

On Plimpton 322. Pythagorean Numbers in Babylonian Mathematics

by

OLAF SCHMIDT*

7039
SCHMIDT, O.
1980

Table 1.

I	II	III	IV
1, 59, 0, 15	1, 59	2, 49	1
1, 56, 56, 58, 14, 50, 6, 15	56, 7	3, 12, 1	2
1, 55, 7, 41, 15, 33, 45	1, 16, 41	1, 50, 49	3
1, 53, 10, 29, 32, 52, 16	3, 31, 49	5, 9, 1	4
.	.	.	.
.	.	.	.
.	.	.	.
1, 23, 13, 46, 40	28	53	15

Since the publication of Plimpton 322 (a text containing Pythagorean numbers) by O. Neugebauer and A. Sachs this particular text has been mentioned in several papers and many attempts to explain the construction of the Pythagorean numbers have been given. Thus it is with some hesitation that the writing of another paper dealing with this Plimpton text is undertaken.

The Plimpton text in its present form (see table 1) contains four columns, I, II, III, and IV, and each column contains 15 numbers. A break at the left hand side of the text indicates that the original tablet was larger. Column IV is the least interesting of the four columns; it enumerates the lines. The integer numbers in column II and III are denoted by b and d respectively, and Neugebauer showed that to each pair of numbers (b, d) on the same line corresponds an integer number l such that

$$l^2 + b^2 = d^2.$$

The number triple (l, b, d) is in other words a Pythagorean number triple. If we for each number triple compute the number

$$\left(\frac{d}{l}\right)^2$$

we obtain the numbers in column I. The numbers b and d in column II

and III respectively are relatively prime, and they vary rather irregularly. It is a remarkable fact, however, that the numbers $\left(\frac{d}{l}\right)^2$ in column I are monotonely decreasing.

In the original discussion by Neugebauer and Sachs two explanations of the construction of the text are given. The first explanation (and the one Neugebauer and Sachs consider most likely) takes as starting point the Pythagorean equation

$$x^2 + y^2 = z^2 \quad (1)$$

and assumes that the text somehow uses that the number triple (x, y, z) given by

$$\begin{aligned} x &= 2pq \\ y &= p^2 - q^2 \\ z &= p^2 + q^2, \end{aligned}$$

where p and q are integers, is a solution to (1). The second explanation takes as starting point the equation

$$z^2 - y^2 = 1. \quad (2)$$

A rational solution to (2) gives an integral solution to (1), and rational solutions to (2) can be obtained from a table of reciprocals by noticing that if (z, y) is a solution of (2) then the pair of numbers

*København Universitet, Matematisk Institut, Universitetsparken 5, DK-2100 København Ø, Denmark

Table 2.

x	$\bar{x}$
2; 24	0; 25
2; 22, 13, 20	0; 25, 18, 45
2; 20, 37, 30	0; 25, 36
2; 18, 53, 20	0; 25, 55, 12
2; 15	0; 26, 40
2; 13, 20	0; 27
2; 9, 36	0; 27, 46, 40
2; 8	0; 28, 7, 30
2; 5	0; 28, 48
2; 1, 30	0; 29, 37, 46, 40
2	0; 30
1; 55, 12	0; 31, 15
1; 52, 30	0; 32
1, 51, 6, 40	0; 32, 24
1, 48	0; 33, 20

$$(z-y, z+y)$$

is a pair of reciprocals. Neugebauer and Sachs actually constructed the first four pairs of such a table of reciprocals (see table 2) from which the numbers in Plimpton 322 can be computed, and E. M. Bruins ("On Plimpton 322", p. 630) has computed the whole table 2 and argued strongly in favour of this second explanation.

