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THE SQUARING OF THE CIRCLE.

AN HISTORICAL SKETCH OF THE PROBLEM FROM THE EARLIEST
TIMES TO THE PRESENT DAY.*

I.

FOR two and a half thousand years, both trained and untrained minds have striven in vain to solve the problem known as the squaring of the circle. Now that geometers have at last succeeded in giving a rigid demonstration of the impossibility of solving the problem with ruler and compasses, it seems fitting and opportune to cast a glance into the nature and history of this very ancient problem. And this will be found all the more justifiable in view of the fact that the squaring of the circle, at least in name, is very widely known outside of the narrow limits of professional mathematicians.

The Proceedings of the French Academy for the year 1775 contain at page 61 the resolution of the Academy not to examine from that time on, any so-called solutions of the quadrature of the circle that might be handed in. The Academy was driven to this determination by the overwhelming multitude of professed solutions of the famous problem, which were sent to it every month in the year,—solutions which of course were an invariable attestation of the ignorance and self-consciousness of their authors, but which suffered collectively from a very important error in mathematics: they were *wrong*. Since that time all professed solutions of the problem received by the Academy find a sure

* From Holtzendorff and Virchow's *Sammlung gemeinverständlicher wissenschaftlicher Vorträge*, Heft 67. Hamburg: Verlagsanstalt, etc.

haven in the waste-basket, and remain unanswered for all time. The circle-squarer, however, sees in this high-handed manner of rejection only the envy of the great towards his grand intellectual discovery. He is determined to meet with recognition, and appeals therefore to the public. The newspapers must obtain for him the appreciation that scientific societies have denied. And every year the old mathematical sea-serpent more than once disports itself in the columns of our papers, that a Mr. N. N., of P. P., has at last solved the problem of the quadrature of the circle.

But what kind of people are these circle-squarers, when examined by the light? Almost always they will be found to be imperfectly educated persons, whose mathematical knowledge does not exceed that of a modern college freshman. It is seldom that they know accurately what the requirements of the problem are and what its nature; they never know the two and a half thousand years' history of the problem; and they have no idea whatever of the important investigations and results which have been made with reference to the problem by great and real mathematicians in every century down to our time.

Yet great as is the quantum of ignorance that circle-squarers intermix with their intellectual products, the lavish supply of conceit and self-consciousness with which they season their performances is still greater. I have not far to go to furnish a verification of this. A book printed in Hamburg in the year 1840 lies before me, in which the author thanks Almighty God at every second page that He has selected him and no one else to solve the 'problem phenomenal' of mathematics, "so long sought for, so fervently desired, and attempted by millions." After the modest author has proclaimed himself the unmasker of Archimedes's deceit, he says: "It thus has pleased our mother nature to withhold this mathematical jewel from the eye of human investigation, until she thought it fitting to reveal truth to simplicity."

This will suffice to show the great self-consciousness of the author. But it does not suffice to prove his ignorance. He has no conception of mathematical demonstration; he takes it for granted that things are so because they seem so to him. Errors of logic,

also, are abundantly found in his book. But apart from this general incorrectness let us see wherein the real gist of his fallacy consists. It requires considerable labor to find out what this is from the turgid language and bombastic style in which the author has buried his conclusions. But it is this. The author inscribes a square in a circle, circumscribes another about it, then points out that the inside square is made up of four congruent triangles, whereas the circumscribed square is made up of eight such triangles; from which fact, seeing that the circle is larger than the one square and smaller than the other, he draws the bold conclusion that the circle is equal in area to six such triangles. It is hardly conceivable that a rational being could infer that something which is greater than 4 and less than 8 must necessarily be 6. But with a man that attempts the squaring of the circle this kind of ratiocination *is* possible.

Similarly in the case of all other attempted solutions of the problem, either logical fallacies or violations of elementary arithmetical or geometrical truths may be pointed out. Only they are not always of such a trivial nature as in the book just mentioned.

Let us now inquire whence the inclination arises which leads people to take up the quadrature of the circle and to attempt to solve it.

Attention must first be called to the antiquity of the problem. A quadrature was attempted in Egypt 500 years before the exodus of the Israelites. Among the Greeks The allurements of the problem. the problem never ceased to play a part that greatly influenced the progress of mathematics. And in the middle ages also the squaring of the circle sporadically appears as the philosopher's stone of mathematics. The problem has thus never ceased to be dealt with and considered. But it is not by the antiquity of the problem that circle-squarers are enticed, but by the allurements which everything exerts that is calculated to raise the individual out of the mass of ordinary humanity, and to bind about his temples the laurel crown of celebrity. It is ambition that spurred men on in ancient Greece and still spurs them on in modern times to crack this primeval mathematical nut. Whether they are competent thereto is a second-

dary consideration. They look upon the squaring of the circle as the grand prize of a lottery that can just as well fall to their lot as to that of any other. They do not remember that—

“Toil before honor is placed by sagacious decrees of Immortals,”

and that it requires years of continued studies to gain possession of the mathematical weapons that are indispensably necessary to attack the problem, but which even in the hands of the most distinguished mathematical strategists have not sufficed to take the stronghold.

But how is it, we must further ask, that it happens to be the squaring of the circle and not some other unsolved mathematical problem upon which the efforts of people are bestowed who have no knowledge of mathematics yet busy themselves with mathematical questions? The question is answered by the fact that the squaring of the circle is about the only mathematical problem that is known to the unprofessional world,—at least by name. Even among the Greeks the problem was very widely known outside of mathematical circles. In the eyes of the Grecian layman, as at present among many of his modern brethren, occupation with this problem was regarded as the most important and essential business of mathematicians. In fact they had a special word to designate this species of activity; namely, *τετραγωνίζειν*, which means to busy one's self with the quadrature. In modern times, also, every educated person, though he be not a mathematician, knows the problem by name, and knows that it is insolvable, or at least, that despite the efforts of the most famous mathematicians it has not yet been solved. For this reason the phrase “to square the circle,” is now used in the sense of attempting the impossible.

But in addition to the antiquity of the problem, and the fact also that it is known to the lay world, we have yet a third factor to point out that induces people to take up with it. This is the report that has been spread abroad for a hundred years now, that the Academies, the Queen of England, or some other influential person, has offered a great prize to be given

About the only
problem known
to the lay world.

Belief that rewards
have been offered.

to the one that first solves the problem. As a matter of fact we find the hope of obtaining this large prize of money the principal incitement to action with many circle-squarers. And the author of the book above referred to begs his readers to lend him their assistance in obtaining the prizes offered.

