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Rediscovering the Archimedean Polyhedra: Piero della Francesca, Luca Pacioli, Leonardo da Vinci, Albrecht Dürer, Daniele Barbaro, and Johannes Kepler

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*Rediscovering the Archimedean Polyhedra:
Piero della Francesca, Luca Pacioli,
Leonardo da Vinci, Albrecht Dürer,
Daniele Barbaro, and Johannes Kepler*

J. V. FIELD

Communicated by C. TRUESDELL

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Introduction

The story of the rediscovery of the Archimedean polyhedra during the Renaissance is not that of the recovery of a ‘lost’ classical text. Rather, it concerns the rediscovery of actual mathematics, and — as so often in the history of mathematics — there is a large component of human muddle in what with hindsight might have been a purely rational process. The pattern of publication, summarised in Table 1 below, indicates very clearly that we do not have a logical progress in which each subsequent text contains all the Archimedean solids found by its author’s predecessors. It accordingly seems as well to provide a preliminary outline of the story before attempting to give a more detailed account of it.

In fact, as far as we know, there was no classical text by ARCHIMEDES to be recovered. The Archimedean solids have that name because in his *Collection*, PAPPUS (fl. AD 300–350) stated that ARCHIMEDES (c. 287–212 BC) had discovered thirteen solids whose faces were regular polygons of more than one kind. PAPPUS then listed the numbers and types of faces of each solid, for instance

telling us that one such solid had four hexagonal faces and four triangular ones.¹ The descriptions do not make it easy to imagine what the bodies would look like, but they are sufficient to establish that these are the solids now known as convex uniform polyhedra. There are in fact exactly thirteen such solids, whose defining properties are that every face is completely visible on the outside of the solid,² and that each vertex of the solid is surrounded by regular polygons arranged in the same way.³ Thus, for the solid with four hexagonal faces and four triangular ones, each solid angle is formed by the meeting of two hexagons and a triangle. All thirteen convex uniform polyhedra are shown in Figures 1 and 2, which are taken from the earliest known mathematical account of the complete set of Archimedean polyhedra in modern times, namely that in JOHANNES KEPLER's *Harmonices mundi libri V*, Linz, 1619, Book 2, Proposition 28. The modern names for the solids are translations of those given them by KEPLER (1571–1630).

KEPLER gives no indication that he is aware that descriptions of some of these solids were to be found in texts more recent than that of PAPPUS. However, he certainly knew the work of DÜRER, and it seems unlikely that he could have been completely ignorant of LUCA PACIOLI's *De divina proportione* (Venice, 1509), so his silence may perhaps best be construed as a comment on the almost complete lack of mathematical discussion of the properties of the Archimedean solids these authors mention. PACIOLI (c. 1445–1514) does, in fact, give brief descriptions of six of the Archimedean solids. His personal friend LEONARDO DA VINCI (1452–1519) provided illustrations for the work. Even in their published form of woodcuts, these illustrations are very striking, and they no doubt played an important part in ensuring the book sold well. They also play an important part in PACIOLI's exposition of his results.

Six Archimedean solids, including four of those later mentioned by PACIOLI, had been discussed in works by PIERO DELLA FRANCESCA (c. 1412–1492), who provided his own illustrations. These works were not printed under PIERO's name until the twentieth century, but there is strong evidence that PACIOLI had access to them in manuscript form.

Seven of the Archimedean solids are described in ALBRECHT DÜRER (1471–1528), *Underweysung der Messung mit dem Zirkel und Richtscheit* (Nuremberg, 1525, new slightly expanded edition 1538; translated as *Institutiones geometricæ*, Paris, 1532). These seven include five of those found in *De divina proportione*, one of which is only to be found in PIERO DELLA FRANCESCA's contribution to the volume. DÜRER acknowledges no sources.

¹ PAPPUS *Collection*, Book V, ed. and trans. in IVOR THOMAS, *Greek Mathematical Works. Selections illustrating the history of Greek Mathematics*, 2 vols, London: Heinemann (Loeb editions), 1939–41. The first printed edition of PAPPUS' *Collection* was in the Latin translation of COMMANDINO (Pesaro, 1588), where the reference to ARCHIMEDES' work appears as if it were part of COMMANDINO's commentary, following Theorem 16, Proposition 17, p. 83 verso.

² That is, the faces surround the centre only once. (This is the definition of 'convex'.)

³ That is, the vertices are equivalent, and therefore lie on a sphere. (This is the definition of 'uniform'.)

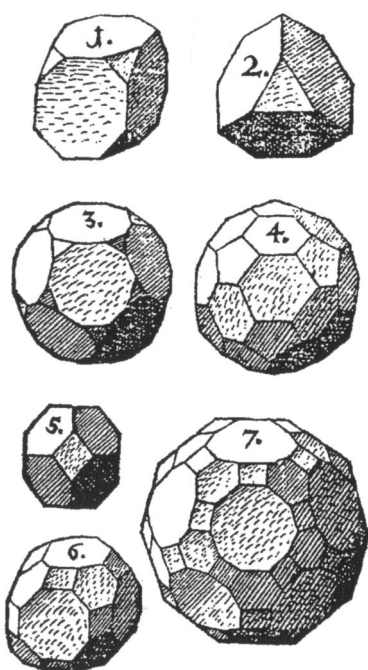


Figure 1

Archimedean polyhedra. The modern names for the solids shown here are 1. truncated cube, 2. truncated tetrahedron, 3. truncated dodecahedron, 4. truncated icosahedron, 5. truncated octahedron, 6. truncated cuboctahedron, 7. truncated icosidodecahedron. From J. KEPLER, *Harmonices mundi libri V*, Linz, 1619, Book 2, p. 64.

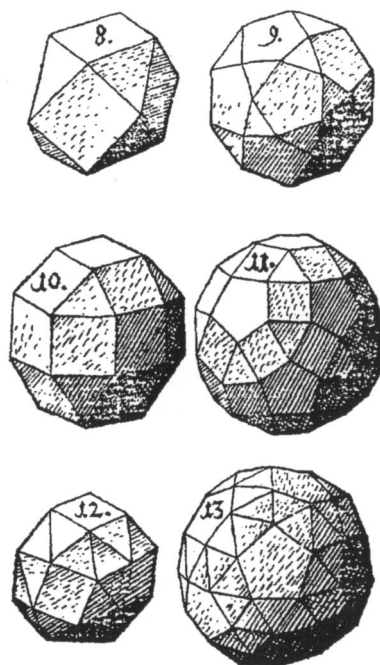


Figure 2

Archimedean polyhedra. The modern names for the solids shown here are 8. cuboctahedron, 9. icosidodecahedron, 10. rhombicuboctahedron, 11. rhombicosidodecahedron, 12. snub cube, 13. snub dodecahedron. From J. KEPLER, *Harmonices mundi libri V*, Linz, 1619, Book 2, p. 63.

Eleven of the Archimedean solids are described in DANIELE BARBARO (1513–1570), *La Pratica della prospettiva* (Venice, 1568, 1569). These eleven include eight which seem to have been taken from either PIERO DELLA FRANCESCA or PACIOLI (without acknowledgement to either), one which BARBARO may have found in DÜRER's *Underweysung* and two more which he presumably discovered for himself, though he makes no claim to have done so. Like PACIOLI and DÜRER, BARBARO keeps mathematical explanation to the minimum.

It is this rather tangled pre-Keplerian part of the story (summarised in Table 1) with which we shall mainly be concerned in this paper. The starting point is not with PAPPUS, whose work does not seem to have been known to PIERO, PACIOLI, DÜRER or BARBARO, but rather with a standard work of the mathematical canon, namely EUCLID's *Elements*.

Table 1

The Archimedean polyhedra in Renaissance sources. The first column shows the number given to the solid in the diagrams in J. KEPLER, *Harmonice mundi* (Linz, 1619), Book 2, Proposition 28 (reproduced in our Figures 1 and 2). The second column gives the modern name of the solid. The last four columns indicate whether the solid appears in the work of PIERO DELLA FRANCESCA (T if in his *Trattato*, L if in *Libellus*), LUCA PACIOLI (*De divina proportione*, 1509), ALBRECHT DÜRER (*Underweysung*, 1525), DANIELE BARBARO (*La Pratica della prospettiva*, 1568, 1569).

