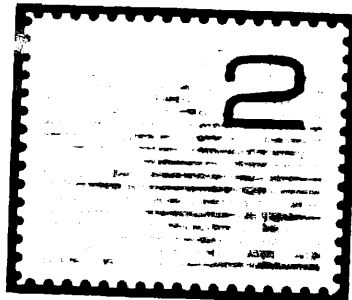


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IN MATHEMATICS
AND SCIENCE



Philosophers and Geometers

600 B.C. – A.D. 400

THE “science” of the ancient Greeks would scarcely qualify as science by modern criteria. Their notions were derived chiefly from speculation and some little experimentation, not from objective observations and measurements (although they did recognize that the earth was a sphere). They toyed with ideas of the nature of matter and its indestructibility, but the atomistic theory of Democritus was quite different from that of Dalton some two thousand years later. Greek medicine did lean largely on experiments by Hippocrates, but Plato’s physics and physiology were anthropomorphic. Aristotle, a keen observer, was more successful in physiology than in physics, but he is chiefly remembered for his logic.

Greek architecture was dignified and impressive, with a grandeur rarely surpassed, but like Greek geometry, it lacked dynamic quality: no flowing movement, only frozen symmetric beauty. This can be seen in the Ionic capital (the top, or crown, of a supporting pillar) at the left in figure 2.1. The other two types of capitals used by the Greeks were the Doric and the Corinthian. The Romans used somewhat similar capital designs. After the fall of Rome, these designs were modified by Gothic architects, but the classic forms were revived during the Renaissance and have survived to the present day. The modern Austrian parliament building at the right in figure 2.1, for instance, is typical of the architecture of ancient Greece.

In addition to their intellectual, philosophic, and aesthetic talents, the Greeks were fairly skillful in handicrafts and technics. They were familiar

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Fig. 2.1. Ionic capital and Greek architecture

with the potter’s wheel and the lathe, and they knew how to solder iron. They used the *gnomon* (sundial) to tell time. They believed that the primary stuff of the physical world consisted of water, earth, air, and fire.

During the course of nearly a thousand years, mathematics flourished among the Greeks. We know that they contributed ideas to the theory of numbers, which they called *arithmetike*, but were much less interested in the art of computation, which they called *logistike*. The former was the concern of philosophers and scholars; the latter was relegated to merchants and artisans. The greatest single contribution of Greek mathematics to posterity was, of course, the concept of deductive reasoning, or logical proof, which was probably introduced by Thales (ca. 600 B.C.) and elaborated on by Euclid and his successors. Because of the association of logic with philosophy, the Greeks regarded geometry as an essential part of a liberal education. (In medieval universities the *quadrivium* consisted of arithmetic, geometry, astronomy, and music, whereas the *trivium* was composed of grammar, rhetoric, and dialectic; together they formed the seven liberal arts.)

We now know that some mathematical knowledge formerly believed to be original with the Greeks must instead be attributed to the Egyptians and to the Babylonians. Nevertheless, the glorious achievements of Greek geometry are not likely to fade even after two thousand years. As G. H. Hardy has so truly said, “The Greeks were the first mathematicians who are still ‘real’ to us today . . . Greek mathematics is the real thing.”

THE RISE OF GREEK SCIENCE

The rise of Greek science began in Ionia, near the Aegean Sea, with Thales and Pythagoras. Of these two men we know very little directly, only

what later writers said of them. It seems that Thales (ca. 600 B.C.) was a man of practical affairs, a shrewd businessman, and a philosopher of sorts, with an interest in astronomy. From his travels in Egypt he brought back some knowledge of Egyptian geometry. He is credited with a few elementary theorems, but it is doubtful if he did more than take the first timid steps toward organizing geometry on a systematic basis.

Even more mystery and legend surround his compatriot Pythagoras (ca. 540 B.C.), for nearly all we know of him has come down through second- and third-hand sources. Without doubt Pythagoras was a mystic and a prophet. In his rather wide travels he gathered considerable knowledge of mathematics and astronomy as well as religious lore. On settling in Croton, a city in southern Italy, he organized a secret society of kindred spirits interested in philosophy and mathematics. In time this Pythagorean brotherhood became firmly established, with elaborate rules and rituals. Pythagoras not only taught the members of this "inner circle" but lectured to the general public as well. Credit for each new discovery was generally given to Pythagoras himself, not to individual members of the brotherhood.