Although the opinion is widely current in the unprofessional world, that professional mathematicians are still The problem among mathematicians. busied with the solution of the problem, this is by no means the case. On the contrary, for some two hundred years, the endeavors of many considerable mathematicians have been solely directed towards demonstrating with exactness that the problem is insolvable. It is, as a rule,—and naturally,—more difficult to prove that something is impossible than to prove that it is possible. And thus it has happened, that up to within a few years ago, despite the employment of the most varied and the most comprehensive methods of modern mathematics, no one succeeded in supplying the wished-for demonstration of the problem's impossibility. At last, Professor Lindemann, of Königsberg, in June, 1882, succeeded in furnishing a demonstration,—and the first demonstration,—that it is impossible by the exclusive employment of ruler and compasses to construct a square that is mathematically exactly equal in area to a given circle. The demonstration, naturally, was not effected with the help of the old elementary methods; for if it were, it would surely have been accomplished centuries ago; but methods were requisite that were first furnished by the theory of definite integrals and departments of higher algebra developed in the last decades; in other words it required the direct and indirect preparatory labor of many centuries to make finally possible a demonstration of the insolvability of this historic problem.

Of course, this demonstration will have no more effect than the resolution of the Paris Academy of 1775, in causing the fecund race of circle-squarers to vanish from the face of the earth. In the future as in the past, there will be people who know nothing, and will not want to know anything of this demonstration, and who believe that they cannot help but succeed in a matter in which others have failed, and that just they have been appointed by Providence to

solve the famous puzzle. But unfortunately the ineradicable passion of wanting to solve the quadrature of the circle has also its serious side. Circle-squarers are not always so self-contented as the author of the book we have mentioned. They often see or at least divine the insuperable difficulties that tower up before them, and the conflict between their aspirations and their performances, the consciousness that they want to solve the problem but are unable to solve it, darkens their soul and, lost to the world, they become interesting subjects for the science of psychiatry.

II.

If we have a circle before us, it is easy for us to determine the length of its radius or of its diameter, which must be double that of the radius; and the question next arises to find the number that represents how many times larger its circumference, that is the length of the circular line, is than its radius or its diameter. From the fact that all circles have the same shape it follows that this proportion will always be the same for both large and small circles. Now, since the time of Archimedes, all civilised nations that have cultivated mathematics, have called the number that denotes how many times larger than the diameter the circumference of a circle is, π ,—the Greek initial letter of the word periphery. To compute π , therefore, means to calculate how many times larger the circumference of a circle is than its diameter. This calculation is called “the numerical rectification of the circle.”

Next to the calculation of the circumference, the calculation of the superficial contents of a circle by means of its radius or diameter is perhaps most important; that is, the computation of how much area that part of a plane which lies within a circle measures. This calculation is called the “numerical quadrature.” It depends, however, upon the problem of numerical rectification; that is, upon the calculation of the magnitude of π . For it is demonstrated in elementary geometry, that the area of a circle is equal to the area of a triangle produced by drawing in the circle a radius, erecting at the extremity of the same a

tangent,—that is, in this case, a perpendicular,—cutting off upon the latter the length of the circumference, measuring from the extremity, and joining the point thus obtained with the centre of the circle. But it follows from this that the area of a circle is as many times larger than the square upon its radius as the number π amounts to.

The numerical rectification and numerical quadrature of the circle based upon the computation of the number π , Constructive rectification and quadrature. are to be clearly distinguished from problems that require a straight line equal in length to the circumference of a circle, or a square equal in area to a circle, to be *constructively* produced out of its radius or its diameter; problems which might properly be called “constructive rectification” or “constructive quadrature.” Approximately, of course, by employing an approximate value for π these problems are easily solvable. But to solve a problem of construction, in geometry, means to solve it with mathematical exactitude. If the value π were exactly equal to the ratio of two whole numbers to one another, the constructive rectification would present no difficulties. For example, suppose the circumference of a circle were exactly $3\frac{1}{2}$ times greater than its diameter; then the diameter could be divided into seven equal parts, which could be easily done by the principles of planimetry with ruler and compasses; then we would produce to the amount of such a part a straight line exactly three times larger than the diameter, and should thus obtain a straight line exactly equal to the circumference of the circle. But as a matter of fact, and as has actually been demonstrated, there do not exist two whole numbers, be they ever so great, that exactly represent by their proportion to one another the number π . Consequently, a rectification of the kind just described does not attain the object desired.

It might be asked here, whether from the demonstrated fact that the number π is not equal to the ratio of two whole numbers however great, it does not immediately follow that it is impossible to construct a straight line exactly equal in length to the circumference of a circle; thus demonstrating at once the impossibility of solving the problem. This question is to be answered in the nega-

tive. For there are in geometry many sets of two lines of which the one can be easily constructed from the other, notwithstanding the fact that no two whole numbers can be found to represent the ratio of the two lines. The side and the diagonal of a square, for instance, are so constituted. It is true the ratio of the latter two magnitudes is nearly that of 5 to 7. But this proportion is not exact, and there are in fact no two numbers that represent the ratio exactly. Nevertheless, either of these two lines can be easily constructed from the other by the sole employment of ruler and compasses. This might be the case, too, with the rectification of the circle; and consequently from the impossibility of representing π by the ratio between two whole numbers the impossibility of the problem of rectification is not inferable.

The quadrature of the circle stands and falls with the problem of rectification. This is based upon the truth above mentioned, that a circle is equal in area to a right-angled triangle, in which one side is equal to the radius of the circle and the other to the circumference. Supposing, accordingly, that the circumference of the circle were rectified, then we could construct this triangle. But every triangle, as is taught in the elements of planimetry, can, with the help of ruler and compasses be converted into a square exactly equal to it in area. So that, therefore, supposing the rectification of the circumference of a circle were successfully performed, a square could be constructed that would be exactly equal in area to the circle.

The dependence upon one another of the three problems of the computation of the number π , of the quadrature of the circle, and its rectification, thus obliges us, in dealing with the history of the quadrature, to regard investigations with respect to the value of π and attempts to rectify the circle as of equal importance, and to consider them accordingly.

We have used repeatedly in the course of this discussion the expression "to construct with ruler and compasses." Conditions of the geometrical solution. It will be necessary to explain what is meant by the specification of these two instruments. When such a number of conditions is annexed to a requirement in geometry to construct a

certain figure that the construction only of *one* figure or a limited number of figures is possible in accordance with the conditions given; such a complete requirement is called a problem of construction, or briefly a problem. When a problem of this kind is presented for solution it is necessary to reduce it to simpler problems, already recognised as solvable; and since these latter depend in their turn upon other, still simpler problems, we are finally brought back to certain fundamental problems upon which the rest are based but which are not themselves reducible to problems less simple. These fundamental problems are, so to speak, the undermost stones of the edifice of geometrical construction. The question next arises as to what problems may be properly regarded as fundamental; and it has been found, that the solution of a great part of the problems that arise in elementary planimetry rests upon the solution of only five original problems. They are :

1. The construction of a straight line which shall pass through two given points.
2. The construction of a circle the centre of which is a given point and the radius of which has a given length.
3. The determination of the point that lies coincidently on two given straight lines extended as far as is necessary,—in case such a point (point of intersection) exists.
4. The determination of the two points that lie coincidently on a given straight line and a given circle,—in case such common points (points of intersection) exist.
5. The determination of the two points that lie coincidently on two given circles,—in case such common points (points of intersection) exist.