JK no	Name	Piero	Pacioli	Dürer	Barbaro
1	truncated cube	L	—	✓	✓
2	tr. tetrahedron	T,L	✓	✓	✓
3	tr. dodecahedron	L	—	—	✓
4	tr. icosahedron	L	✓	—	✓
5	tr. octahedron	L	✓	✓	✓
6	tr. cuboctahedron	—	—	✓	✓
7	tr. icosidodecahedron	—	—	—	✓
8	cuboctahedron	T	✓	✓	✓
9	icosidodecahedron	—	✓	—	✓
10	rhombicuboctahedron	—	✓	✓	✓
11	rhombicosidodecahedron	—	—	—	✓
12	snub cube	—	—	✓	—
13	snub dodecahedron	—	—	—	—

Polyhedra in Euclid and pseudo-Euclid

Five regular polyhedra (shown in Figure 3) were known to the Ancient Greeks. They are often known as the 'Platonic' or 'cosmic' solids because they are discussed in PLATO's dialogue *Timæus*, where each is associated with one of the five elements — the cube with earth, the icosahedron with water, the octahedron with air, the tetrahedron with fire and the dodecahedron (more or less by default) with ether, the material of the heavens. There is a brief reference to this association in the otherwise markedly non-mathematical commentary on *Timæus* by CALCIDIUS, which was known throughout the Middle Ages.⁴

There is very little technical discussion of the geometrical properties of the regular polyhedra in *Timæus*, and still less in CALCIDIUS' commentary. These works might serve to introduce Natural Philosophers to the solids, and may well have provided an important incentive for studying them, but for the relevant geometry one had to turn to EUCLID, specifically to Book 13 of the *Elements*. This is now known to be the last book of EUCLID's work. However,

⁴ CALCIDIUS, whose name is sometimes spelled CHALCIDIUS, was a Christian philosopher who lived in the fourth century AD. The first printed edition of his commentary, entitled *Timæi Platonis traductio*, is dated 1520.

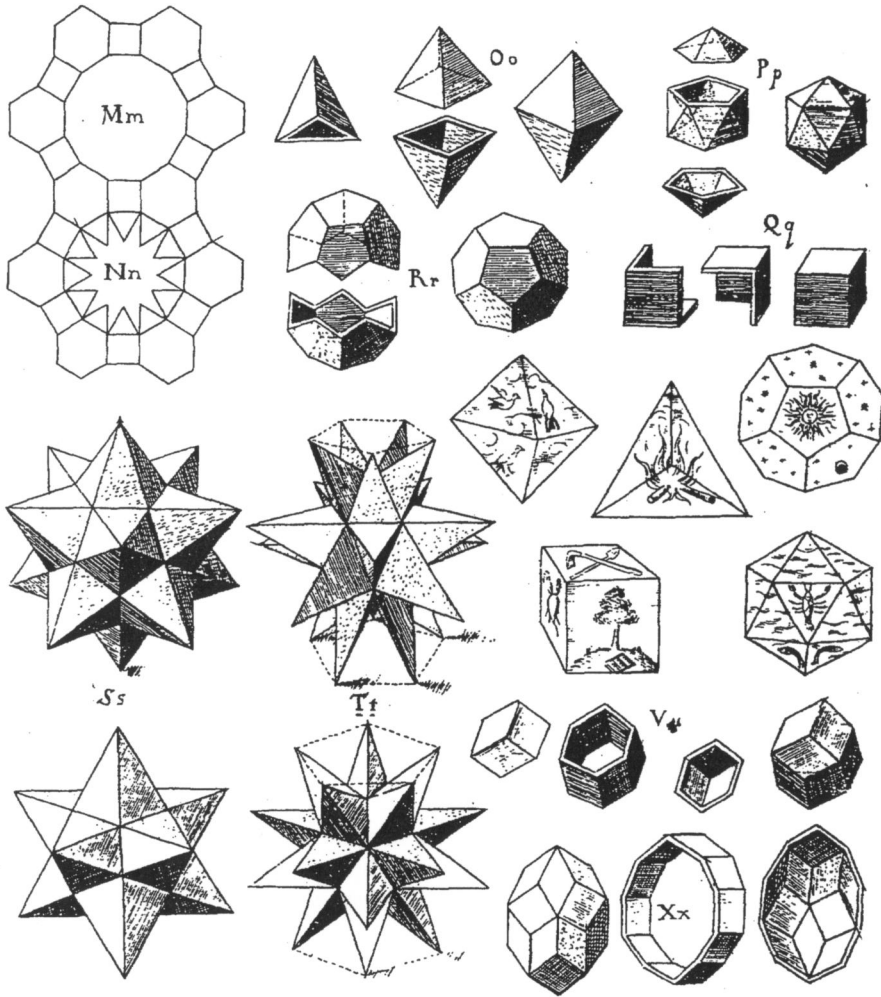


Figure 3

The five regular solids known since Ancient times are shown upper right. These are the five regular polyhedra that are convex, that is, those whose faces are completely visible on the outside of the solid. At the lower left, marked Ss and Tt, are drawings of the two star polyhedra discovered by KEPLER (see Appendix). The illustration is a plate from

JOHANNES KEPLER, *Harmonices mundi libri V*, Liniz, 1619, Book 2.

Medieval mathematicians accepted as genuine two pseudo-Euclidean books, known as *Elements* 14 and 15, which are also partly concerned with the regular polyhedra, and were probably regarded by their authors as supplements to Book 13. Book 14 begins with a theorem relating the sides of regular polygons inscribed in the same circle, goes on to discuss 'mean and extreme ratio' (the proportion sometimes called 'divine' in the Renaissance, and now usually called the Golden Section) and then considers metrical relations between regular

polyhedra inscribed in the same sphere. Book 15 is less substantial. It is made up of a disjointed series of propositions in which a regular solid is inscribed inside another regular solid, for instance by selected vertices of the first becoming vertices of the second. The inspiration for this work is probably EUCLID's construction of a dodecahedron by adding pieces on to a cube so as to construct additional vertices lying on the same sphere as those of the cube (*Elements* 13, proposition 17).

These two pseudo-Euclidean books were certainly present in the versions of the *Elements* known to both PIERO DELLA FRANCESCA and LUCA PACIOLI. Neither questions their authenticity, though PACIOLI must have known ZAMBERTI's version of the *Elements* (Venice, 1505) in which Books 14 and 15 are ascribed to HYPsicLES. Modern scholars accept this attribution for Book 14,⁵ but describe Book 15 merely as 'Late Antique'. In any case, for our present purposes it is not the authorship of *Elements* 14 and 15 that is of importance but rather the content of the books. This seems the likeliest source for PIERO DELLA FRANCESCA's rediscovery of some of the Archimedean solids.

Piero della Francesca

PIERO DELLA FRANCESCA, who is now best remembered not as a mathematician but as one of the finest painters of the fifteenth century, was probably the oldest surviving son of a fairly prosperous merchant from the town of Borgo San Sepolcro in the valley of the river Tiber, in what is now Tuscany.⁶ His circumstances made it natural that he should be taught to read and write, at least in the vernacular, and that he should also be taught some mathematics. In fact, GIORGIO VASARI (1511–1574) tells us in his *Life of the painter* that in his youth PIERO showed unusual ability in mathematics, and that he went on to write 'many' mathematical treatises of his own.⁷ Three of these have now been recovered: an 'abacus treatise' (*Trattato d'abaco*), a treatise on drawing in perspective (*De prospectiva pingendi*), and a short work on geometry part of which is devoted to polyhedra (*Libellus de quinque corporibus regularibus*). None of these works was printed under PIERO's name in the Renaissance, but all are

⁵ ZAMBERTI prints the ascription to Hypsicles as occurring at the beginning of his Greek manuscript of the book.

⁶ Recent research in notarial archives has thrown much light on PIERO's early life. It has also shown that although the usual form of his family name was 'DEI FRANCESCHI' the form 'DELLA FRANCESCA' was used by his grandfather. See JAMES R. BANKER, 'Piero della Francesca as assistant to Antonio d'Anghiari in the 1430s: some unpublished documents', *Burlington Magazine*, 135, 1993, 16–21; and the same author's paper presented to a conference in Urbino in October 1992 and to be published in the Proceedings (ed. M. DALAI EMILIANI *et al.*).

⁷ G. VASARI, *Le Vite dei piu eccellenti architetti, pittori e scultori italiani da Cimabue a tempi nostri*, Florence, 1550, 1568, ed. P. BAROCCHI, R. BETTARINI, Florence, 1971.

available in modern printed editions.⁸ Here we shall be concerned only with the first and third of these works, since, unlike many later treatises on perspective, PIERO's has nothing to say about any regular polyhedron except the cube.⁹

PIERO's *Trattato d'abaco* contains an unusually large amount of geometry, unusual that is in comparison with what is normally found in textbooks of the time. The weight given to geometry presumably reflects PIERO's own preferences, since the work was apparently written at the request of friends rather than for use in a school. The geometrical part begins conventionally enough, however, with problems relating to finding the heights of triangles and then their areas. As is usual in schoolbooks of the time, instruction proceeds entirely by series of worked examples, with very little discursive text. The attention to triangles is easily explained by its practical usefulness: taxes were assessed according to the area under cultivation, so it was advisable to be able to make one's own estimates rather than having to rely upon the official ones. Among other things, PIERO's father dealt in the vegetable dyestuff indigo, so PIERO probably had direct experience of surveying the fields in which the crop was grown.

