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WARRACK, G.
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MUSIC AND MATHEMATICS

By GUY WARRACK

How often has it been said in conversation that "Music and Mathematics go together"? It is probably no more absurd than most other such generalizations; but it is certainly no less so, and in a hundred people who make such an assertion it would be lucky to find a single one who could substantiate it with any conviction. Certainly many mathematicians are interested in music, and many musicians in mathematics, but that proves nothing. It would be easy to point to at least as many tone-deaf mathematicians and to as many musicians whose idea of mathematics is making an income-tax return (probably getting it wrong). It would be very difficult to find one man who had done first-rate work in both spheres. On the whole the credit-balance seems to lie with the mathematicians, for some of them have at least written books about musical subjects, whereas I know of no instance of a musician writing on mathematical subjects.

However, writing about music or mathematics does not make the writer a musician or a mathematician. Indeed, very few writers on music can in any sense be called musicians, and as to mathematics, "The function of a mathematician is to do something, to prove new theorems, to add to mathematics, and not to talk about what he or other mathematicians have done". So wrote G. H. Hardy, in 'A Mathematician's Apology', and all musicians will agree that, *mutatis mutandis*, his words apply equally to them.

Yet the writer on musical subjects cannot be dismissed quite so lightly, for though he makes no direct contribution to the art of music, he may advance the science of it, and the scientific advance will clear the path for artistic advance. Probably, then, the most useful writings on music have not been by musicians but by mathematicians, or at any rate by mathematical physicists, for they have researched into acoustics, and it is the science of acoustics that has brought our musical instruments to the present high state of efficiency without which the playing of music as we know it would be impossible.

These acousticians are truly the "back-room boys" of music. Some have been pure mathematicians of the first rank, some, probably the majority, have been more physicists than mathematicians, and a few have had more ingenuity than scientific training. None, as far as I know, has been a member of the front-rank. Among the first of the categories we should have to include such names as Pythagoras, Mersenne, Descartes, Brook Taylor, Euler, d'Alembert and Lagrange. Many of these may be considered to be mathematical physicists as well as pure mathematicians, but the distinction between pure and applied mathematics was formerly less rigid than it is to-day.

Every schoolboy knows, not always very affectionately, the name of Pythagoras in connection with one of the many theorems with which he enriched mathematics, that which relates the length of the hypotenuse of a right-angled triangle to the lengths of the other two sides. In actual fact many of his theorems were more important and more beautiful than this one, but the fate of many to be remembered popularly by works which are neither their best nor their most characteristic. In the middle of the sixth century B.C. Pythagoras turned his mind to acoustics. Indeed, he is often regarded as the first acoustician, for he was the first

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to realize that if a string sounding a certain note is halved in length, the new resultant sound will be exactly an octave higher than the original one. He proceeded further to divide his string into other fractions, so laying the foundation-stone of the theory of Harmonic Series, the basis of all acoustics. He is commemorated in acoustics to-day by the "Pythagorean comma" which is the difference between two enharmonically related notes.

If Pythagoras is popularly connected with a theorem that is not his finest, Pierre Mersenne (1588–1648) is chiefly remembered by a mathematical assertion which has since been proved to be definitely incorrect. In his 'Cogitata Physico-Mathematica', published four years before his death, he stated that if $N=2^p-1$, the only values of p not greater than 257 that make N prime are 1, 2, 3, 5, 7, 13, 17, 19, 31, 67, 127 and 257. Later research has shown that 61, 89 and 107 should have been included in the list, and that 67 and 257 should not. It is not yet known whether or not N is prime when $p=157, 167, 193, 199, 227$ and 229. Besides being a mathematician, Mersenne was a theologian, philosopher and musicologist. In this last capacity he brought out his 'Traité de l'harmonie universelle' in 1627, which was extended into the 'Harmonie universelle' nine years later. In these works he gives much valuable information about the musical instruments of his time and deals with the increasingly urgent problems of temperament.

René Descartes was *par excellence* a creative mathematician. His Analytical Geometry is one of the greatest and most far-reaching inventions in the whole history of mathematics. He was a pioneer of the undulation theory in acoustics and, like Mersenne, exercised with the question of temperament. His contributions to musical theory are contained in his letters and his 'Compendium Musicae', which appeared in 1650.