The present explanation leads to a construction of Plimpton 322 which is based upon a table of reciprocals, and in this respect it coincides with the construction given by E. M. Bruins and with the construction rejected by Neugebauer and Sachs. Our explanation deviates however from the explanations by either Neugebauer, Sachs or Bruins in that it does not take either equation (1) or equation (2) as starting point. It takes as starting point an identity (12) (see *infra*) which can be derived from a normal form of a quadratic equation, i.e. a system of equations

$$x + y = a \quad (3)$$

$$x \cdot y = b, \quad (4)$$

where a and b are rational numbers. *A priori* this system has nothing to do with Pythagorean numbers. But by giving a detailed description of how such a system is solved, and by requiring that the solutions x and y be rational for given a and b we arrive at an identity (12) which may lead to Pythagorean numbers. In detail this is seen as follows. In the text material a solution (x, y) to (3) and (4) is found by computing

$$\begin{aligned} &\text{first: } \frac{a}{2} \\ &\text{then } \left(\frac{a}{2}\right)^2 \\ &- \left(\frac{a}{2}\right)^2 - b \\ &- \sqrt{\left(\frac{a}{2}\right)^2 - b} \\ &- \frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - b} \\ &\text{and } \frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 - b}. \end{aligned} \quad (5)$$

The solution (x, y) is given by the last two numbers. To this procedure for finding the solutions to a quadratic equation given on a normal form we may add the following comments. Nowhere does the Babylonian text material reveal how these solutions were obtained. By the following simple reasoning we may however justify the procedure that leads to the solution. First we assume that x and y are equal. Then each of them is equal to

$$\frac{a}{2},$$

and thus equation (3) is satisfied. In order to decide whether equation (4) is satisfied we compute.

$$\frac{a}{2} \cdot \frac{a}{2} = \left(\frac{a}{2}\right)^2$$

and if this is equal to b the problem is solved. If not we assume that

$$\begin{aligned} x &= \frac{a}{2} + \text{"something"} \\ y &= \frac{a}{2} - \text{"something"}. \end{aligned} \quad (6)$$

This rather unconventional way of writing x and y has been used in

order to emphasize that an apparently sophisticated mathematical table like Plimpton 322 can be worked out by very elementary reasonings and without the use of any mathematical notation. The critical reader may object that in writing "x" and "y" for the unknowns a modern notation has already been used. It is, however, a simple matter to make the explanation free from modern notation by replacing x and y by e.g. "length" and "width", terms sometimes used in this connection by the texts. The problem now is to adjust "something" so that the equation

$$x \cdot y = b$$

is satisfied. To this end we compute $x \cdot y$ and find

$$\left(\frac{a}{2}\right)^2 - (\text{"something"})^2.$$

This must be equal to b, and hence

$$(\text{"something"})^2 = \left(\frac{a}{2}\right)^2 - b \quad (7)$$

and from this we find

$$\text{"something"}$$

by extracting the square root, and thus we have explained the procedure used by the text.

We have given a detailed exposition of this procedure in order to point out that it is almost certain that the attentive user of this procedure noticed that the computations may get to a stand still at (5) or (7) unless a and b are chosen properly, i.e. unless a and b are chosen such that

$$\left(\frac{a}{2}\right)^2 - b = \square \quad (8)$$

where $\square$ denotes the square of a rational number. And now the question: how was such a choice made, naturally suggests itself. We shall of course never know for certain how the Babylonians managed to choose the numbers a and b so that (8) is satisfied. An obvious guess is that they did not start the system of equations (3) and (4) by choosing a and b. They rather started by choosing the solutions x and y, and then the numbers a and b were computed by putting

and

$$\begin{aligned} a &= x + y \\ b &= x \cdot y. \end{aligned} \quad (9)$$

It is now evident that if a and b are chosen as in (9), then (8) is satisfied. In other words it is evident that for any rational numbers x and y

$$[1/2 (x+y)]^2 - xy = \square. \quad (10)$$

We may even assume that it also was known that the $\square$ in (10) is the same as ("something")² in (7), and therefore ("something")² or $\square$ can be found from (6) by a subtraction so that

$$\square = [1/2 (x-y)]^2.$$

Thus the Babylonian computer who was familiar with a normal form of a quadratic equation most likely would know that

$$[1/2 (x+y)]^2 - xy = [1/2 (x-y)]^2. \quad (11)$$

When we say that (11) was known we mean that it was known in the form that half the sum of two numbers raised to the second power minus the product of the numbers is equal to half the difference of the two numbers raised to the second power. Next we consider a special case of a normal form of a quadratic equation, namely a system in which $b = 1$. Such a system actually occurs in the text material (Neugebauer, *Mathematische Keilschrift-Texte*, vol. I, p. 106. The system is