After triangles, PIERO turns to solid geometry, first the regular tetrahedron, then the cube, with references to *Elements* 13, and then, after some discussion of the sphere, still by means of worked examples, to the problems of inscribing the two polyhedra in a sphere. The problems are given in numerical form. For instance for the tetrahedron we have

There is a spherical body whose diameter is 7; I want to put in it a figure with 4 equilateral triangular faces, so that the corners [of the figure] touch the circumference [of the sphere]. I ask what its edges are.¹⁰

⁸ PIERO DELLA FRANCESCA, *Trattato d'abaco: Dal Codice Ashburnhamiano 280 (359*.291*) della Biblioteca Medicea Laurenziana di Firenze*, ed. G. ARRIGHI, Pisa, 1970; PIERO DELLA FRANCESCA, *De prospectiva pingendi*, ed. G. NICCO FASOLA, Florence, 1942 (reprint Florence, 1984); PIERO DELLA FRANCESCA, 'L'Opera "De corporibus regularibus" di Pietro dei Franceschi detto della Francesca, usurpata da Fra' Luca Pacioli', ed. G. MANCINI, *Memorie della R. Accademia dei Lincei*, series 5, 14.8B, 1916, 441–580. This last, elderly, edition has now been replaced by a new one based on the single surviving manuscript (Vatican Codex Urbinas 632): PIERO DELLA FRANCESCA, *Libellus de quinque corporibus regularibus*, eds M. DALAI EMILIANI, C. GRAYSON, C. MACCAGNI *et al.*, Florence: Giunti, 1995. This publication, which I have not yet been able to see, is the first volume in a projected, and very welcome, new edition of all of PIERO's writings.

⁹ As far as is known, PIERO's was the first treatise on perspective. Most later treatises made extensive use of it, through manuscript copies or the substantial extracts printed in DANIELE BARBARO's, *La Pratica della prospettiva*, Venice, 1568, 1569, see Appendix, p. 277 ff.

¹⁰ PIERO, *Trattato*, ed. Arrighi (note 9), p. 229.

Then, with no warning, and still in numerical form, we have a problem involving a new solid:

There is a spherical body whose diameter is 6; I want to put in it a body with 8 faces, 4 triangular and 4 hexagons. I ask what its edge is.¹¹

By way of supplementing this description PIERO supplies a diagram (see Figure 4), in which a circle indicates the circumsphere. The diagram may at first seem to be in 'correct perspective', but it can be shown that this is probably not so. In fact, PIERO uses a variety of styles in the illustrations to his work on polyhedra (see Appendix).

There is no hint of how the new solid may have been discovered. In fact, PIERO's silence about any source for his problem is not necessarily to be taken as an indication of originality, since almost all of PIERO's algebraic examples (though not always their solutions) are derived from earlier work.¹² However, in view of PIERO's numerous references to EUCLID, and his repeated references to Ancient authors of the humanist canon in his treatise on perspective, it seems likely that if PIERO knew that this example had a classical antecedent he would have mentioned it. There is some evidence that PIERO knew not only the name but also something of the work of ARCHIMEDES.¹³ All the same, it seems he did not associate this name with the solid whose faces were four triangles and four hexagons.

PIERO's apparent independence of even PAPPUS' brief descriptions of the Archimedean solids is confirmed by his treatment of the cuboctahedron, which first appears in his next problem:

There is a spherical body, whose diameter is 6 *bracci*; I want to put in it a figure with fourteen faces, 6 square and 8 triangular, with equal edges. I ask what each edge will be.

Such a figure as this is cut out from the cube, because it [the cube] has 6 faces and 8 corners; which, cutting off its 8 corners, makes 14 faces, that is thus. You have the cube ABCD.EFGH, divide each side in half: AB in the point I, and CD in the point L, BD in the point K, AC in the point M, . . .¹⁴

The accompanying diagram is shown in our Figure 5. PIERO has again used a circle to indicate the presence of the circumsphere (a more detailed discussion

¹¹ PIERO, *Trattato*, ed. Arrighi (note 9), p. 230.

¹² See R. FRANCI, L. TOTI RIGATELLI, 'Towards a History of Algebra from Leonardo of Pisa to Luca Pacioli', *Janus*, 72, 1985, 17-82; and M. D. DAVIS, *Piero della Francesca's mathematical treatises: The 'Trattato d'abaco' and 'Libellus de quinque corporibus regularibus'*, Ravenna, 1977.

¹³ See MARSHALL CLAGGETT, *Archimedes in the Middle Ages*, Philadelphia, 1973.

¹⁴ PIERO ed. Arrighi pp. 231-32. The Florentine *braccio* was of length about 58.36 cm. The usual form of the plural is *braccia*, though PIERO seems generally to use the form given here.

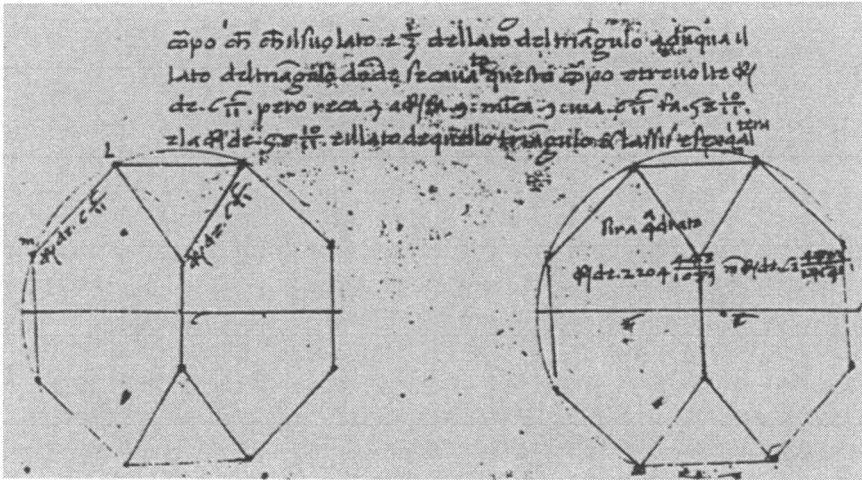


Figure 4

Drawings of the truncated tetrahedron, each diameter about 48 mm. From PIERO DELLA FRANCESCA, *Trattato d'abaco*, Florence, Biblioteca Medicea-Laurenziana, Codice Ashburnhamiano 280 (359*), carta 107 verso. This manuscript is autograph, and it is probable that the illustrations to it were actually drawn by PIERO himself.

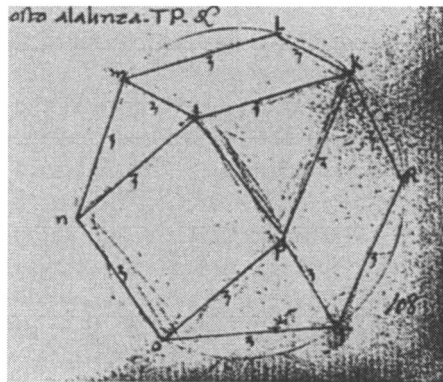


Figure 5

Diagram of cuboctahedron, diameter about 58 mm. From PIERO DELLA FRANCESCA, *Trattato d'abaco*, Florence, Biblioteca Medicea-Laurenziana, Codice Ashburnhamiano 280 (359*), carta 108 recto. This manuscript is autograph, and it is probable that the illustrations to it were actually drawn by PIERO himself.

of the conventions used in this diagram is given in our Appendix). For this solid, PIERO has given us a hint, at least, as to how we may construct it for ourselves, namely by cutting off the corners of a cube, symmetrically, through the mid-points of the sides. His diagram, however, omits the initial cube.

Possibly the reader is expected to draw that for himself. PIERO's instructions might, in any case, refer either to making the shape in three dimensions or to making a two-dimensional drawing of it.

Perhaps, to a mathematician with PIERO's exceptional talent for drawing, the two possibilities we have mentioned do not seem so different as they do to the rest of us. We may notice, however, that in constructing the truncated tetrahedron he has slid over another problem, namely that of deciding where one must cut each triangular face so that removing its three corners turns it into a regular hexagon. The solution is very simple: one cuts at the points of trisection of the sides; see Figure 6. There is no such problem in constructing the cuboctahedron, since the cut is made to the mid-points of the edges, thereby making a square face into a smaller face of the same shape. Either when writing the *Trattato* or at least at some later time, PIERO must have been aware that the two methods of cutting were significantly different, since *Elements* 15, in a proposition to which PIERO refers in the *Libellus* (almost certainly written after the *Trattato* — see below), describes the result of carrying out the second form of truncation on a tetrahedron. Figure 7 shows the form the accompanying diagram takes in the printed version of PIERO's quotation of it that appeared in PACIOLI's *De divina proportione* in 1509. The proposition, in *Elements* 15, and in PIERO's *Libellus*, is to construct an octahedron inside a tetrahedron, and the task is accomplished by joining the mid-points of the edges (thereby turning each triangular face into a smaller face of the same shape).

In the *Libellus*, PIERO works through the regular polyhedra in order of their number of faces, and in the *Trattato* also the tetrahedron is presented before the cube. Thus, at the formalisation stage if not in the first heuristic stage, PIERO might reasonably be supposed to have noticed the procedure of cutting to the mid points of the sides on the tetrahedron and then asked himself what would happen if the same procedure were applied to a cube. This provides a possible method of discovery of the cuboctahedron.

The above is a somewhat elaborate intellectual reconstruction, but it may none the less be a necessary one: imagining what happens when one cuts the corners off even a familiar shape is a much less easy task than it may sound.¹⁵ It would seem that PIERO's unusual degree of skill, even for a painter, in handling pictorial compositions in space as well as in the picture plane, shows up as a capacity for three-dimensional visualisation that stands him in good stead as a mathematician.