Brook Taylor (1685–1731) opened new mathematical doors with the key of the Calculus of Finite Differences, first expounded in 1715 in his 'Methodus incrementorum directa et inversa'. His researches enabled him to determine the form of movement of a vibrating string, and he found that the rate of the string's vibration varied directly as the weight stretching it, and inversely as its own length and weight—a discovery of the first importance. It was in the 'Methodus' that he first enunciated the well-known theorem bearing his name which led to such important developments as Newton's Binomial Theorem.

Leonhard Euler (1707–1783) turned his magnificent analytical brain to almost every conceivable sort of problem that could be attacked mathematically, whether the problem was in itself important or comparatively trivial, such as moving a knight over every square of a chess-board or going for a walk over the seven bridges of Königsberg without walking along the same road twice. His mathematical discoveries of importance are too many even to summarize here, but in 1739 he published a 'Tentamen novae theoriae musicae' in which he used logarithms for calculating the pitch of notes for the first time. His attitude to music was that no conglomeration of sounds can be satisfactory unless the law of their arrangement can be perceived—clearly a mathematician's attitude rather than a musician's. Leibnitz went still farther when he described music as an unconscious act of calculation.

Jean le Rond d'Alembert (1717–1783) was not, perhaps, in the very first flight of mathematicians, as were Pythagoras, Descartes, Taylor and Euler. Still, he made his contribution to the lore, and his postulate that any algebraic equation has at least one solution, real or complex, is important. It took half a century to prove. His writings on musical

subjects greatly exceeded (in bulk anyhow) those of his mathematical predecessors. On the partly physical side there are his 'Recherches sur la courbe que forme une corde tendue mise en vibration', written in 1747, and, fourteen years later, his 'Recherches sur la vitesse du son' and 'Recherches sur les cordes sonores'. On the more aesthetic side we note the 'Éléments de musique théorique et pratique suivant les principes de M. Rameau' and 'Fragments sur l'opéra' (1752), in which he defends Gluck *contra mundum*. Later came his treatise 'De la liberté de la musique'.

Taylor's theorem has already been mentioned, but oddly enough its importance was not spotted for nearly sixty years, when Joseph Louis Lagrange (1736–1813) described it as "le principal fondement du calcul différentiel". Lagrange contributed some elegant theorems to the Theory of Numbers. He is famous for the postulate (which he did not succeed in proving) that every prime of the form $4n-1$ is the sum of a prime of the form $4n+1$, and of double another prime, also of the form $4n+1$. And whereas Euler had proved that any quadratic irrationality can be represented by a periodic continued fraction, Lagrange proved the converse, that any such periodic continued fraction represents a solution of a quadratic equation. Besides being a pure mathematician of a high order, Lagrange also did acoustical work and, in or about the year 1758, published his 'Recherche sur la nature et la propagation du son'.

These seven mathematicians make a strong team. Of those who were physicists rather than mathematicians Hermann von Helmholtz (1821–1894) was probably the most distinguished; but our present preoccupation is with mathematicians and musicians, not with physicists. Some ingenious musicians have sought to improve their instruments, and must have had a fair notion of mathematical physics to do so, though they need not have been, and almost certainly were not, mathematicians in Hardy's sense of the word any more than Taylor or Euler were musicians in ours. Among these men—to cite merely two of many—there were Stölzel, a Breslau horn player in the early nineteenth century, who developed the newly invented valve-horn, and Theobald Boehm (1793–1881), a Munich flautist, who revolutionized the fingering of many woodwind instruments beside the flute. These were great benefactors, and it might not be far from the mark to say that without their researches 'Till Eulenspiegel' and 'Daphnis et Chloë' could not have been written.

Composers as a class are not much concerned with the scientific side of their art. The obvious exception is Rameau, who delved deeply into the theory of music and studied the acoustical writings of Mersenne and Descartes. He had the advantage over them that he was a far better musician, and could refute some of their theories on aesthetic grounds, while turning others, which had hitherto remained only theories, to artistic account. His findings were published in his 'Traité de l'harmonie' (1722), 'Nouveau Système' (1726), 'Génération harmonique' (1737), 'Démonstration' (1750) and 'Nouvelles réflexions' (1752).