For the solution of the three last of these five problems the eye alone is needed, while for the solution of the two first problems, besides pencil, ink, chalk, and the like, additional special instruments are required : for the solution of the first problem a ruler is most generally used, and for the solution of the second a pair of compasses. But it must be remembered that it is no concern of geometry what mechanical instruments are employed in the solution of the five problems mentioned. Geometry simply limits itself to

the presupposition that these problems are solvable, and regards a complicated problem as solved if, upon a specification of the constructions of which the solution consists, no other requirements are demanded than the five above mentioned. Since, accordingly, geometry does not itself furnish the solution of these five problems, but rather exacts them, they are termed *postulates*.* All problems of planimetry are not reducible to these five problems alone. There are problems that can be solved only by assuming other problems as solvable which are not included in the five given; for example, the construction of an ellipse, having given its centre and its major and minor axes. Many problems, however, possess the property of being solvable with the assistance solely of the five postulates above formulated, and where this is the case they are said to be "constructible with ruler and compasses," or "elementarily" constructible.

After these general remarks upon the solvability of problems of geometrical construction, which an understanding of the history of the squaring of the circle makes indispensably necessary, the significance of the question whether the quadrature of the circle is or is not solvable, that is elementarily solvable, will become intelligible. But the conception just discussed of elementary solvability only gradually took clear form, and we therefore find among the Greeks as well as among the Arabs, endeavors, successful in some respects, that aimed at solving the quadrature of the circle with other expedients than the five postulates. We have also to take these endeavors into consideration, and especially so as they, no less than the unsuccessful efforts at elementary solution, have upon the whole advanced the science of geometry, and contributed much to the clarification of geometrical ideas.

* Usually geometers mention only two postulates (Nos. 1 and 2). But since to geometry proper it is indifferent whether only the eye, or additional special mechanical instruments are necessary, the author has regarded it more correct in point of method to assume five postulates.

III.

In the oldest mathematical work that we possess we find a rule that tells us how to make a square which is equal in The Egyptian quadrature. area to a given circle. This celebrated book, the Papyrus Rhind of the British Museum, translated and explained by Eisenlohr (Leipzig, 1887), was written, as it is stated in the work, in the thirty-third year of the reign of King Ra-a-us, by a scribe of that monarch, named Ahmes. The composition of the work falls accordingly into the period of the two Hiksos dynasties, that is, in the period between 2000 and 1700 B.C. But there is another important circumstance attached to this. Ahmes mentions in his introduction that he composed his work after the model of old treatises, written in the time of King Raenmat; whence it appears that the originals of the mathematical expositions of Ahmes, are half a thousand years older yet than the Papyrus Rhind.

The rule given in this papyrus for obtaining a square equal to a circle, specifies that the diameter of the circle shall be shortened one ninth of its length and upon the shortened line thus obtained a square erected. Of course, the area of a square of this construction is only approximately equal to the area of the circle. An idea may be obtained of the degree of exactness of this original, primitive quadrature by our remarking, that if the diameter of the circle in question is one metre in length, the square that is supposed to be equal to the circle is a little less than half a square decimetre larger; an approximation not so accurate as that computed by Archimedes, yet much more correct than many a one later employed. It is not known how Ahmes or his predecessors arrived at this approximate quadrature; but it is certain that it was handed down in Egypt from century to century, and in late Egyptian times it repeatedly appears.

Besides among the Egyptians, we also find in pre-Grecian antiquity an attempt at circle-computation among the The Biblical and Babylonian quadratures. Babylonians. This is not a quadrature; but aims at the rectification of the circumference. The Babylonian mathematicians had discovered, that if the radius of a circle be successively

inscribed as chord within its circumference, after the sixth inscription we arrive at the point of departure, and they concluded from this that the circumference of a circle must be a little larger than a line which is six times as long as the radius, that is three times as long as the diameter. A trace of this Babylonian method of computation may even be found in the bible; for in 1 Kings vii. 23, and 2 Chron. iv. 2, the great laver is described, which under the name of the "molten sea" constituted an ornament of the temple of Solomon; and it is said of this vessel that it measured ten cubits from brim to brim, and thirty cubits round about. The number 3 as the ratio between the circumference and the diameter is still more plainly given in the Talmud, where we read that "that which measures three lengths in circumference is one length across."

With regard to the earlier Greek mathematicians,—as Thales and Pythagoras,—we know that they acquired the foundations of their mathematical knowledge in Egypt. But nothing has been handed down to us which shows that they knew of the old Egyptian quadrature, or that they dealt with the problem at all. But tradition says, that, subsequently, the teacher of Euripides and Pericles, the great philosopher and mathematician Anaxagoras, whom Plato so highly praised, "drew the quadrature of the circle" in prison, in the year 434. This is the account of Plutarch in the seventeenth chapter of his work "De Exilio." The method is not told us in which Anaxagoras had supposedly solved the problem, and it is not said whether knowingly or unknowingly he accomplished an approximate solution after the manner of Ahmes. But at any rate, to Anaxagoras belongs the merit of having called attention to a problem that bore great fruit, in having incited Grecian scholars to busy themselves with geometry, and thus more and more to advance that science.

Again, it is reported that the mathematician Hippias of Elis invented a curved line that could be made to serve a double purpose: first, to trisect an angle, and second, to square the circle. This curved line is the *τετραγωνίζουσα*

so often mentioned by the later Greek mathematicians, and by the Romans called "quadratrix." Regarding the nature of this curve we have exact knowledge from Pappus. But it will be sufficient, here, to state that the quadratrix is not a circle nor a portion of a circle, so that its construction is not possible by means of the postulates enumerated in the preceding section. And therefore the solution of the quadrature of the circle founded on the construction of the quadratrix is not an elementary solution in the sense discussed in the last section. We can, it is true, conceive a mechanism that will draw this curve as well as compasses draw a circle; and with the assistance of a mechanism of this description the squaring of the circle is solvable with exactitude. But if it be allowed to employ in a solution an apparatus especially adapted thereto, every problem may be said to be solvable. Strictly taken, the invention of the curve of Hippias substitutes for one insuperable difficulty another equally insuperable. Some time afterwards, about the year 350, the mathematician Dinostratus showed that the quadratrix could also be used to solve the problem of rectification, and from that time on this problem plays almost the same rôle in Grecian mathematics as the related problem of quadrature.

As these problems gradually became known to the non-mathematicians of Greece, attempts at solution at once ^{The Sophists' solution.} sprang up that are worthy of a place by the side of the solutions of modern amateur circle-squarers. The Sophists, especially, believed themselves competent by seductive dialectic to take a stronghold that had defied the intellectual onslaughts of the greatest mathematicians. With verbal nicety, amounting to puerility, it was said that the squaring of the circle depended upon the finding of a number which represented in itself both a square and a circle; a square by being a square number, a circle in that it ended with the same number as the root number from which, by multiplication with itself, it was produced. The number 36, accordingly, was, as they thought, the one that embodied the solution of the famous problem.