¹⁵ I find that most people give up very rapidly — or equally rapidly come up with the wrong answer — when asked to imagine that they are holding a cube with finger and thumb on opposite vertices, so that the corresponding diagonal is vertical, and then imagine that they cut horizontally straight through the middle of the solid, and are then asked what shape will be formed in the plane of section. Mathematicians seem to do better at this than art historians, perhaps because knowledge is helping out the visual imagination.

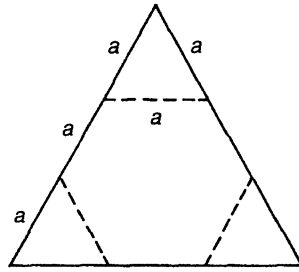


Figure 6

Cutting the corners off an equilateral triangle to make it into a regular hexagon.
Diagram by J. V. FIELD.

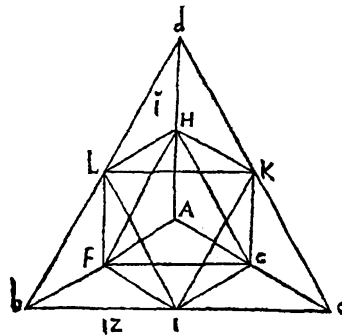


Figure 7

Drawing an octahedron in a tetrahedron. From [PIERO DELLA FRANCESCA, *Libellus de quinque corporibus regularibus*] in LUCA PACIOLI, *De divina proportione*, Venice, 1509. This diagram also occurs in pseudo-Euclid *Elements* 15, proposition 2. Photo courtesy of the Trustees of the British Library.

PIERO'S *Libellus de quinque corporibus regularibus* describes the five solids that are obtained by cutting corners off the five regular solids in the manner employed for the tetrahedron in the *Trattato*, that is turning each face from a regular n -gon into a regular $2n$ -gon. As we remarked, for the truncation of the tetrahedron, actually performing these truncations, in a drawing or on a solid shape, would involve calculating where the cut should be made. However, in the *Libellus*, as in the *Trattato*, PIERO introduces the new solids in problems that relate the edge of the solid to the diameter of its circumsphere. Thus the initial procedure concerning each face is to find the side of the $2n$ -gon in terms of the diameter of its circumcircle, though where the faces of the original solid are triangles (that is, for the truncations of the icosahedron, octahedron and tetrahedron)¹⁶ PIERO does say at once that the hexagonal faces are formed by

¹⁶ PIERO DELLA FRANCESCA, *Libellus de quinque corporibus regularibus*, Tractatus 4, Casus 2, 4, 6, PIERO ed. Mancini (note 8), pp. 559–60, 562. 564–5.

dividing edges into three equal parts. For the truncated dodecahedron, however, which PIERO treats second, after the truncated icosahedron,¹⁷ we begin with a long calculation of the side of the decagon, and for the truncation of the cube there is a similar calculation for the octagon.¹⁸ Some of PIERO's constructions are thus more a matter of principle than of practicality. The *Libellus* is, in fact, addressed to a patron, GUIDOBALDO DA MONTEFELTRO, rather than to an apprentice. The dedicatee of the work was the son of FEDERIGO DA MONTEFELTRO and his wife BATTISTA SFORZA, whose portraits by PIERO, showing them in profile against a panoramic landscape background, are now in the Uffizi Gallery, Florence. The dedicatory letter tells us that the work has been written as a companion to PIERO's perspective treatise, and the two volumes were indeed for some time shelved side by side in the Ducal Library at Urbino.¹⁹

GUIDOBALDO's date of accession as Duke of Urbino on his father's death, in 1482, gives a *terminus post quem* for the dedication of the single surviving manuscript of PIERO's *Libellus*, which is in Latin (Vatican Codex Urbinas 632). Since PIERO's mathematical works, like most of his painted ones, are almost impossible to date, this is useful information. However, it cannot be taken as decisive evidence for the date of composition of the work. PACIOLI tells us that PIERO wrote his work on perspective in Italian, and a Latin translation was made by his friend the great (that is, it would seem, locally famous) scholar MATTEO DAL BORGO.²⁰ The implication is that PIERO could not, or at least did not, himself write in Latin.²¹ Thus the Latin copy of the *Libellus* is presumably a translation from a vernacular original, and we cannot be sure that it was not merely this new version that was made for GUIDOBALDO. The date at which the

¹⁷ The solids appear in PIERO DELLA FRANCESCA, *Libellus de quinque corporibus regularibus*, Tractatus 4, Casus 2 and 3, PIERO ed. Mancini (note 8), pp. 559–60, 560–1.

¹⁸ PIERO DELLA FRANCESCA, *Libellus de quinque corporibus regularibus*, Tractatus 4, Casus 5, PIERO ed. Mancini (note 8), pp. 563–4.

The procedure for turning a square into an octagon seems to have been a standard exercise in 'practical geometry', perhaps because it was used in establishing ground plans. The procedure was to draw the diagonals of the square, take a compass opening equal to half a diagonal, and taking the corners of the square as centres, strike this length off along each pair of sides. The resulting points are the corners of the required regular octagon. This method is very neat — and is in fact also mathematically correct.

¹⁹ Since these books were in the library at Urbino, they were presumably accessible in the following century to the great editor and translator of Greek mathematical texts, FEDERICO COMMANDINO (1509–1575), but his translation of PAPPUS' *Collection*, published posthumously, in 1588, has no introduction and there is no evidence that COMMANDINO was in fact acquainted with PIERO's work.

²⁰ LUCA PACIOLI, *Summa de arithmetica, geometria, proportioni e proportionalità*, Venice, 1494, Dist. VI, tratt. I, art. II. p. 68 verso, 1.28ff. On the identification of MATTEO see JAMES R. BANKER, 'Piero della Francesca's friend and translator: Maestro Matteo di Ser Paolo d'Anghiari', *Rivista d'Arte*, 4th series, vol. 8, 1992, 331–340.

²¹ On PIERO's intervention in the translation of his perspective treatise see J. V. FIELD, 'Piero della Francesca and the "distance point method" of perspective construction', *Nuncius*, 10.2, 1995, 509–530.

manuscript was presented to GUIDOBALDO accordingly gives us no more than a *terminus ante quem* for the work it contains. Thus, what we actually know is that PIERO had discovered the polyhedra before a date after 1482.

By following his chosen method of truncation in the *Libellus*, PIERO arrives at the first five solids shown in Figure 1, the solids now known by the KEPLERian names truncated tetrahedron, truncated cube, truncated octahedron, truncated icosahedron, truncated dodecahedron.

The cuboctahedron, described in the *Trattato d'abaco*, does not appear in the *Libellus de quinque corporibus regularibus*. There is no good evidence for an absolute date for either work. However, some of the algebraic and geometrical problems from the *Trattato d'abaco* reappear in a more developed or neater form in the *Libellus*, which strongly suggests that the latter was written after the former. Accordingly, the omission of the cuboctahedron from the *Libellus* is presumably deliberate. PIERO may perhaps be seen as setting out the results of one particular geometrical method, rather than displaying as many new solids as possible. He makes no comment on the completeness of the set of solids he has presented.

Luca Pacioli and Leonardo da Vinci

As we have seen, PIERO DELLA FRANCESCA's work on polyhedra is written in a style that provides the reader with very little information about how the author may have come to his results. Rather the same style is found in PACIOLI's *De divina proportione*, but with two additional complications. The first is that the diagrams accompanying the part of the work that deals with what we may call PACIOLI's Archimedean polyhedra, diagrams known to have been drawn by LEONARDO DA VINCI,²² contain much information not given in the text, and the reader is explicitly referred to them for further details. The second complication is that PACIOLI, while acknowledging no such debt in his preface, has clearly had access to a copy of PIERO DELLA FRANCESCA's book on the regular solids: an Italian version of it appears as the final part of the printed text of *De divina proportione* — and with a dedication that implies the piece is in PACIOLI's gift. The illustrations to this part of the book are placed in the margins (LEONARDO's illustrations, which are for the earlier part, are given as separate plates) and they show a number of different representational styles, all of which are to be found in PIERO's *Trattato* and the unique Latin copy of his *Libellus*. However, the actual illustrations that appear in print are not exactly the same as those in the Latin *Libellus*. The illustrations in earlier parts of *De divina proportione*, other than those supplied by LEONARDO, are clumsy and in a uniformly old-fashioned (non-perspectival) style,²³ so it seems possible that

²² PACIOLI says in his preface that they were done 'by the divine left hand of my friend Lionardo of Florence'.

²³ An example is reproduced in the Appendix, Figure A1d.

PACIOLI gave his printer a copy of the original vernacular text of PIERO's work (now lost), with PIERO's own illustrations (or copies based on them). Neither the Italian of 1509 nor the Latin of the manuscript is so obviously idiomatic or clumsy as to be readily identifiable as an original or a translation. In both texts there seem to be problems with technical vocabulary. If in search of elegance, concision and clarity one is well advised to turn to PIERO's painting rather than to his mathematical texts.