Besides the works of Mersenne and Descartes, Rameau had read the theoretical writings of Gioseffo Zarlino (1517–1590). These include the 'Institutioni armoniche' (1558), 'Dimonstrazioni armoniche' (1571) and 'Sopplimenti musicali' (1588). Zarlino, like Rameau, and unlike Mersenne and Descartes, was a musician. He was choirmaster at St. Mark's, Venice, and a composer of great note in his day. As a theorist he was exceedingly advanced for his time and in dealing with problems of temperament hit on the device of illustrating his points with

diagrams, a method which was taken up by later acousticians, including Descartes.

Enough has been said to show that some of the greatest mathematicians have interested themselves in music and have added some knowledge to its science. Those whom I have mentioned as being musicians were not great mathematicians, and those who were great mathematicians were not musicians; yet they have all played an important part in the development of the art. I do not think that any musician has made a contribution to mathematics comparable with the mathematicians' contribution, even if it is an indirect one, to music.

Certainly one or two musicians have had mathematical ability. François André Philidor (1726–1795), whose real name was Danican, was evidently a highly original composer and operatic innovator. I regret to say that I do not know his operas, but he introduced such novel devices as the “air descriptif”, unaccompanied quartets and a duet formed of two apparently incongruous melodies. His operas were many in number and are said to be superior to those of Grétry. He was a considerable mathematician, but his abilities were devoted to chess, of which he was probably the most famous player of his day. He wrote a notable work, ‘Analyse du jeu des échecs’, and astonished the London Chess Club by simultaneously beating three first-class players “blindfold”. It used to be said that the late Sir Walter Parratt could play several games of chess and a Bach fugue at the same time. I know nothing of either Philidor's or Parratt's mathematics (Grove tells us that the former had “a natural gift for abstruse calculations”), but they were great chess players, and Hardy says—

every chess-player can recognize and appreciate a “beautiful” game or problem. Yet a chess problem is *simply* an exercise in pure mathematics (a game not entirely, since psychology also plays a part), and everyone who calls a problem “beautiful” is applauding mathematical beauty, even if it is beauty of a comparatively lowly kind. Chess problems are the hymn-tunes of mathematics.

One other name comes into my mind at this point—the name of a man who was a professional mathematician before becoming a professional musician: Ernest Ansermet. He was professor of mathematics in the University of Lausanne until he became a conductor of international repute. His mathematical reputation was, I imagine, largely local, and I do not know that he made any striking additions to mathematics. As a conductor he has an uncanny memory and conducts the works by Stravinsky without a score, works in which, as a wit once put it, “the time-signatures read like the morning trains to London”.

I have been drawing attention to isolated cases where music and mathematics meet in individuals, fully realizing that such cases constitute no proof that in general music and mathematics “go together”. No doubt as many instances could be cited where they clearly do not. As far as I know, Schoenberg has never published ‘Vorträge über ausgewählte Fragen der Zahlentheorie’, we have not yet read Vaughan Williams on Diophantine Equations, nor has Sir Thomas Beecham conquered new worlds with ‘Functions of a Complex Variable’. On the whole I think that these gentlemen are wise to stick to making music. But if it is true that in certain cases the two arts and sciences (for music is a science as well as an art, and mathematics an art as well as a science) do “go together”, then we shall want to know the reason why. I suggest that when two apparently incongruous interests are united in one man, it may be for one of two opposite reasons. Either the two interests are not so incongruous as they might appear superficially, and have, in fact, something in common which commends them both to that same man;

or else they are so opposed to each other as to be mutually complementary, and then, when a man's mind is fatigued with one, he escapes for relief to the other. Furthermore, these alternative opposites are not irreconcilable, for the two interests may have a common substratum, but widely different superficial manifestations. I suspect that this is just how it is with music and mathematics.