Contrasted with this twisting of words the speculations of Bryson and Antiphon, both contemporaries of Socrates, though inexact,

appear in high degree intelligent. Antiphon divided the circle into four equal arcs, and by joining the points of division obtained a square; he then divided each arc again into two equal parts and thus obtained an inscribed octagon; thence he constructed an inscribed dodecagon, and perceived that the figure so inscribed more and more approached the shape of a circle. In this way, he said, one should proceed, until there was inscribed in the circle a polygon whose sides by reason of their smallness should coincide with the circle. Now this polygon could, by methods already taught by the Pythagoreans, be converted into a square of equal area; and upon the basis of this fact Antiphon regarded the squaring of the circle as solved.

Nothing can be said against this method except that, however far the bisection of the arcs is carried, the result must still remain an approximate one.

The attempt of Bryson of Heraclea was better still; for this scholar did not rest content with finding a square that was very little smaller than the circle, but obtained by means of circumscribed polygons another square that was very little larger than the circle. Only Bryson committed the error of believing that the area of the circle was the arithmetical mean between an inscribed and a circumscribed polygon of an equal number of sides. Notwithstanding this error, however, to Bryson belongs the merit, first, of having introduced into mathematics by his emphasis of the necessity of a square which was too large and one which was too small, the conception of maximum and minimum "limits" in approximations; and secondly, by his comparison with a circle of the inscribed and circumscribed regular polygons, the merit of having indicated to Archimedes the way by which an approximate value for π was to be reached.

Not long after Antiphon and Bryson, Hippocrates of Chios treated the problem, which had now become more and more famous, from a new point of view. Hippocrates was not satisfied with approximate equalities, and searched for curvilinearly bounded plane figures which should be mathematically equal to a rectilinearly bounded figure, and therefore

could be converted by ruler and compasses into a square equal in area. First, Hippocrates found that the crescent-shaped plane figure produced by drawing two perpendicular radii in a circle and describing upon the line joining their extremities a semicircle, is exactly equal in area to the triangle that is formed by this line of junction and the two radii; and upon the basis of this fact the endeavors of the untiring scholar were directed towards converting a circle into a crescent. Naturally he was unable to attain this object, but by his efforts to this end he discovered many a new geometrical truth; among others the generalised form of the theorem mentioned, which bears to the present day the name of "Lunulae Hippocratis," the lunes of Hippocrates. Thus it appears, in the case of Hippocrates, in the plainest light, how the very insolvable problems of science are qualified to advance science; in that they incite investigators to devote themselves with persistence to its study and thus to fathom its depths.

Following Hippocrates in the historical line of the great Grecian geometers comes the systematist Euclid, Euclid's avoidance of the problem. whose rigid formulation of geometrical principles has remained the standard presentation down to the present century. The Elements of Euclid, however, contain nothing relating to the quadrature of the circle or to circle-computation. Comparisons of surfaces which relate to the circle are indeed found in the book, but nowhere a computation of the circumference of a circle or of the area of a circle. This palpable gap in Euclid's system was filled by Archimedes, the greatest mathematician of antiquity.

Archimedes was born in Syracuse in the year 287 B. C., and devoted his life, there spent, to the mathematical and Archimedes's calculations. the physical sciences, which he enriched with invaluable contributions. He lived in Syracuse till the taking of the town by Marcellus, in the year 212 B. C., when he fell by the hand of a Roman soldier whom he had forbidden to destroy the figures he had drawn in the sand. To the greatest performances of Archimedes the successful computation of the number π unquestionably belongs. Like Bryson he started with regular inscribed and circumscribed polygons. He showed how it was possible, beginning with

the perimeter of an inscribed hexagon, which is equal to six radii, to obtain by way of calculation the perimeter of a regular dodecagon, and then the perimeter of a figure having double the number of sides of the preceding one. Treating, then, the circumscribed polygons in a similar manner, and proceeding with both series of polygons up to a regular 96-sided polygon, he perceived on the one hand that the ratio of the perimeter of the inscribed 96-sided polygon to the diameter was greater than $6336 : 2017\frac{1}{4}$, and on the other hand, that the corresponding ratio with respect to the circumscribed 96-sided polygon was smaller than $14688 : 4673\frac{1}{2}$. He inferred from this, that the number π , the ratio of the circumference to the diameter, was greater than the fraction $\frac{6336}{2017\frac{1}{4}}$ and smaller than $\frac{14688}{4673\frac{1}{2}}$. Reducing the two limits thus found for the value of π , Archimedes then showed that the first fraction was greater than $3\frac{1}{7}$, and that the second fraction was smaller than $3\frac{1}{7}$, whence it followed with certainty that the value sought for π lay between $3\frac{1}{7}$ and $3\frac{1}{7}$. The larger of these two approximate values is the only one usually learned and employed. That which fills us most with astonishment in the Archimedean computation of π , is, first, the great acumen and accuracy displayed in all the details of the computation, and then the unwearied perseverance that he must have exercised in calculating the limits of π without the advantages of the Arabian system of numerals and of the decimal notation. For it must be considered that at many stages of the computation what we call the extraction of roots was necessary, and that Archimedes could only by extremely tedious calculations obtain ratios that expressed approximately the roots of given numbers and fractions.

With regard to the mathematicians of Greece that follow Archimedes, all refer to and employ the approximate value of $3\frac{1}{7}$ for π , without however, contributing anything essentially new or additional to the problems of quadrature and of cyclometry. Thus Heron of Alexandria, the father of surveying, who flourished about the year 100 B. C., employs for purposes of practical measurement sometimes the value $3\frac{1}{7}$ for π and sometimes even the rougher approximation $\pi = 3$. The astronomer

The later mathematicians of Greece.

Ptolemy, who lived in Alexandria about the year 150 A. D., and who was famous as being the author of the planetary system universally recognised as correct down to the time of Copernicus, was the only one who furnished a more exact value; this he designated, in the sexagesimal system of fractional notation which he employed, by 3, 8, 30,—that is 3 and $\frac{8}{60}$ and $\frac{30}{3600}$, or as we now say 3 degrees, 8 minutes (*partes minutae primae*), and 30 seconds (*partes minutae secundae*). As a matter of fact, the expression $3 + \frac{8}{60} + \frac{30}{3600} = 3\frac{17}{20}$ represents the number π more exactly than $3\frac{1}{4}$; but on the other hand, is, by reason of the magnitude of the numbers 17 and 120 as compared with the numbers 1 and 7, more cumbersome.

IV.

In the mathematical sciences, more than in any other, the Romans stood upon the shoulders of the Greeks. Indeed, with respect to cyclometry, they not only did not Among the Romans add anything to the Grecian discoveries, but often evinced even that they either did not know of the beautiful result obtained by Archimedes, or at least did not know how to appreciate it. For instance, Vitruvius, who lived during the time of Augustus, computed that a wheel 4 feet in diameter must measure $12\frac{1}{2}$ feet in circumference; in other words, he made π equal to $3\frac{1}{8}$. And, similarly, a treatise on surveying, preserved to us in the Gudian manuscript of the library at Wolfenbüttel, contains the following instructions to square the circle: Divide the circumference of a circle into four parts and make one part the side of a square; this square will be equal in area to the circle. Aside from the fact that the rectification of the arc of a circle is requisite to the construction of a square of this kind, the Roman quadrature, viewed as a calculation, is more inexact even than any other computation; for its result is that $\pi = 4$.