On the other hand, PIERO's *Libellus* reads as a very tough-minded piece of mathematics if one comes to it after PACIOLI's work on polyhedra. PACIOLI very successfully avoids almost all mathematical discussion by simply referring the reader to the relevant propositions in EUCLID (for the regular polyhedra) and to LEONARDO's diagrams for the new ones. For instance, his account of the cuboctahedron is in a chapter devoted to the cube, which includes the new solid as a 'cut' version of it — 'exacedro scapezzo o abscisus planus'. The reference to the solid being 'plane' is because we shall next be given an 'elevated' (*elevatus*) version, that is one with an equilateral pyramid attached to each face. All solids are shown in both solid and skeletal form. The published woodcut version of LEONARDO's diagram for the skeletal version of the cuboctahedron is shown in Figure 8.

PACIOLI's description of the solid is

The cut hexahedron . . . has 24 lines which make around it 48 plane angles, of which 24 are right angles and the others acute angles. And it has 12 solid angles and is contained by 14 planes or bases, that is by 6 squares and 8 triangles. And all the said lines belong equally to the squares and the triangles because those 6 squares joined together at their angles necessarily cause 8 triangles as did the hexagons in the cut tetrahedron. And [this solid] comes from the cube cut uniformly in the mid point of each of its sides, as is shown to the eye in its proper material shape [in the figures].²⁴

The inventory of edges and faces is presumably intended to provide help in reading the diagram, whose naturalistic perspective style is very unusual for the time.²⁵ The reference to the solid being made up by joining its faces together is probably a reminiscence of PLATO's method of constructing the regular solids from their faces in *Timæus*. The first section of PACIOLI's treatise, the part dealing with 'divine proportion' proper, contains many references to *Timæus*, in particular to explain that the proportion is called divine on account of its connection with the dodecahedron (EUCLID uses extreme and mean ratio to construct the solid), which is the polyhedron associated with the heavens in *Timæus*. Since we have had occasion to notice KEPLER's silence about *De divina*

²⁴ LUCA PACIOLI, *De divina proportione*, Venice, 1509, part 1, Chapter 49, p. 14 verso.

²⁵ It is easy for a twentieth-century reader to forget that what we should describe as naturalistic perspective was a new-fangled form of representation in mathematics in the Renaissance. See Appendix.

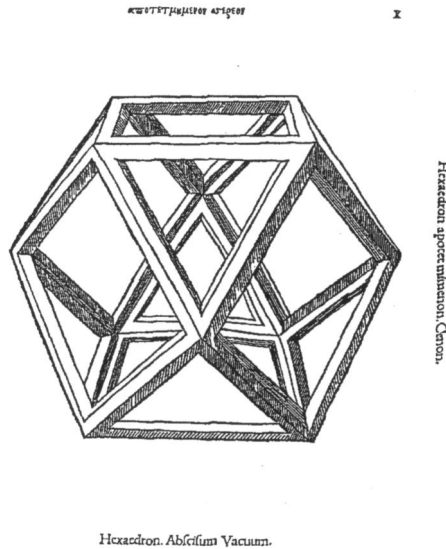


Figure 8

The solid now called a cuboctahedron. The woodcut is based on a drawing by LEONARDO DA VINCI. From LUCA PACIOLI, *De divina proportione*, Venice, 1509, plate 10. Photo courtesy of the Trustees of the British Library.

proportione, it may be as well to point out that there is no need to take KEPLER's similar method of constructing polyhedra from their faces, in *Harmonice mundi*, Book 2, as an indication that he had read PACIOLI. KEPLER was a dedicated reader of *Timæus*.²⁶

The end part of PACIOLI's description of the cuboctahedron is very close to what PIERO says in the *Trattato*.²⁷ However, it will be noted that in making a comparison with the truncated tetrahedron — a comparison confirmed by the analogous Latin names given to the two solids in LEONARDO's plates: *tetraedron abscisum* and *hexaedron abscisum* — PACIOLI has conflated the two kinds of truncation that PIERO appears to have distinguished. What is now called a truncated cube, that is a body whose faces are six octagons and eight triangles (the solid labelled 1 in Figure 1), does not appear in PACIOLI's work, except in the printing of PIERO's at the end of it.

Truncating a tetrahedron to the mid points of its edges would give an octahedron (see Figure 7 above), but that does not really explain why PACIOLI is

²⁶ See J. V. FIELD, *Kepler's Geometrical Cosmology*, London and Chicago, 1988; and J. V. FIELD, 'Le platonisme de Johannes Kepler', *Enrahonar*, núm. 23, 1995, 7–33.

²⁷ For PACIOLI's extensive use of the arithmetical and algebraic problems of the *Trattato*, see S. A. JAYWARDENE, 'The *Trattato d' abaco* of Piero della Francesca', in *Cultural Aspects of the Italian Renaissance: Essays in Honour of Paul Oskar Kristeller*, Manchester, 1976, 229–243.

not interested in the case in which a cube is truncated in the same way as the tetrahedron, that is in a way that doubles the number of sides of the face. Given LEONARDO's reputation, it may seem tempting to hypothesise that the solid in question was omitted because he simply never got round to drawing the requisite diagram. However, for reasons which are not explained, the difference that PACIOLI makes in truncating the tetrahedron and the cube reappears in his truncations of the icosahedron and the dodecahedron. The former is truncated so as to make its twenty triangular faces become hexagons, giving the solid with twenty hexagonal faces and twelve pentagonal ones that is also in PIERO's *Libellus*, and is now called a truncated icosahedron (see Figure 1, number 4).²⁸ However, for the dodecahedron PACIOLI truncates by cutting to the mid-points of the edges, which gives him the solid now known as the icosidodecahedron. LEONARDO's illustration of the solid is shown in Figure 9, and one of his sketches of it, showing the process of truncation, appears in Figure 14. This solid is not described in PIERO's *Libellus*, where the solid obtained by truncating the dodecahedron has twelve decagonal faces and twenty triangular ones. That is, it is the solid now known as a truncated dodecahedron (see Figure 1, number 3), a body which does not appear in PACIOLI's work, except in the printing of PIERO's treatise at the end of *De divina proportione*.

PACIOLI's reason for not employing truncation to the mid-points of sides for the icosahedron might have been that he had noticed that doing so would give him the same solid he had already obtained by his truncation of the dodecahedron. Similarly, truncation to the mid points of edges of the octahedron (which PACIOLI also does not show) would have given him the cuboctahedron. In the order of presentation in *De divina proportione*, we obtain the cuboctahedron from the cube, which is treated before the octahedron, but the icosahedron is treated before the dodecahedron. If PACIOLI's reasoning was that truncation to mid-points of edges was not going to get him anything new, his order of working must have been different from the order of presentation in the treatise — which is, of course, a perfectly plausible suggestion.

There the matter might rest, with PACIOLI having omitted two solids described by PIERO but having added one not in PIERO — although it can be obtained by a method of truncation PIERO had employed in his *Trattato d'abaco* (which PACIOLI certainly knew). There are, however, a number of facts that prevent the historian from coming to rest in this moderately comfortable position of being able to award prizes all round. The most glaring is that in *De divina proportione* PACIOLI also describes another Archimedean solid, not in fact obtainable by PIERO's methods, though PACIOLI appears to think otherwise. Moreover, PACIOLI, or possibly one of his patrons, ex-pupils or admirers acting on his behalf, lays claim to this particular solid by showing it as his attribute in

²⁸ This solid has recently achieved fame as giving the shape of the molecule C_{60} . See H. W. KROTO, J. B. HEATH, S. C. O'BRIEN, R. F. CURL & R. E. SMALLEY, ' C_{60} : Buckminsterfullerene', *Nature*, 318, 1985, 162–163 (issue number 6042, 14 November 1985).

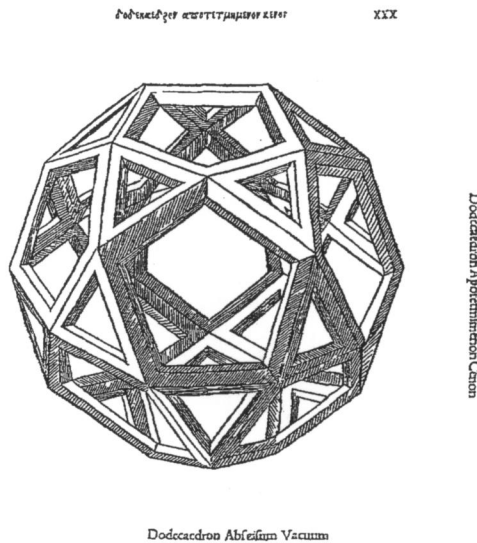


Figure 9

The solid now called an icosidodecahedron. The woodcut is based on a drawing by LEONARDO DA VINCI. From LUCA PACIOLI, *De divina proportione*, Venice, 1509, plate 30.

Photo courtesy of the Trustees of the British Library.

a portrait.²⁹ LEONARDO's skeletal version of the body in question is shown in Figure 10.