$$\begin{aligned} x + y &= a \\ x \cdot y &= 1 \end{aligned}$$

and denoting the reciprocal of a number x by $\bar{x}$ this system may be written in the form

$$\begin{aligned} x + \bar{x} &= a \\ x \cdot \bar{x} &= 1 \end{aligned}$$

and the identity (11) now becomes

$$[1/2 (x+\bar{x})]^2 - 1 = [1/2 (x-\bar{x})]^2. \quad (12)$$

in this respect our explanation agrees better with the text than the two previous explanations. Implicitly the number l does of course enter into our explanation; it is readily seen that l is the greatest common divisor of the integers obtained from (13) by a multiplication of a suitable power of 60. A remark by Neugebauer that l most likely was computed in a column prior to column I is not confirmed by our explanation. Our explanation rather leads to a text which in its complete form contains the following columns:

[−IV], [−III], [−II], [−I], [0], I, II, III, IV

where columns I, . . . , IV are the actually preserved columns, and columns [−IV], . . . , [0] are reconstructed columns. (See table 3.)

Columns [−IV] and [−III] contain 15 pairs of numbers

x and $\bar{x}$

given in table 2.

Columns [−II] and [−I] contain 15 pairs of numbers

$\frac{1}{2} (x - \bar{x})$ and $\frac{1}{2} (x + \bar{x})$

computed from the two preceding columns.

Columns [0] and I contain 15 pairs of numbers

$[\frac{1}{2} (x - \bar{x})]^2$ and $[\frac{1}{2} (x + \bar{x})]^2$

computed from the two preceding columns. The identity (12) shows that the difference between two numbers from columns I and [0] is 1. Thus if the number in column I, line 2 is

1; 56,56,58,14,50,6,15

then the number in column [0], line 2 is

0; 56,56,58,14,50,6,15

and having noticed this very simple rule, there is hardly any need of both columns [0] and I. One of them will do.

Columns II and III contain numbers proportional to the numbers in column [−II] and [−I], and reduced in the sense explained above. The computation of the numbers in columns II and III may be illustrated by the following example. The numbers in columns [−II] and [−I], line 3 multiplied by 60^3 are equal to

57,30,45 and 1,23,6,45.

The greatest common divisor of these numbers is regular (see *supra*). By dividing the numbers by 3·3·5 we obtain the numbers

1,16,41 and 1,50,49.

These numbers do not have any regular common divisor and hence they are relatively prime, and they are the numbers in columns II and III, line 3.

Thus it is possible to construct a text of altogether 9 columns (or 8 columns if column [0] which deviates from column I by one is left out) of which the last four are the columns appearing in the fragment of Plimpton 322. We notice that this reconstructed text does not have any obvious relation to a right triangle. In the heading of columns II and III we do find, however, the words “width” and “diagonal”. In order to explain this fact we refer to the identity (12) which shows that the difference between the squares of $\frac{1}{2} (x + \bar{x})$ and $\frac{1}{2} (x - \bar{x})$ is a square, and this remains true if the numbers are reduced, i.e. multiplied by a proper constant, and so the pairs of numbers obtained in this way (i.e. the numbers in columns II and III) deserve the names “diagonal” and “width” (of a rectangle).

BIBLIOGRAPHY

- E. M. Bruins, “On Plimpton 322. Pythagorean numbers in Babylonian Mathematics”, *Koninklijke Nederlandsche Akademie van Wetenschappen. Proceedings*, Vol. 52 (1949), 191–194.
 E. M. Bruins, “Reciprocals and Pythagorean Triads”, *Physica*, Vol. 9 (1967), 371–392.
 O. Neugebauer and A. Sachs, *Mathematical Cuneiform Texts*, New Haven, Connecticut, 1945. The text Plimpton 322 is published and discussed on p. 38–41.
 O. Neugebauer, *Mathematische Keilschrift-Texte*, Vol. 1–3. Reprint, Berlin, 1973.