Much of what was once regarded as contributions of Pythagoras, including the so-called Pythagorean theorem itself, had actually been known earlier by the Babylonians. It is even possible that the Chinese may have been familiar with the Pythagorean relation (see *Was Pythagoras Chinese?* by Frank J. Swetz and T. I. Kao [University Park, Pa.: Pennsylvania State University and NCTM, 1977]).

Yet the Pythagorean school gave mathematics in Greece an entirely new complexion. Mathematics was henceforth regarded as closely related to philosophy and wisdom in general. And arithmetic (number theory, not computation) was regarded by the Pythagoreans as even more significant than geometry. Pythagoras is supposed to have said, "Number rules all." In fact, the Pythagoreans were among the first to use geometry to express relations between quantities and numbers. They developed a theory of proportion that served them well until they encountered numbers that could not be expressed as the ratio of two whole numbers, such as $\sqrt{2}$ and $\sqrt{40}$.

In any discussion of the Pythagoreans, it is difficult to separate legend from fact. However, the most notable aspect of the Pythagorean influence is the belief it fostered in the universal importance of numbers. Clinging to number mysticism and worship, they assigned all sorts of characteristics to numbers. Odd numbers had male attributes; even numbers, female traits. The number two was the first female number; three was the first male number. Four was the number of justice; five represented marriage; and so on. This sort of idolatry led to numerology and astrology, which still enjoy popularity today. Of far greater significance mathematically was the

Pythagorean discussion of figurate numbers, which leads not only to interesting geometric patterns but also to infinite progressions of numbers (see fig. 2.2).

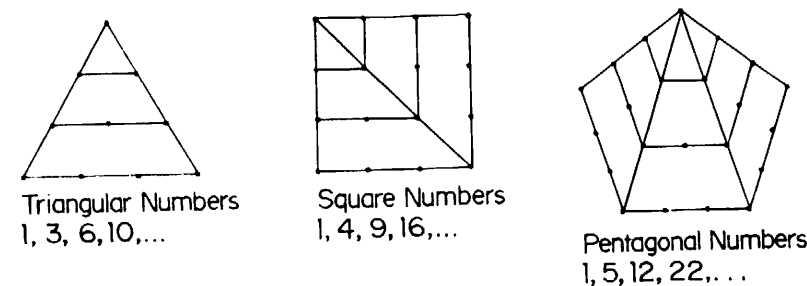


Fig. 2.2

Of interest also is the five-pointed star, or pentagram, which was the badge of the brotherhood and which led to a greater knowledge of geometry. This included the golden section, which refers to the division of a line segment into two parts such that the ratio of the length of the entire segment is to the length of the greater part as the greater part is to the smaller part (fig. 2.3). Either intersection point (e.g., *B* or *C*) on any diagonal of a regular pentagon divides that diagonal into the golden section.

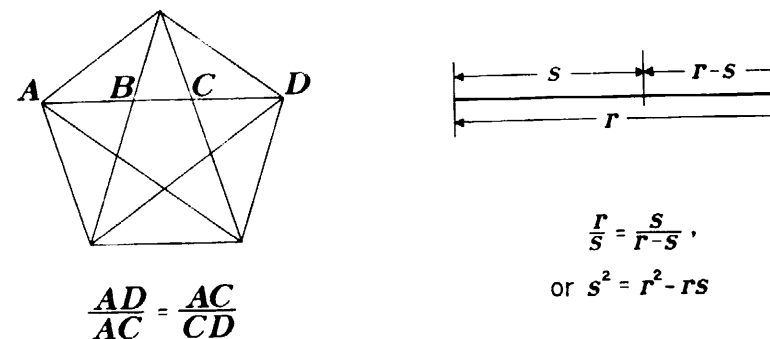


Fig. 2.3

We come now to the famous right-triangle theorem itself. Both Pythagoras and his theorem have been honored on stamps. At the left in figure 2.4 is an artistic representation of the theorem for the special case of a 3:4:5 right triangle; the stamp at the right depicts a Greek coin from the island of Samos, birthplace of Pythagoras, showing Pythagoras presumably consulting an oracle; in the center another stamp shows a geometric