The mathematical performances of the Hindus were not only greater than those of the Romans, but in certain Among the Hindus. directions even surpassed those of the Greeks. In the most ancient source for the mathematics of India that we know of, the *Culvasûtras*, which date back to a little before our chronological era, we do not find, it is true, the squaring of the

circle treated of, but the opposite problem is dealt with, which might fittingly be termed the circling of the square. The half of the side of a given square is prolonged one third of the excess in length of half the diagonal over half the side, and the line thus obtained is taken as the radius of the circle equal in area to the square. The simplest way to obtain an idea of the exactness of this construction is to compute how great π would have to be if the construction were exactly correct. We find out in this way that the value of π upon which the Indian circling of the square is based, is about from five to six hundredths smaller than the true value, whereas the approximate π of Archimedes, $3\frac{1}{7}$, is only from one to two thousandths too large, and the old Egyptian value exceeds the true value by from one to two hundredths. Cyclometry very probably made great advances among the Hindus in the first four or five centuries of our era; for Aryabhatta, who lived about the year 500 after Christ, states, that the ratio of the circumference to the diameter is 62832 : 20000, an approximation that in exactness surpasses even that of Ptolemy. The Hindu result gives 3.1416 for π , while π really lies between 3.141592 and 3.141593. How the Hindus obtained this excellent approximate value is told by Ganeça, the commentator of Bhâskara, an author of the twelfth century. Ganeça says that the method of Archimedes was carried still farther by the Hindu mathematicians; that by continually doubling the number of sides they proceeded from the hexagon to a polygon of 384 sides, and that by the comparison of the circumferences of the inscribed and circumscribed 384-sided polygons they found that π was equal to 3927 : 1250. It will be seen that the value given by Bhâskara is identical with the value of Aryabhatta. It is further worthy of remark that the earlier of these two Hindu mathematicians does not mention either the value $3\frac{1}{7}$ of Archimedes or the value $3\frac{17}{10}$ of Ptolemy, but that the later knows of both values and especially recommends that of Archimedes as the most useful one for practical application. Strange to say, the good approximate value of Aryabhatta does not occur in Bramagupta, the great Hindu mathematician who flourished in the beginning of the seventh century; but we find the curious information in this author that the area of a circle is exactly equal to the

square root of 10 when the radius is unity. The value of π as derivable from this formula,—a value from two to three hundredths too large,—has unquestionably arisen upon Hindu soil. For it occurs in no Grecian mathematician; and Arabian authors, who were in a better position than we to know Greek and Hindu mathematical literature, declare that the approximation which makes π equal to the square root of 10, is of Hindu origin. It is possible that the Hindu people, who were addicted more than any other to numeral mysticism, sought to find in this approximation some connection with the fact that man has ten fingers; and ten accordingly is the basis of their numeral system.

Reviewing the achievements of the Hindus generally with respect to the problem of the quadrature, we are brought to recognise that this people, whose talents lay more in the line of arithmetical computation than in the perception of spatial relations, accomplished as good as nothing on the pure geometrical side of the problem, but that the merit belongs to them of having carried the Archimedean method of computing π several stages farther, and of having obtained in this way a much more exact value for it—a circumstance that is explainable when we consider that the Hindus are the inventors of our present system of numeral notation, possessing which they easily outdid Archimedes, who employed the awkward Greek system.

With regard to the Chinese, this people operated in ancient times with the Babylonian value for π , or 3; but ^{among the Chinese} possessed knowledge of the approximate value of π ~~Archimedes~~ at least since the end of the sixth century. Besides this, there appears in a number of Chinese mathematical treatises an approximate value peculiarly their own, in which $\pi = 3\frac{7}{6}$; a value, however, which notwithstanding it is written in larger figures, is no better than that of Archimedes. Attempts at the *constructive* quadrature of the circle are not found among the Chinese.

Greater were the merits of the Arabians in the advancement and development of mathematics; and especially in ^{Among the Arabs.} virtue of the fact that they preserved from oblivion both Greek and Hindu mathematics, and handed them down to the

Christian countries of the West. The Arabians expressly distinguished between the Archimedean approximate value and the two Hindu values the square root of 10 and the ratio 62832 : 20000. This distinction occurs also in Muhammed Ibn Musa Alchwarizmî, the same scholar who in the beginning of the ninth century brought the principles of our present system of numerical notation from India and introduced the same into the Mohammedan world. The Arabians, however, did not study the numerical quadrature of the circle only, but also the constructive ; as, for instance, Ibn Alhaitam, who lived in Egypt about the year 1000 and whose treatise upon the squaring of the circle is preserved in a Vatican codex, which has unfortunately not yet been edited.

Christian civilisation, to which we are now about to pass, produced up to the second half of the fifteenth century
In Christian times. extremely insignificant results in mathematics. Even with regard to our present problem we have but a single important work to mention ; the work, namely, of Frankos Von Lüttich, upon the squaring of the circle, published in six books, but only preserved in fragments. The author, who lived in the first half of the eleventh century, was probably a pupil of Pope Sylvester II, himself a not inconsiderable mathematician for his time, and who also wrote the most celebrated book on geometry of the period.

Greater interest came to be bestowed upon mathematics in
Cardinal Nicolaus De Cusa. general, but especially on the problem of the quadrature of the circle, in the second half of the fifteenth century, when the sciences again began to revive. This interest was especially aroused by Cardinal Nicolaus De Cusa, a man highly esteemed on account of his astronomical and calendrical studies. He claimed to have discovered the quadrature of the circle by the employment solely of compasses and ruler, and thus attracted the attention of scholars to the now historic problem. People believed the famous Cardinal, and marvelled at his wisdom, until Regiomontanus, in letters which he wrote in 1464 and 1465 and which were published in 1533, rigidly demonstrated that the Cardinal's quadrature was incorrect. The construction of Cusa was as follows. The radius of a circle is prolonged a distance equal to the

side of the inscribed square ; the line thus obtained is taken as the diameter of a second circle and in the latter an equilateral triangle is described ; then the perimeter of the latter is equal to the circumference of the original circle. If this construction, which its inventor regarded as exact, be considered as a construction of approximation, it will be found to be more inexact even than the construction resulting from the value $\pi = 3\frac{1}{4}$. For by Cusa's method π would be from five to six thousandths smaller than it really is.

In the beginning of the sixteenth century a certain Bovillius appears, who announced anew the construction of Cusa ; meeting however with no notice. But about the middle of the sixteenth century a book was published which the scholars of the time at first received with interest. It bore the proud title "De Rebus Mathematicis Hactenus Desideratis." Its author, Orontius Finaeus, represented that he had overcome all the difficulties that had ever stood in the way of geometrical investigators ; and incidentally he also communicated to the world the "true quadrature" of the circle. His fame was short-lived. For soon afterwards, in a book entitled "De Erratis Orontii," the Portuguese Petrus Nonius demonstrated that Orontius's quadrature, like most of his other professed discoveries, was incorrect.