In KEPLER's illustration, shown in Figure 2, the solid is numbered 10. Its modern name, following KEPLER, is rhombicuboctahedron. KEPLER obtains it, as he obtains all the other Archimedean, by considering fitting faces together round a vertex, a method he has clearly borrowed from *Timæus*. It is possible that PACIOLI, who, as we have seen, does seem to refer to this method, also proceeded in a similar way. What he says, however, is that the solid can be obtained by truncation of his *hexaedron abscisus* (our cuboctahedron). PACIOLI begins by describing the new solid as having 26 faces, then proceeds, as for other solids, to list the numbers of edges and angles and so on. Then he says

And the origin of this [solid] is from the hexahedron uniformly cut (*secto*), sliced (*tagliato*) similarly all round as is shown to the eye in material form [in the Figure].³⁰

²⁹ There is doubt about the painter responsible for this picture, which is now in Naples, but the identification of the sitter is reasonably secure, since one of the books that is shown in the picture is clearly intended to be the book for which PACIOLI was and is best known, namely his *Summa de arithmetica, geometria, proportioni e proportionalità*, Venice 1494.

³⁰ LUCA PACIOLI, *De divina proportione*, Venice, 1509, part 1, Chapter 53, p. 15 verso.

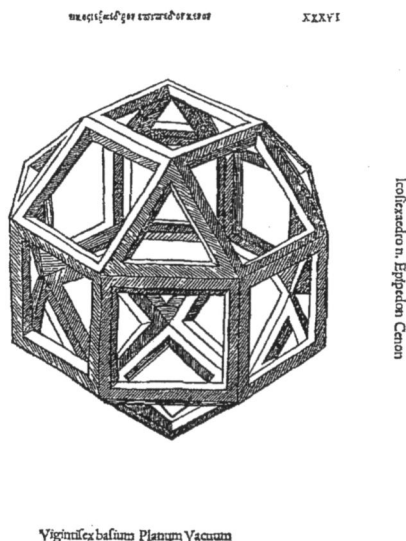


Figure 10

The solid now called a rhombicuboctahedron. The woodcut is based on a drawing by LEONARDO DA VINCI. From LUCA PACIOLI, *De divina proportione*, Venice, 1509, plate 36. Photo courtesy of the Trustees of the British Library.

He adds that the shape will be particularly useful to architects — a prophecy that seems to have remained unfulfilled.

Unfortunately, PACIOLI's mathematical derivation of the solid can, at best, be described as incomplete. As can be seen from the faces of the new solid, which are squares and triangles, PACIOLI must envisage his truncation process as taking us to the mid points of the edges of the original solid. This solid, the cuboctahedron (PACIOLI's *hexaedron abscisum*) shown in Figure 2, number 8, in Figure 5 and in Figure 8, has vertices surrounded by triangles and squares (two of each arranged alternately). Thus the new face formed by cutting off a corner of the solid will have four sides, and its edges, since we are cutting to the mid points of the edges of the cuboctahedron, will be the lines joining mid points of the sides of the squares and triangles. Diagrams of the faces concerned are shown in Figure 11. As can be seen in the Figure, if the side of the cuboctahedron is $2a$, joining the mid-points of the sides of the triangular face will give us a triangle, of side a . Joining the mid-points of the sides of the square face will give us a square of side $\sqrt{2}a$. The new faces produced by the truncation will be rectangles, with sides in the ratio $1:\sqrt{2}$, while the other faces will be equilateral triangles of side a and squares of side $\sqrt{2}a$. Many years after PACIOLI's time, this solid was illustrated in WENTZEL JAMNITZER (1508–1585), *Perspectiva corporum regularium* (Nuremberg, 1568), which seems to be concerned with a mathematical form of alchemy that is indebted to *Timæus*

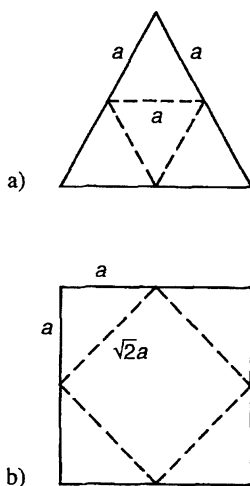


Figure 11

Truncating the faces of a cuboctahedron of side $2a$. a) equilateral triangle of side $2a$ becomes equilateral triangle of side a ; b) square of side $2a$ becomes square of side $\sqrt{2}a$.

Diagram by J. V. FIELD.

(see Figure 12).³¹ The solid in question is the bottom one on the left. A cardboard model of the same solid is shown on the left in Figure 13a. In contrast, the faces of the solid drawn by LEONARDO are all squares or equilateral triangles. However the solid shown by LEONARDO can be obtained from the true truncated form of the cuboctahedron by a process mathematicians now call distortion, though PACIOLI or any other Renaissance mathematician would surely have rebelled at the use of such a term for a procedure that, in this case, is designed to make rectangular faces become square. Such a process would

³¹ JAMNITZER does not seem to be concerned with mathematics as such. Most of his work consists of plates like the one shown in our Figure 12. These plates are in five sets, each with a decorative frontispiece illustrating the element that PLATO describes as corresponding to the polyhedron concerned (which is usually shown at the top left of the plate). In the absence of linking text, one can only guess at the thought underlying the groupings of solids. Some plates appear to show successive truncations, but JAMNITZER is apparently concerned with the symmetry of the truncation procedure rather than with whether it produces regular faces. In fact, he repeatedly shows solids whose faces are not regular, and the Archimedean polyhedra he shows are only those that can be derived by simple truncation, without subsequent distortion. Most of these appear in DÜRER (in his *Underweysung* of 1525, see below), but some, such as the icosidodecahedron (in JAMNITZER's plate F VI), are found only in PACIOLI's *De divina proportione*. However, given his interest in truncation, it seems possible that JAMNITZER may have found these solids independently.

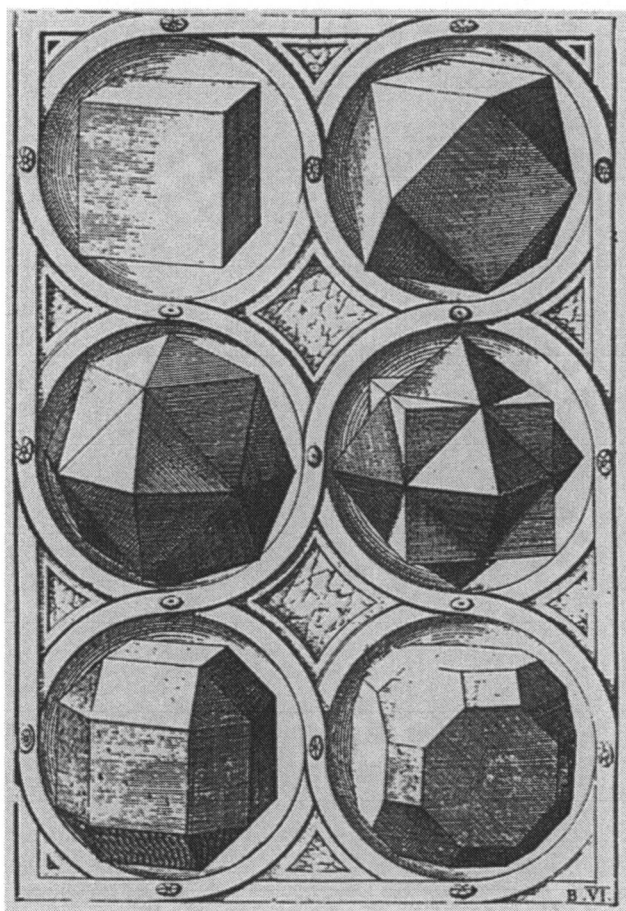


Figure 12

Polyhedra, including, bottom left, the one obtained by truncation of a cuboctahedron to the centre of its edges. From WENTZEL JAMNITZER, *Perspectiva corporum regularium*, Nuremberg, 1568. plate B VI. Preliminary drawings were made by Jamnitzer himself, using a model and a sighting device, and the engraving for the plates was done by his chief engraver, JOST AMMAN (1539–1591).

surely have been seen as ‘perfecting’ the body. What one has to imagine is a three-way compression, along the mutually perpendicular axes that lie parallel to the long sides of the original rectangles. The truncated solid and the result obtained when it is subjected to one such compression is shown in Figure 13a. Figure 13b shows the effect of a second compression.

PACIOLI does not provide an illustration even of the first of the solids shown in Figure 13, so the demands he is making on his reader’s visual imaginations, or their willingness to take his word for the mathematical truth of what he says,

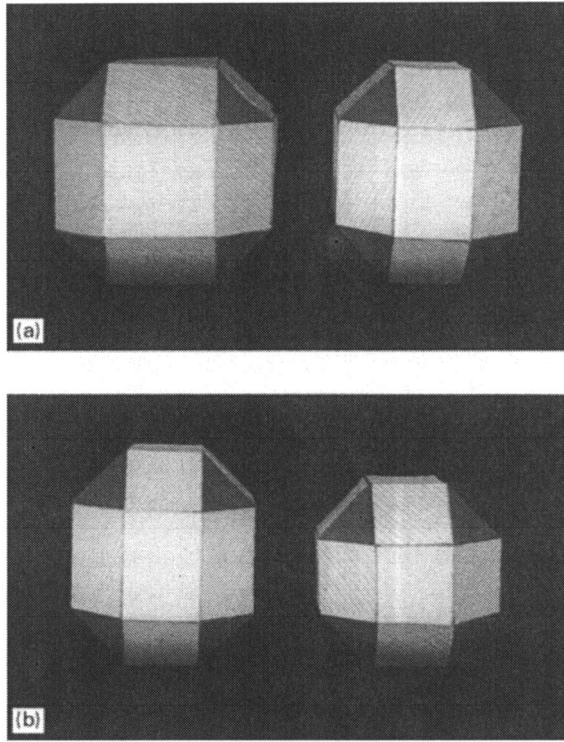


Figure 13

a) PACIOLI's *hexahedron abscisum* truncated to the mid points of its sides (left), and the result of compressing this new solid horizontally so that one set of rectangular faces become squares (right). b) Solid from the right hand side of Fig. 13a (left), and the result of compressing it vertically so that another set of rectangular faces become squares (right).