Superficially they appear opposites, the one appealing to the emotions through the senses, the other to the intellect. I doubt, however, whether this view will bear close scrutiny. For one thing, much music-making, whether composing or performing, is laborious, detailed, highly technical spade-work akin to calculation; for another, the finest mathematical work is as creative and often as intuitive as the writing of a symphony. Beauty and truth not only pervade both processes, they are the chief aim of both.

Non-mathematicians have occasionally been puzzled by the term "creation" as applied to mathematics. "The material is all there", they will argue; "all the mathematician has to do is to work it out". This is to some extent justifiable, but only to the same extent as it is in music. The notes are "there"—a very limited number of them are available for practical purposes, when all is said and done—and all the composer has to do is to work them into nice patterns. But no one would deny that this arranging of notes into patterns is in every sense creation. What does the mathematician actually do? Hardy gives us the answer:

A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with *ideas*. . . . The mathematicians' patterns, like the painter's or the poets, must be *beautiful*; the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics.

Although the comparison is with the sister-arts of poetry and painting, it is but a small step to apply it to music: indeed the writer almost takes this step himself when he talks of the ideas fitting in a "harmonious" way. We can, then, expect the creative mathematician to find just the same aesthetic joy in his work as the musician.

But Hardy goes on to show the great difference. "Music", he truly says, "can be used to stimulate mass emotion, while mathematics can not." Certainly one cannot imagine that if Cantor's theory of Linear Aggregates were expounded in the Albert Hall it would be greeted with the tempestuous applause that is called forth by Beethoven's fifth Symphony. But does this affect the creator? Not much, I think. Cantor worked at his theories for the same reason as Beethoven worked at his symphonies, because each was forced by his nature to express his personality through the medium that happened to suit it best. Every composer, whether of symphonies or theorems works primarily to satisfy himself and his artistic conscience. (Of course this applies to all artists, but we are not concerned with the others.) True, there may be extrinsic incentives, money, fame, or what you will, but the work, once undertaken, is guided solely by this conscience.

In both arts creative excitement has occasionally outrun discretion and accuracy of detail. Beethoven composed at such white heat that he could not stop to notice that at one point in the seventh Symphony he gave sub-dominant harmony to the wind and tonic harmony to the strings. Naturally this in no way affects the greatness of the symphony, and every conductor corrects the slip. More serious are Beethoven's artistic lapses—it is probably generally agreed that a few of his compositions are as banal as others are sublime. Mathematicians have also had their lapses.

Fermat, for instance, searching for an algebraical formula which would generate primes, gave in 1640 as a solution $2^{2^n} + 1$. His assertion was disproved a century later by Euler when he showed that $2^{2^5} + 1$, or $2^{32} + 1$, or 4294967297 was divisible by 641. It must be remembered that in the seventeenth century there was less polished machinery for testing primality than there is now, but even so it seems odd to us that Fermat should have made this positive statement which could be disproved by taking such a comparatively early value of n as 5. As a matter of fact, during the last seventy years, $2^{2^n} + 1$ has been shown to be composite for another dozen values of n , namely 6, 7, 8, 9, 11, 12, 15, 18, 23, 36, 38 and 73. In the meanwhile Karl Friedrich Gauss (1777–1855) had by chance found that the “Fermat Numbers”, so far from being an isolated fact in the Theory of Numbers, had a direct bearing on the problem of inscribing regular polygons within a circle. It must be stated in Fermat’s favour that he issued no “proof” of his assertion (which was arrived at by the dangerous path of induction). A false assertion is only rash, while a false proof is an artistic sin. Although Fermat’s false assertion has perhaps attracted undue attention, he lives through his great work, ‘*De maximis et minimis*’, the theorem showing that if p is prime and a is prime to p , then $a^{p-1} - 1$ is divisible by p , and his statement that the equation $x^n + y^n = z^n$ is impossible for values of n greater than 2, x , y and z being integers (not yet generally proved, though Fermat claimed to have discovered “a truly beautiful proof”), just as Beethoven is remembered by ‘*Fidelio*’ and the symphonies, and not by ‘*Die Weihe des Hauses*’ and ‘*Wellingtons Sieg*’.