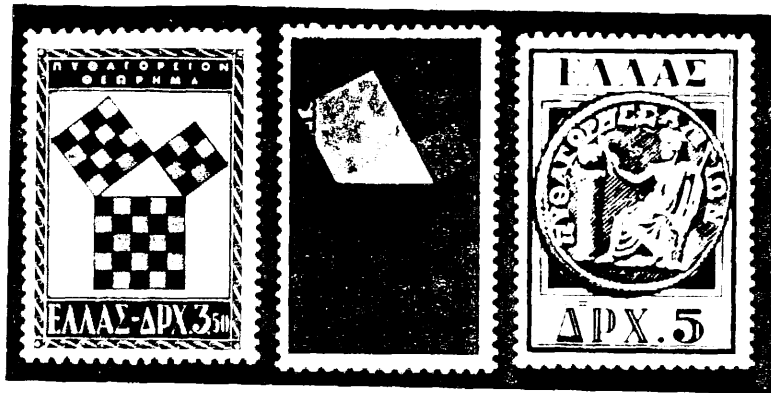


Fig. 2.4. Pythagorean theorem

diagram of the Pythagorean relation. A statement of the theorem, $A^2 + B^2 = C^2$, is given on the stamp in figure 2.5, which nicely illustrates Greek architecture once more.



Fig. 2.5. Law of Pythagoras

It is not clear how the Pythagoreans justified the theorem, which, as we have already said, was known to the early Babylonians and Egyptians. What "geometric proof," if any, Pythagoras gave is not known, but Euclid and others gave several proofs. Since the days of Greek geometry, over a hundred mathematical proofs of the theorem have been given, some of them geometric, others largely algebraic. Two are suggested in figure 2.6; readers might like to see if they can figure them out by themselves.

Next, we turn to the stage of Greek mathematics known as the "Athenian school," which ranged over the fifth and fourth centuries B.C. Thales and Pythagoras had opened the way, laying the foundations of geometry and theoretical arithmetic. Now the city of Athens was thriving

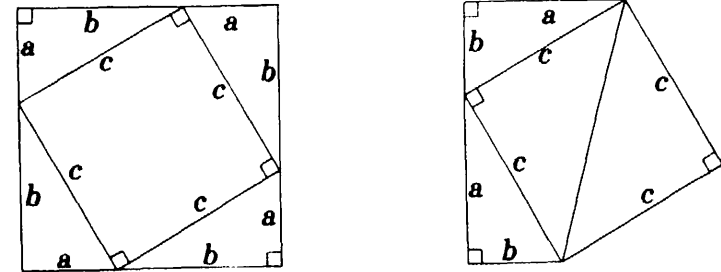


Fig. 2.6.

as the intellectual center of the Greek world. Philosophers, mathematicians, and astronomers gathered in Athens, among them Democritus, Plato, and Aristotle.

Democritus struggled with the problem of dividing a solid such as a cone or a cylinder into many thin slices, an idea that anticipated the development of the calculus many centuries later. His thinking suggested the doctrine of atomism. According to this notion, all physical phenomena could be explained in terms of extremely small but finite, hard atoms, incapable of subdivision and moving about in empty space. He believed these atoms were unchangeable and indestructible. The atoms differed from one another in size and form and properties, which accounted for the variation in the properties of different substances.

Democritus is honored on the stamp at the right in figure 2.7, which also bears the contemporary symbol of atomic structure; it was issued to commemorate the inauguration of the Democritus Nuclear Research Center at Aghia Paraskevi, in Greece. Democritus's concept was remarkably similar to modern theories of the structure of matter, but whereas contemporary physics and chemistry are based on quantitative experiments and



Fig. 2.7. Aristotle and Democritus

rigorous mathematical analysis, the doctrine of Democritus grew out of intuition and philosophic speculation. Although his atomism was decried by Socrates and others, his views never died out completely. They were revived, so to speak, about 1800 by John Dalton, the English chemist, who built his theory on a century and a half of chemical experimentation.