In the period following this the number of circle-squarers so increased that we shall have to limit ourselves to those whom mathematicians recognise. And particularly is Simon Van Eyck to be mentioned, who towards the close of the sixteenth century published a quadrature which was so approximate that the value of π derived from it was more exact than that of Archimedes ; and to disprove it the mathematician Peter Metius was obliged to seek a still more accurate value than $3\frac{1}{4}$. The erroneous quadrature of Van Eyck was thus the occasion of Metius's discovery that the ratio $355:113$, or $3\frac{16}{113}$, varied from the true value of π by less than one one-millionth, eclipsing accordingly all values hitherto obtained. Moreover, it is demonstrable by the theory of continued fractions, that, admitting figures to four places only, no two numbers more exactly represent the value of π than 355 and 113.

In the same way the quadrature of the great philologist Joseph Scaliger led to refutations. Like most circle-squarers who believe in their discovery, Scaliger also was little versed in the elements of geometry. He solved, however,—at least in his own opinion he did,—the famous problem; and published in 1592 a book upon it, which bore the pretentious title “Nova Cyclometria” and in which the name of Archimedes was derided. The worthlessness of his supposed discovery was demonstrated to him by the greatest mathematicians of his time; namely, Vieta, Adrianus Romanus, and Clavius.

Of the erring circle-squarers that flourished before the middle of the seventeenth century three others deserve particular mention—Longomontanus of Copenhagen, who rendered such great services to astronomy, the Neapolitan John Porta, and Gregory of St. Vincent. Longomontanus made $\pi = 3\frac{14185}{10000}$, and was so convinced of the correctness of his result that he thanked God fervently, in the preface to his work “Inventio Quadraturæ Circuli,” that He had granted him in his high old age the strength to conquer the celebrated difficulty. John Porta followed the initiative of Hippocrates, and believed he had solved the problem by the comparison of lunes. Gregory of St. Vincent published a quadrature, the error of which was very hard to detect but was finally discovered by Descartes.

Of the famous mathematicians who dealt with our problem in the period between the close of the fifteenth century and the time of Newton, we first meet with Peter Metius, before mentioned, who succeeded in finding in the fraction $355 : 113$ the best approximate value for π involving only small numbers. The problem received a different advancement at the hands of the famous mathematician Vieta. Vieta was the first to whom the idea occurred of representing π with mathematical exactness by an infinite series of continuable operations. By comparison of inscribed and circumscribed polygons, Vieta found that we approach nearer and nearer to π if we allow the operations of the extraction of the square root of $\frac{1}{2}$, and of addition and of multiplication to succeed each other in a certain manner, and that π must come out exactly,

if this series of operations could be indefinitely continued. Vieta thus found that to a diameter of 10000 million units a circumference belongs of 31415 million and from 926535 to 926536 units of the same length.

But Vieta was outdone by the Netherlander Adrianus Romanus, who added five additional decimal places to the ten Adrianus Romanus, Ludolf Van Ceulen. of Vieta. To accomplish this he computed with unspeakable labor the circumference of a regular circumscribed polygon of 1073741824 sides. This number is the thirtieth power of 2. Yet great as the labor of Adrianus Romanus was, that of Ludolf Van Ceulen was still greater; for the latter calculator succeeded in carrying the Archimedean process of approximation for the value of π to 35 decimal places, that is, the deviation from the true value was smaller than one one-thousand quintillionth, a degree of exactness that we can hardly have any conception of. Ludolf published the figures of the tremendous computation that led to this result. His calculation was carefully examined by the mathematician Griemberger and declared to be correct. Ludolf was justly proud of his work, and following the example of Archimedes, requested in his will that the result of his most important mathematical performance, the computation of π to 35 decimal places, be engraved upon his tombstone; a request which is said to have been carried out. In honor of Ludolf, π is called to-day in Germany the Ludolfian number.

Although through the labor of Ludolf a degree of exactness for cyclometrical operations was now obtained that was The new method of Snell. Huygens's verification of it. more than sufficient for any practical purpose that could ever arise, neither the problem of constructive rectification nor that of constructive quadrature was thereby in any respect theoretically advanced. The investigations conducted by the famous mathematicians and physicists Huygens and Snell about the middle of the seventeenth century, were more important from a mathematical point of view than the work of Ludolf. In his book "Cyclometricus" Snell took the position that the method of comparison of polygons, which originated with Archimedes and was employed by Ludolf, need by no means be the best method of at-

taining the end sought; and he succeeded by the employment of propositions which state that certain arcs of a circle are greater or smaller than certain straight lines connected with the circle, in obtaining methods that make it possible to reach results like the Ludolfian with much less labor of calculation. The beautiful theorems of Snell were proved a second time, and better proved, by the celebrated Dutch promoter of the science of optics, Huygens (*Opera Varia*, p. 365 et seq.; "Theoremata De Circuli et Hyperbolae Quadratura," 1651), as well as perfected in many ways. Snell and Huygens were fully aware that they had advanced only the problem of numerical quadrature, and not that of the constructive quadrature. This, in Huygens's case, plainly appeared from the vehement dispute he conducted with the English mathematician James Gregory. This controversy has some significance for the history of our problem, from the fact that Gregory made the first attempt to prove that the squaring of the circle with ruler and compasses must be impossible. The result of the controversy, to which we owe many valuable treatises, was, that Huygens finally demonstrated in an incontrovertible manner the incorrectness of Gregory's proof of impossibility, adding that he also was of opinion that the solution of the problem with ruler and compasses was impossible, but nevertheless was not himself able to demonstrate this fact. And Newton, later, expressed himself to a similar effect. As a matter of fact it took till the most recent period, that is over 200 years, until higher mathematics was far enough advanced to furnish a rigid demonstration of impossibility.

The controversy between Huygens and Gregory.

v.

Before we proceed to consider the promotive influence which the invention of the differential and the integral calculus had upon our problem, we shall enumerate a few at least of that never-ending line of mistaken quadrators who delighted the world by the fruits of their ingenuity from the time of Newton to the present period; and out of a pious and sincere consideration for the contemporary world, we shall entirely omit in this to speak of the circle-squarers of our own time.

First to be mentioned is the celebrated English philosopher Hobbes. In his book "De Problematis Physicis," ^{Hobbes's quadrature.} in which he chiefly proposes to explain the phenomena of gravity and of ocean tides, he also takes up the quadrature of the circle and gives a very trivial construction that in his opinion definitively solved the problem, making $\pi = 3\frac{1}{8}$. In view of Hobbes's importance as a philosopher, two mathematicians, Huygens and Wallis, thought it proper to refute Hobbes at length. But Hobbes defended his position in a special treatise, in which to sustain at least the appearance of being right, he disputed the fundamental principles of geometry and the theorem of Pythagoras; so that mathematicians could pass on from him to the order of the day.