Models made by J. V. FIELD. Photographs by N. JACHEC, Birkbeck College.

are somewhat large. Moreover, it is not clear whether he is knowingly marking either of these demands. That is, it is not clear if he has himself perceived the difficulties with which his text confronts a mathematically alert reader (or one whose hindsight has been sharpened by reading KEPLER). As PACIOLI in fact side-steps all but the simplest mathematical explanation throughout *De divina proportione*, the overall style of the work would excuse the absence of mathematical explanation of the origin of the new solid. Doubt must, however, remain as to how PACIOLI himself had come to recognise its existence.

Various rational reconstructions can be made, but all involve the exercise of visual imagination to a degree which nothing else in PACIOLI's work leads us to suppose he possessed. His illustrator for this treatise was, however, certainly endowed with a visual imagination that amounted to at least the equivalent of one of today's most sophisticated computer-aided design programs. LEONARDO's surviving drawings again and again show him drawing something and then

manipulating it in various ways.³² Among the objects subjected to such modifications are numerous drawings of polyhedra.³³ Like most of LEONARDO's drawings, these pictures of polyhedra are usually tiny, which makes them difficult to interpret precisely. However, it is very tempting to see some of them as offering possible records of visual explorations that could have led to a 'perfected' version of the solid with some rectangular faces that PACIOLI would have obtained by his truncation of the *hexaedron abscisum*. See, for instance, the drawings reproduced in our Figure 14.

Perhaps LEONARDO himself first thought of the solid and the idea of its coming from PACIOLI's truncated version of the cube (or rather one of PIERO's two versions) was a later realisation. Discovering the new solid, one not produced merely by cutting corners off another, seems to require a sense of symmetry in three dimensions that is more in accord with what we know of LEONARDO than with what we know of PACIOLI.

In any case, the disjunction between text and illustrations in *De divina proportione* is very marked. This is one of the first printed books, if not the first printed book, in which the illustrator seems to have an edge over the writer of the text. PACIOLI has in fact told us, or rather reminded the dedicatee of the work (LUDOVICO GONZAGA for the printed edition), that he (PACIOLI) had made models of all the solids he discusses. This passage in the manuscript versions of *De divina proportione* has been left to stand in the printed edition, in which LEONARDO's drawings are to take the place of the models as visual aids.³⁴ LEONARDO's drawings have every appearance of being made after actual physical models: first because there are changes in the drawings, resulting from changes in the viewpoint, visible in some of the manuscript versions. Such changes could not have been made straight into the finished drawing, as they seem to have been, if a perspective construction was used, for a new viewpoint would have necessitated a complete new construction (no slight undertaking).³⁵ A second reason for supposing LEONARDO worked from actual physical models, probably using some kind of sighting device, is provided by the actual viewpoints selected in the 'final' versions, the ones that were printed. As can be seen in Figure 8,

³² See M. J. KEMP, *Leonardo da Vinci: The Marvellous Works of Nature and Man*, London, 1981.

³³ On the basis of their subject these are usually dated to the years when LEONARDO was close to PACIOLI, but other evidence for their date is lacking. See KIM VELTMAN, *Studies on Leonardo da Vinci I: Linear Perspective and the Visual Dimensions of Science*, Munich, 1986, p. 170 ff, for illustrations of these drawings. The accompanying interpretations misuse some technical mathematical terms. Moreover, they seem to me to be largely anachronistic and in places very tendentious, though (as the present essay will make clear) I am inclined to agree with the main thrust of Veltman's argument, which is that LEONARDO's mathematical abilities may have been underrated.

³⁴ LUCA PACIOLI, *De divina proportione*, Venice, 1509, Chapter 2, p. 2 *recto*, 1.2 ff.

³⁵ Such changes of viewpoint are visible in the manuscript preserved in Geneva. I am grateful to Professor MARTIN KEMP for passing on to me his observations on this manuscript.

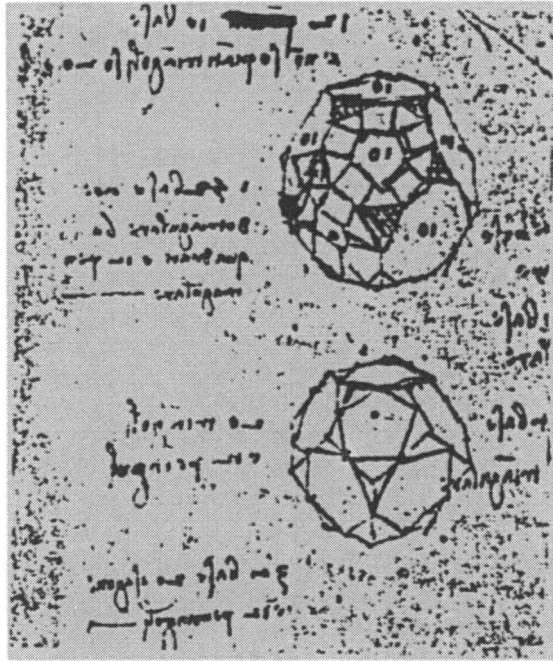


Figure 14

LEONARDO DA VINCI, drawings of polyhedra, detail from Codex Atlanticus, folio 735 verso (formerly 272 verso b). The lower solid is an icosidodecahedron, showing it as produced by truncation to the mid points of the edges of a regular dodecahedron. The icosidodecahedron was first illustrated in PACIOLI (1509), using a drawing by LEONARDO. The upper solid is not one of the Archimedean polyhedra, but it resembles the rhombicosidodecahedron, which was first illustrated in KEPLER (1619), see number 11 in Fig. 2 above. KEPLER presents this solid as analogous to the rhombicuboctahedron (first shown by PACIOLI), and LEONARDO may have conceived it in the same way, though the solid shown in his drawing is not possible in mathematical terms (unless some faces are not regular). The drawing is vivid testimony to LEONARDO's ability to imagine forms in space.

LEONARDO's viewpoint has been chosen so that the forward-pointing vertex of the solid is shown exactly against the edge behind it. This alignment, which makes the drawing easier to read, could have been arrived at by calculation, but it seems much more likely to be the result of an artist having a sharp eye for noticing a good viewpoint. Drawing the solids must have given LEONARDO ample time for looking at them and appreciating their symmetries, so we can even suggest a plausible context for his coming up with some geometrical ideas of his own.

As we have seen, PACIOLI did not take enough care in his description of his new polyhedron to make it certain that he would receive full credit for its discovery. At the least, he omitted to mention that truncation alone was not enough to derive the solid from his truncated version of the cube. In fact,

a little later in *De divina proportione*, in Chapter 55, he says that an infinite number of new bodies can be produced by cutting parts away from those already described. It is implied that the new bodies will also all have regular faces, like those described in the preceding chapters. PACIOLI says

It does not seem to me, most noble Duke, that I should extend my discussion of these bodies, aware as I am that their progression goes on indefinitely (*lor processo tenda in infinito*) by the continued and successive cutting off, one after the other, of their solid angles, and according to this [process of cutting] their differing shapes come to multiply.³⁶

The implication of this passage would seem to be that PACIOLI has not taken account of the distortion (or in Renaissance terms 'perfecting') that needs to be carried out to produce a new body with regular faces when the solid angle of the corner cut off from the previous body is contained by faces of more than one kind. Moreover, the passage strongly suggests that PACIOLI did not know PAPPUS' report of ARCHIMEDES' having discovered only thirteen solids with regular faces.³⁷ If he had known it, a reference to it would surely have seemed apposite here, particularly since, in his own list of properties of extreme and mean ratio he actually chose to stop at thirteen, claiming that the Duke held this number in particular esteem.³⁸

PACIOLI's account of the first five of the six Archimedean solids in *De divina proportione* derives each from one of the five regular polyhedra found in EUCLID's *Elements*. He uses both the methods of truncation that he found in PIERO DELLA FRANCESCA's *Trattato d'abaco*, choosing to cut triangular faces so that they become hexagons — thereby obtaining the truncated tetrahedron and the truncated octahedron already described by PIERO — and choosing to cut the square faces of the cube and the pentagonal faces of the dodecahedron to the mid-points of their sides, a method which produces the cuboctahedron (already described by PIERO) and the icosidodecahedron (which is not in PIERO). There is no obvious mathematical reason for choosing two different methods of truncation and then failing to apply both to all the regular solids — though as we have already noted, some of the results of doing so would not give one anything new, *e.g.* truncating a tetrahedron to the mid-points of its sides will give an octahedron (see Figure 7). The clue to PACIOLI's choice seems more

³⁶ LUCA PACIOLI, *De divina proportione*, Venice, 1509, p. 16 verso, Chapter 55, first lines.