To return: the composer is directed by his artistic conscience and must be satisfied as to the essential truth of his work. What exactly constitutes aesthetic truth is a difficult question. I cannot attempt to answer it fully, but I am prepared to suggest a few contributory factors. One is ordered fitness, or the logical procession from each stage to the next. It is perhaps easier to appreciate this in mathematics than in music, but anyone who knows, say, Mozart’s G minor Symphony will recognize this property as readily in it as in Euclid’s proof that there is an infinity of primes. The ideas grow and unfold themselves in such an inevitable manner that it is impossible to imagine even an unexpected turn going any other way. The importance or significance of the ideas embodied in a work form another criterion of its aesthetic truth, though this importance is easier to recognize than to define. Truth is never trivial. Again, economy is a necessary element, for truth is not, or need never be, prolix. Contrast the symphonies of Brahms with those of his lesser contemporaries such as Raff. Most of us will agree that Brahms’s ideas are more fundamentally important than Raff’s, without possibly being able to say why. We are on surer ground when we maintain that Brahms’s thought moves constantly forward and nothing is allowed to disturb its progress, whereas Raff’s work is marred by endless repetitions and redundancies. (By an odd coincidence, I had just written these last words when I chanced to come across a letter of Tchaikovsky’s: “Played Brahms. It irritates me that this self-conscious mediocrity should be recognized as a genius. In comparison with him Raff was a giant, not to speak of Rubinstein, who was a much greater man. And Brahms is so chaotic, so dry and meaningless!” If you side with Tchaikovsky—not many will nowadays—all you have to do is to re-read my last sentence, for “Brahms” reading “Raff” and for “Raff” reading “Brahms.” The general point is unaffected.)

The three characteristics which we may summarize as inevitability,

importance and economy are necessary conditions in all great music, and they are equally necessary in all great mathematics. It would be easy to go on adding to them at will, but there are few, if any, such conditions claimed as necessary by the musician which the mathematician would not claim too.

Many parallels could be drawn between the historical developments of music and mathematics, but such parallels must always be weakened by the more general possibility of equating the histories of almost any two arts.

Innovations do not spring out of nothing, though they may gain impetus very quickly. They do not happen at all until the world is ready for them. The symphony did not start with a jerk in the hands of Haydn and Mozart any more than analytical geometry started with a jerk in the hands of Descartes or the calculus in the hands of Newton or Leibnitz. Men like Stamitz and C.P.E. Bach prepared the ground for the great symphonists, just as Appolonius of Perga, Omar Khayyám and Fermat foreshadowed Descartes' *proles sine matre creata*, as Chasles quite wrongly called it. Stamitz and C. P. E. Bach made the symphony a necessity: Mozart and Haydn supplied it.

This preparation of the world by its lesser men for an important new development leads both in mathematics and in music to a remarkably frequent phenomenon—the almost simultaneous arrival in the firmament of twin stars, or greater constellations, of the first magnitude. In music we have our Handel and Bach, our Haydn and Mozart, the Schumann-Brahms-Dvořák constellation, the Russian nationalist group and countless others. In mathematics Descartes, Fermat and Pascal shone together; Newton and Leibnitz independently but contemporaneously shed their light on what was to become the calculus, as did Cantor and Dedekind on the continuum. Examples could easily be multiplied: unfortunately they prove little or nothing, but they do form a fascinating field for speculation.

Indeed, to prove that music and mathematics either do or do not go together seems about as easy as to prove Fermat's theorem. All we can hope to say is *why* they do *when* they do, and this is difficult enough in all conscience. To epitomize my own tentative suggestions, musical composition and mathematical creation are each the expression of the personality of an individual: the standards governing the pattern-making of both kinds of practitioner have much in common. This is the common stem: the branches diverge widely, and music makes a sensuous appeal where mathematics has none, for there it is the argument that pleases, not the curve of an integral sign.

I have been talking particularly about practitioners, but the arguments apply to all who have a more passive attitude: those who follow and appreciate good mathematics, and those who listen to good music with pleasure and understanding. If it be true that the two arts we have been surveying have in truth a sturdy stem in common, while, on leaving the stem, they branch in opposite directions, does it not explain why one man is often disposed towards both, but can still turn with relief from one to the other?