In the last century France especially was rich in circle-squarers. We will mention: Oliver de Serres, who by means ^{French quadrators of the Eighteenth Century.} of a pair of scales determined that a circle weighed as much as the square upon the side of the equilateral triangle inscribed in it, that therefore they must have the same area, an experiment in which $\pi = 3$; Mathulon, who offered in legal form a reward of a thousand dollars to the person who would point out an error in his solution of the problem, and who was actually compelled by the courts to pay the money; Basselin, who believed that his quadrature must be right because it agreed with the approximate value of Archimedes, and who anathematised his ungrateful contemporaries, in the confidence that he would be recognised by posterity; Liger, who proved that a part is greater than the whole and to whom therefore the quadrature of the circle was child's play; Clerget, who based his solution upon the principle that a circle is a polygon of a definite number of sides, and who calculated, also, among other things, how large the point is at which two circles touch.

Germany and Poland also furnish their contingent to the army of circle-squarers. Lieutenant-Colonel Corsonich pro- ^{Germany and Poland.} duced a quadrature in which π equalled $3\frac{1}{8}$, and promised fifty ducats to the person who could prove that it was incorrect. Hesse of Berlin wrote an arithmetic in 1776, in which a true quadrature was also "made known," π being exactly equal to $3\frac{1}{8}$. About the same time Professor Bischoff of Stettin defended

a quadrature previously published by Captain Leistner, Preacher Merkel, and Schoolmaster Böhm, which made π *implicite* equal to the square of $\frac{62}{35}$, not even attaining the approximation of Archimedes.

From attempts of this character are to be clearly distinguished
Constructive ap-
proximations. constructions of approximation in which the inventor
Euler. Kochansky. is aware that he has not found a mathematically exact construction, but only an approximate one. The value of such a construction will depend upon two things—first, upon the degree of exactness with which it is numerically expressed, and secondly on the fact whether the construction can be more or less easily made with ruler and compasses. Constructions of this kind, simple in form and yet sufficiently exact for practical purposes, have for centuries been furnished us in great numbers. The great mathematician Euler, who died in 1783, did not think it out of place to attempt an approximate construction of this kind. A very simple construction for the rectification of the circle and one which has passed into many geometrical text books, is that published by Kochansky in 1685 in the *Leipziger Berichte*. It is as follows: “Erect upon the diameter of a circle at its extremities perpendiculars; with the centre as vertex, mark off upon the diameter an angle of 30° ; find the point of intersection with the perpendicular of the line last drawn, and join this point of intersection with that point upon the other perpendicular which is at a distance of three radii from the base of the perpendicular. The line of junction thus obtained is then very approximately equal to one-half of the circumference of the given circle.” Calculation shows that the difference between the true length of the circumference and the line thus constructed is less than $\frac{3}{100000}$ of the diameter.

Although such constructions of approximation are very inter-
Inutility of con-
structive approxi-
mations. esting in themselves, they nevertheless play but a subordinate rôle in the history of the squaring of the circle; for on the one hand they can never furnish greater exactness for circle-computation than the thirty-five decimal places which Ludolf found, and on the other hand they are not adapted to advance in any way the question whether the exact quadrature of the circle with ruler and compasses is possible.

The numerical side of the problem, however, was considerably advanced by the new mathematical methods perfected by Newton and Leibnitz, commonly called the differential and the integral calculus. And about the middle of the seventeenth century, some time before Newton and Leibnitz represented π by series of powers, the English mathematicians Wallis and Lord Brouncker, Newton's predecessors in a certain sense, succeeded in representing π by an infinite series of figures combined by the first four rules of arithmetic. A new method of computation was thus opened. Wallis found that the fourth part of π is represented more exactly by the regularly formed product

The researches of
Newton, Leib-
nitz, Wallis, and
Brouncker.

$$\frac{2}{3} \times \frac{4}{3} \times \frac{4}{5} \times \frac{6}{5} \times \frac{6}{7} \times \frac{8}{7} \times \frac{8}{9} \times \text{etc.}$$

the farther the multiplication is continued, and that the result always comes out too small if we stop at a proper fraction but too large if we stop at an improper fraction. Lord Brouncker, on the other hand, represents the value in question by a continued fraction in which all the denominators are equal to 2 and the numerators are odd square numbers. Wallis, to whom Brouncker had communicated his elegant result without proof, demonstrated the same in his "Arithmetic of Infinites."

The computation of π could hardly be farther advanced by these results than Ludolf and others had carried it, though of course in a more laborious way. However, the series of powers derived by the assistance of the differential calculus of Newton and Leibnitz furnished a means of computing π to hundreds of decimal places.

Gregory, Newton, and Leibnitz next found that the fourth part of π was equal exactly to

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \dots$$

Other
calculations.

if we conceive this series, which is called the Leibnitzian, indefinitely continued. This series is indeed wonderfully simple, but is not adapted to the computation of π , for the reason that entirely too many members have to be taken into account to obtain π accurately to a few decimal places only. The original formula, however, from which this series is derived, gives other formulas which

are excellently adapted to the actual computation. This formula is the general series :

$$\alpha = a - \frac{1}{3}a^3 + \frac{1}{5}a^5 - \frac{1}{7}a^7 + \dots,$$

where α is the length of the arc that belongs to any central angle in a circle of radius 1, and where a is the tangent to this angle. From this we derive the following :

$$\begin{aligned} \frac{\pi}{4} = (a + b + c + \dots) - \frac{1}{3}(a^3 + b^3 + c^3 + \dots) \\ + \frac{1}{5}(a^5 + b^5 + c^5 + \dots) - \dots, \end{aligned}$$

where $a, b, c \dots$ are the tangents of angles whose sum is 45° . Determining, therefore, the values of $a, b, c \dots$, which are equal to small and easy fractions and fulfil the condition just mentioned, we obtain series of powers which are adapted to the computation of π . The first to add by the aid of series of this description additional decimal places to the old 35 in the number π was the English arithmetician Abraham Sharp, who following Halley's instructions, in 1700, worked out π to 72 decimal places. A little later Machin, professor of astronomy in London, computed π to 100 decimal places; putting, in the series given above, $a = b = c = d = \frac{1}{5}$ and $e = -\frac{1}{239}$, that is employing the following series :

$$\begin{aligned} \frac{\pi}{4} = 4 \cdot \left[\frac{1}{5} - \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} - \frac{1}{7 \cdot 5^7} + \dots \right] \\ - \left[\frac{1}{239} - \frac{1}{3 \cdot 239^3} + \frac{1}{5 \cdot 239^5} - \dots \right] \end{aligned}$$

In the year 1819, Lagny of Paris outdid the computation of Machin, determining in two different ways the first 127 decimal places of π . Vega then obtained as many as 140 places, and the Hamburg arithmetician Zacharias Dase went as far as 200 places. The latter did not use Machin's series in his calculation, but the series produced by putting in the general series above given $a = \frac{1}{2}$, $b = \frac{1}{5}$, $c = \frac{1}{8}$. Finally, at a recent date, π has been computed to 500 places.

