios. To describe a ratio as *epimorios* is precisely to designate it as one where there is a 'part in addition'. Individual epimoric ratios are expressed by such words as *epitritos*, *epogdoos*, and so on, meaning 'a third in addition' (4:3), 'an eighth in addition' (9:8). The expression indicates the fraction of the smaller term by which the greater exceeds

⁴⁷ This condition is not stated explicitly for the simple concords, but is plainly presupposed in the derivation of the ratios of the fifth and the fourth at *Harm* 15.29-16.6. See also 16.12-25, noting especially the references to halves and thirds at lines 20-21.

it. Ptolemy's treatment of these ratios as those in which the difference between the terms is a 'simple part' of the smaller term is straightforwardly in line with this usage.

Secondly, we must ask why this feature of epimoric ratios should give them any special importance, why it is essential that melodic ratios should have this feature, and why 'closeness to equality' is appropriately measured by the relation between the smaller term and the difference. Here Ptolemy's stance is of great interest and is thoroughly characteristic of his general approach to harmonic theory. The principles enunciated are ones that make consistent mathematical sense, and to that extent they are 'rational'. But in explaining why they should be preferred to any other, equally intelligible considerations, Ptolemy makes no appeal to further rational principles of a higher order. Instead, he leaves their credentials to emerge from reflections about the capacities of perception. Summarily, his position is this. From a perceptual point of view, the finest relations between sounds are those that are most clearly and accurately recognizable.⁴⁸ Now perception is the more reliable in 'detecting the amounts by which differing things exceed one another' where 'the amounts in question consist in larger parts of the things to which they belong.' (*Harm* 4.12-13) As the examples that follow show (*Harm* 4.21 - 5.2), these 'parts' are to be understood as 'simple parts'; and the sense is that perception can judge the size of the difference between two quantities, relative to the sizes of those quantities themselves, the more accurately when the difference is a larger factor of the terms. Hence we judge the relation between two objects whose magnitudes stand to one another as 16 to 15, for example, less reliably than that between magnitudes related as 4 to 3. It is implied that the relation is still harder to recognize when the difference is not a simple part of the terms at all and hence is not a common measure of them. Perception cannot hope to judge accurately the relation between the difference and the terms when the latter are magnitudes standing to one another as 7 to 5, for instance.⁴⁹

48 Ptolemy gives no direct statement of this thesis. But it seems to lie behind the reflections of *Harm* I.1, and is required as a presupposition to link them with the discussions in Book I.7 and 15.

49 Hearing does not of course grasp intervals in their character as ratios between the quantities bounding them; but the ratio is the 'form and cause' of the modification (*pathos*) of the air which the senses perceive qualitatively. See *Harm* I.1 and 3, II.3

Whether or not Ptolemy's thesis here is persuasive (and it is certainly not implausible), the important point is the way it matches his later remarks about epimoric ratios. The principles enunciated there are to be adopted, not because their mathematical or metaphysical status can be shown on rational grounds to exceed all others, but because they give a mathematical interpretation to forms of judgement that have been found relevant to harmonics at the perceptual level. It remains possible that other kinds of mathematical excellence exist, grounded in other principles and relevant to other features of our experience. What the analysis shows is that a coherent account of one form of excellence can be built up from the notions of 'commensurable excesses' and 'closeness to equality', and that this is the one whose dominant role in the aesthetic assessment of quantitative relations, including those in harmonics, is assured by the nature of the perceptual judgements on which such assessments must draw.

Our next step, evidently, should be to investigate the ways in which Ptolemy applies his principles in the detailed business of harmonic division. But I shall not attempt that here. His intricate methods of derivation, for which further principles are also required, demand at least an essay-length study to themselves.⁵⁰ Perhaps, however, the rather general and preliminary remarks I have offered may suggest one respect (and there are others) in which his approach was capable of breathing new life into mathematical harmonics, a science which in his time looked set to wither into an arid scholasticism. Ptolemy insists that any hypothesis in harmonics must satisfy both rational and perceptual tests, and in particular that if divisions generated from allegedly rational principles fail to satisfy the judgements of the ear, this shows, not that reason as such is an inappropriate criterion of correctness, but that the principles have been wrongly chosen or applied.⁵¹

What is it, then, for a principle to be wrongly chosen or applied unsuitably, if it and the use made of it are rationally intelligible? Such a question cannot readily be answered by writers in the strictly 'rationalistic' tradition that stems from Plato. Archytas' theory of proportions is rationally intelligible, and his application of it is neither less nor more

50 There are further discussions of the principles and their application (but not of the details of the process of division) in Barker (1990). See also Düring (1934), 197-200.

51 See especially *Harm* I.2 and the opening of I.7.

intelligible than that of Plato; yet their conclusions differ, and others, different again, might be propounded consistently with the principles involved. Which application is to be preferred, and why? How can the principles adopted by Archytas and Plato be judged against those of Ptolemy or against any other logically consistent principle that a theorist might choose? Abstract mathematics alone seems to suggest no answer. Ptolemy insists, certainly, on the intelligibility of his principles: it is the scientist's task to demonstrate the rationality and orderliness characterizing everything that is perceptibly beautiful (see especially *Harm* I.2). But his choice of principles is governed by the conviction that they represent mathematically the form taken by the corresponding perceptual judgements.

Hence challenges to Ptolemy's analyses can be mounted on the basis of stateable criteria, and the resulting controversies might even, in principle, be decidable. It is not merely that a critic is entitled to object to Ptolemy's results, as Ptolemy frequently emphasizes, on the grounds that the divisions he constructs are perceptually unconvincing. He may also offer criticisms of the principles from which the divisions arise, not because he finds them rationally inadequate (in which case his reasons would probably be impressionistic and undefinable), but because he believes that they involve misinterpretations of the forms of perceptual judgements. If it could be shown empirically, for example, that gradations of perceived 'harmoniousness' in relations between sounds do not correspond, after all, to degrees of 'closeness to equality' in the sense outlined above, that would be enough to call Ptolemy's whole system into question.

5. Conclusion

We have looked, fairly cursorily, at three kinds of approach to harmonic division. It has become clear that in various respects they overlap, and might in some instances be used to complement one another. First, the results of the simple procedures of Thrasyllus and the *Sectio Canonis* are identical with those of Plato's more complex strategy. In view of their shared preoccupation with the ratios of concordant intervals, this is not surprising; and Thrasyllus, as we have noted, was in any case working in a thoroughly Platonic context. Here the most significant point, for Plato himself at any rate, lay in the capacity of the proportional procedure to underpin the claims of the 'method of concordance', and of its geometrical counterpart in the division of a line, by explicating the

rational grounds of the coherence and unity of the structures they generated. Plato's procedure overlaps, in turn, with that of Archytas, who deploys the theory of proportions in ways less simple, and hence perhaps less metaphysically appealing than those of Plato, but thereby makes some progress towards the goal of locating rational coherence in the attunements of real musical practice. His approach overlaps again with that of Ptolemy, both in its determination to address the perceptible data (which for Plato were only of marginal interest), and more specifically in the status it assigns to epimoric ratios. It remained for Ptolemy to identify the features of these ratios which made their special status rationally intelligible. Paradoxically, the success of his strategy lies in its integration of their rational credentials with an analysis of the ways in which harmonious relations are perceived and appreciated by the irrational ear.