A generation or so after Democritus, two other philosophers helped to mold Greek thought—Plato (ca. 380 B.C.) and Aristotle (ca. 340 B.C.). Plato was more deeply interested in *moral* philosophy than in *natural* philosophy (i.e., physical science), which he regarded as somehow inferior. He viewed mathematics as a more or less lofty, abstract form of thought, remote from the mundane affairs of everyday life. The single exception that Plato conceded was the relation of mathematics to astronomy. In his view, the heavens reflected the perfection of abstract mathematics, and this conviction prevailed for centuries until the time of Johannes Kepler (1570–1630).

Aristotle is honored on the left-hand stamp in figure 2.7. A great philosopher and an able biologist, he is also known for his systematic treatment of deductive logic—that is, reasoning from accepted statements to necessary conclusions. This elaborate Aristotelian *verbal* logic exerted profound influence until the nineteenth century, when the modern mathematician George Boole introduced the study of *symbolic* logic to mathematics. Aristotle tended toward experimental science; he rejected the atomistic speculations of Democritus. Aristotle's philosophy in his own day was not as influential as that of Plato, but the Arabs later "rediscovered" him, and for centuries thereafter Europe regarded the authority of Aristotle as the last word.

THE GOLDEN AGE

The period from 300 to 200 B.C. is often called the golden age of Greek mathematics. During this time creativity reached its zenith. Three men towered high above their contemporaries as well as above those who had preceded them or who were to follow: Euclid, Archimedes, and Apollonius.

Euclid

With the founding of the city of Alexandria about 330 B.C., mathematical activity shifted from Greece to Egypt. Although Alexander the Great did not live to see it, the metropolis that he founded at the mouth of the Nile blossomed into a renowned center of learning to which both Greek and Oriental scholars flocked in great numbers. In the course of time, the university at Alexandria boasted more than 700 000 volumes. One of the first teachers at the university was Euclid. His influence was felt a generation later in the brilliant achievements of two other scholars at the

university, namely, Archimedes and Apollonius, probably the greatest mathematicians of antiquity.

Euclid's greatest contribution was his work entitled the *Elements*, in reality a series of thirteen books. In this he collected and systematized all the mathematics known at the time (about 300 B.C.). Books 1, 2, and 4 deal with lines and plane figures (due chiefly to Pythagoras); Book 3 deals with circles (Hippocrates); Book 5 is devoted to the theory of proportion (Eudoxus); Book 6, areas and similar figures; Books 7, 8, and 9, the theory of numbers (mostly Pythagoras); Book 10, irrational quantities; Book 11, solid geometry; Book 12, the method of exhaustion; and Book 13, proofs and constructions for the five regular solids (Plato).

Despite certain logical weaknesses, this monumental work survived virtually unchanged for more than two thousand years. Indeed, the *Elements* was so revered that in England many centuries later, instead of studying geometry, one simply "read Euclid." The logical weaknesses alluded to chiefly concern the postulates, particularly the so-called parallel postulate. These shortcomings were eventually remedied in the middle of the nineteenth century, but that story must wait until a later chapter.

Archimedes

Although Archimedes (ca. 250 B.C.) was without doubt the greatest scientist and mathematician of ancient times, he also achieved fame for his "practical" discoveries and ingenious mechanical devices. He was the first to explain the law of the lever and was so thrilled by its usefulness that he is supposed to have exclaimed, "Give me a place to stand on and I can move the world." Actually, in recognizing that the weights and lever arms are in inverse proportion, he applied quantitative measurement to observations, anticipating the science of mechanics by nearly two thousand years.

Archimedes is honored on the two stamps shown in figure 2.8: at the right is a reproduction of a painting by the seventeenth-century artist José



Fig. 2.8. Archimedes and his law