The computation to so many decimal places may serve as an illustration of the excellence of the modern method as contrasted with those anciently employed, but otherwise it has neither a theoretical nor a practical value. That the computation of π to say 15

decimal places more than sufficiently satisfies the subtlest requirements of practice may be gathered from a concrete example of the degree of exactness thus obtainable. Imagine a circle to be described with Berlin as centre, and the circumference to pass through Hamburg ; then let the circumference of the circle be computed by multiplying its diameter with the value of π to 15 decimal places, and then conceive it to be actually measured. The deviation from the true length in so large a circle as this even could not be as great as the 18 millionth part of a millimetre.

Idea of exactness obtainable with the approximate values of π .

An idea can hardly be obtained of the degree of exactness produced by 100 decimal places. But the following example may possibly give us some conception of it. Conceive a sphere constructed with the earth as centre, and imagine its surface to pass through Sirius, which is $134\frac{1}{2}$ million million kilometres distant from us. Then imagine this enormous sphere to be so packed with microbes that in every cubic millimetre millions of millions of these diminutive animalcula are present. Now conceive these microbes to be all unpacked and so distributed singly along a straight line, that every two microbes are as far distant from each other as Sirius from us, that is $134\frac{1}{2}$ million million kilometres. Conceive the long line thus fixed by all the microbes, as the diameter of a circle, and imagine the circumference of it to be calculated by multiplying its diameter with π to 100 decimal places. Then, in the case of a circle of this enormous magnitude even, the circumference thus calculated would not vary from the real circumference by a millionth of a millimetre.

This example will suffice to show that the calculation of π to 100 or 500 decimal places is wholly useless.

Before we close this chapter upon the evaluation of π , we must mention the method, less fruitful than curious, which Professor Wolff of Zurich employed some decades ago to compute the value of π to 3 places. The floor of a room is divided up into equal squares, so as to resemble a huge chess-board, and a needle exactly equal in length to the side of each of these squares, is cast haphazard upon the floor. If we calculate, now, the probabilities of the needle so falling as to lie wholly within

Professor Wolff's curious method.

one of the squares, that is so that it does not cross any of the parallel lines forming the squares, the result of the calculation for this probability will be found to be exactly equal to $\pi - 3$. Consequently, a sufficient number of casts of the needle according to the law of large numbers must give the value of π approximately. As a matter of fact, Professor Wolff, after 10000 trials, obtained the value of π correctly to 3 decimal places.

Fruitful as the calculus of Newton and Leibnitz was for the evaluation of π , the problem of converting a circle into a square having exactly the same area was in no wise advanced thereby. Wallis, Newton, Leibnitz, and their immediate followers distinctly recognised this. The quadrature of the circle could not be solved ; but it also could not be proved that the problem was insolvable with ruler and compasses, although everybody was convinced of its insolubility. In mathematics, however, a conviction is only justified when supported by incontrovertible proof ; and in the place of endeavors to solve the quadrature there accordingly now come endeavors to prove the impossibility of solving the celebrated problem.

The first step in this direction, small as it was, was made by the French mathematician Lambert, who proved in the year 1761 that π was neither a rational number nor even the square root of a rational number ; that is, that neither π nor the square of π can be exactly represented by a fraction the denominator and numerator of which are whole numbers, however great the numbers be taken. Lambert's proof showed, indeed, that the rectification and the quadrature of the circle could not be possibly accomplished in the particular way in which its impossibility was demonstrated, but it still did not exclude the possibility of the problem being solvable in some other more complicated way, and without requiring further aids than ruler and compasses.

Proceeding slowly but surely it was next sought to discover the essential distinguishing properties that separate problems solvable with ruler and compasses, from problems the construction of which is elementarily impossible, that is by solely employing the postulates. Slight reflection showed,

Mathematicians
now seek to prove
the insolubility
of the problem.

Lambert's contri-
bution.

The conditions of
the demonstra-
tion.

that a problem elementarily solvable, must always possess the property of having the unknown lines in the figure relating to it connected with the known lines of the figure by an equation for the solution of which equations of the first and second degree alone are requisite, and which may be so disposed that the common measures of the known lines will appear only as integers. The conclusion was to be drawn from this, that if the quadrature of the circle and consequently its rectification were elementarily solvable, the number π , which represents the ratio of the unknown circumference to the known diameter, must be the root of a certain equation, of a very high degree perhaps, but in which all the numbers that appear are whole numbers; that is, there would have to exist an equation, made up entirely of whole numbers, which would be correct if its unknown quantity were made equal to π .

Since the beginning of this century, consequently, the efforts of a number of mathematicians have been bent upon proving that π generally is not algebraical, that is, Final success of
Prof. Lindemann. that it cannot be the root of any equation having whole numbers for coefficients. But mathematics had to make tremendous strides forward before the means were at hand to accomplish this demonstration. After the French Academician, Professor Hermite, had furnished important preparatory assistance in his treatise “*Sur la Fonction Exponentielle*,” published in the seventy-seventh volume of the “*Comptes Rendus*,” Professor Lindemann, at that time of Freiburg, now of Königsberg, finally succeeded, in June 1882, in rigorously demonstrating that the number π is not algebraical,* thus

* For the benefit of my mathematical readers I shall present here the most important steps of Lindemann's demonstration, M. Hermite in order to prove the transcendental character of

$$e = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \dots$$

developed relations between certain definite integrals (*Comptes Rendus* of the Paris Academy, Vol. 77, 1873). Proceeding from the relations thus established, Professor Lindemann first demonstrates the following proposition: If the coefficients of an equation of n th degree are all real or complex whole numbers and the n roots of this equation $z_1, z_2, \dots, z_n$ are different from zero and from each other it is impossible for

$$e^{z_1} + e^{z_2} + e^{z_3} + \dots + e^{z_n}$$

supplying the first proof that the problems of the rectification and the squaring of the circle, with the help only of algebraical instruments like ruler and compasses are insolvable. Lindemann's proof appeared successively in the Reports of the Berlin Academy (June, 1882), in the "Comptes Rendus" of the French Academy (Vol. 115. pp. 72 to 74), and in the "Mathematischen Annalen" (Vol. 20. pp. 213 to 225).

"It is impossible with ruler and compasses to construct a square equal in area to a given circle." These are the words of the final determination of a controversy which is as old as the history of the human mind. But the race of circle-squarers, unmindful of the verdict of mathematics, that most infallible of arbiters, will never die out so long as ignorance and the thirst for glory shall be united.

HERMANN SCHUBERT.

to be equal to $\frac{a}{b}$, where a and b are real or complex whole numbers. It is then shown that also between the functions

$$e^{rz_1} + e^{rz_2} + e^{rz_3} + \dots + e^{rz_n},$$

where r denotes an integer, no linear equation can exist with rational coefficients variant from zero. Finally the beautiful theorem results: If z is the root of an irreducible algebraic equation the coefficients of which are real or complex whole numbers, then e^z cannot be equal to a rational number. Now in reality $e^{\pi\sqrt{-1}}$ is equal to a rational number, namely, -1 . Consequently, $\pi\sqrt{-1}$, and therefore π itself, cannot be the root of an equation of n th degree having whole numbers for coefficients, and therefore also not of such an equation having rational coefficients. The property last mentioned, however, π would have if the squaring of the circle with ruler and compasses were possible.