³⁷ See note 1 (p. 242).

³⁸ This reversal of the usual aversion for the number is mentioned in PACIOLI's Chapter 23. The number 13 is said to be esteemed by the Duke as the number of Christ plus his disciples, namely the number of people at the Last Supper, an assemblage that the Duke so honours that he has caused LEONARDO to paint it in Santa Maria delle Grazie (*De divina proportione*, Venice, 1509, p. 8 recto). As scenes showing the Last Supper were very common in refectories, this sign of the Duke's favour presumably relates to his choice of LEONARDO rather than a less famous or skilful artist.

likely to lie in the purely practical matter of his having made models to illustrate his text. If he wanted to make a matching set of solids, so that, for instance, the model of the truncated tetrahedron was exactly the size one would obtain by cutting the corners off the model tetrahedron, then it would be very convenient that the new hexagonal face was so easy to derive from the previous triangular one (see Figure 11a). Similarly, the new square faces of the *hexaedron abscisum* would be easy to derive from those of the original cube (see Figure 11b). The octagonal faces required for PIERO's truncated cube (Figure 1, number 1) would be rather less easy to construct. And the decagonal faces for PIERO's truncated dodecahedron (Figure 1, number 3) would have been more awkward still. Since PACIOLI apparently believed that he was in any case only showing a selection from an infinite series of solids with regular faces, he may well have chosen to display the ones that could be described most adequately, namely those for which it was possible to make convincing models.

PACIOLI did not at first contemplate publishing *De divina proportione*. It was intended to be presented to his patron as a manuscript. There would accordingly have been no impediment to its illustrations taking the form they did, that of a set of models with accompanying labels. A glance at the normal style of illustration in mathematical texts explains PACIOLI's choice. The usual methods of drawing solids in such texts follow conventions that might make it difficult to provide an adequate illustration of an unfamiliar shape.³⁹ Some standard forms are shown in the Figures in our Appendix. These forms are entirely appropriate for texts in which, as in EUCLID's *Elements*, the author's words are the dominant component in the exposition. In *De divina proportione* PACIOLI is moving into another, characteristically Renaissance, mode of expression in which illustrations can carry information not found elsewhere. PIERO DELLA FRANCESCA had used this mode too, but it had apparently not inhibited his mathematical explorations. That it does seem to have inhibited PACIOLI's accordingly suggests that all the solids produced simply by truncation were discovered (or found in PIERO's work)⁴⁰ before PACIOLI met LEONARDO, in Milan in 1496.⁴¹ The 26-faced solid derived from the *hexaedron abscisum*, which (as we have seen) requires more than simple truncation, is given a separate chapter, at the end of the series

³⁹ In fact, even as late as the beginning of the nineteenth century makers of mathematical equipment, such as the firm of GEORGE ADAMS, supplied sets of models that were solid forms of the illustrations required for the last three books of EUCLID's *Elements*. One such set, made of boxwood, is preserved in the Museum of the History of Science in Oxford.

⁴⁰ The icosidodecahedron, shown in Figure 2, number 9 and Figure 9, is not found in any of PIERO's extant works, but it is difficult to believe that he failed to discover the solid, since it is obtained by using the method of truncation he employed in obtaining the cuboctahedron and he did find both of the other solids derived by truncation of the icosahedron and the dodecahedron.

⁴¹ This may not in fact have been their first meeting, but in his introduction to *De divina proportione* PACIOLI implies that the two did not get to know one another until they met in Milan.

of solids with regular faces. In principle, the solid should have been put with the others derived from the cube, so this isolation may perhaps reflect a later discovery, in which the heuristic trigger was not a physical model but a solid seen in LEONARDO's mind's eye.

Albrecht Dürer

DÜRER travelled to Italy to learn about perspective, and it is probable that while he was there he met LUCA PACIOLI, either in Venice or Bologna in 1507 or 1508.⁴² In any case, almost everything in DÜRER's account of polyhedra could have been derived from the published text of *De divina proportione*. Of the seven Archimedean solids described in the *Underweysung* of 1525 four could have been taken from PACIOLI's own text, one could have come from the printed version of PIERO's *Libellus* at the end of the book, and two appear to have been discovered by DÜRER himself. The four that may have come from PACIOLI are the truncated tetrahedron (Figure 1, number 2 and Figure 4), the cuboctahedron (Figure 2, number 8, Figure 5 and Figure 8), the truncated octahedron (Figure 1, number 5) and the rhombicuboctahedron (Figure 2, number 10, and Figure 10). The two solids which appear to be new to the literature are the truncated cuboctahedron (Figure 1, number 6) and the snub cube (Figure 2, number 12). In the posthumous German edition of DÜRER's work (Nuremberg, 1538), two more Archimedean solids appear. The accounts of them have been taken from DÜRER's manuscripts. The bodies in question are the truncated icosahedron (Figure 1, number 4) and the icosidodecahedron (Figure 2, number 9). Both these solids are to be found in PACIOLI's *De divina proportione*, though the first appears only in the part written by PIERO DELLA FRANCESCA. Each of the seven solids in the *Underweysung* is shown only in the form of a 'net'. This form of illustration, which seems to have originated with DÜRER, shows the solid as if a cardboard model of it had been cut along selected edges until the piece could be made flat. DÜRER in fact refers to the diagrams as if they were capable of being made into a solid shape. For instance, he says of the truncated cuboctahedron

The seventh and following body as it lies open has six eight-cornered and eight six-cornered and twelve four-cornered planes [*felder*: literally 'fields'] and as one puts it together [*zusammen leget*] so it acquires forty-eight corners and seventy-two ridge edges.⁴³

⁴² For a recent discussion of DÜRER's mathematics and its sources, see ALBRECHT DÜRER, *The Painter's Manual* [a facsimile reprint of *Underweysung der Messung mit dem Zirkel und Richtscheyt*, Nuremberg, 1525] with translation and commentary by WALTER L. STRAUSS, New York: Abaris Books, 1977; and ALBRECHT DÜRER, *La Géométrie*, translated from the German and with an introduction by J. Peiffer, Paris: Éditions du Seuil, 1995.

⁴³ ALBRECHT DÜRER, *Underweysung der Messung mit dem Zirkel und Richtscheyt*, Nuremberg, 1525, sig. Nij verso, ed. STRAUSS (note 42), p. 340 (translation J. V. FIELD).

Unfortunately, that is exactly all that DÜRER chooses to tell us about the solid concerned. The accounts of other solids are equally laconic. As a result, one is yet again forced back upon rational reconstruction in trying to decide how DÜRER may have discovered the two solids which he could not have found in *De divina proportione*.

DÜRER introduces polyhedra, at the beginning of his fourth book, as solids that can be drawn with straightedge and compasses.⁴⁴ The first five solids to be shown are the five regular polyhedra, each of which is shown as a net, in plan and in perspective. DÜRER states that these solids are to be found in EUCLID, but does not give a precise reference. The solids are presented in the order tetrahedron, octahedron, icosahedron, cube, dodecahedron, that is, DÜRER seems to take account of the number of sides of the faces as well as the number of faces of the solids. After the dodecahedron, we have a sphere, shown divided into gores by cutting along meridians. DÜRER then mentions that each of the regular solids can be inscribed in a sphere so that each of its corners rests on the surface of the sphere, and adds that

Also many decorative (*hübscher*) bodies can be made which also fit into a hollow sphere with all their corners on it but they have different faces.⁴⁵

From what follows it becomes clear that 'different faces' (*ungleicher felder*) means that not all the faces of any solid are of a single shape. Since each solid has a circumsphere it is clear (to a twentieth-century reader) that all its vertices must be the same, that is, the solid is uniform. One must surely presume that DÜRER also recognized this, and so we have a criterion for accepting a solid. However, the only method of generating candidates appears to be the net. There is no mention of a three-dimensional process except in the second paragraph of the brief preamble, where DÜRER says that one can add a pyramid to each face, with an apex at a greater or lesser height above the face in question. This is an extended version of the process already described by PACIOLI and illustrated by LEONARDO.⁴⁶ DÜRER's written style is here (as elsewhere) far from lucid, but there appears to be no reference to a process of truncation. As the net seems to be DÜRER's preferred method of presentation for less familiar solids, it seems possible that it was also a tool in the process of discovery.

The first of the two new Archimedean solids to be presented is the snub cube, described as having six square faces and thirty-two triangular ones, making twenty-four (solid) angles and sixty ridge edges. DÜRER shows a net whose layout emphasises the symmetry of the solid concerned (see Figure 15). Now,

⁴⁴ ALBRECHT DÜRER, *Underweysung der Messung mit dem Zirkel und Richtscheit*, Nuremberg, 1525, sig. Mij verso, ed. STRAUSS (note 42), p. 316.

⁴⁵ ALBRECHT DÜRER, *Underweysung der Messung mit dem Zirkel und Richtscheit*, Nuremberg, 1525, sig. Mvi verso, ed. cit. in note 42, p. 328 (translation J. V. FIELD).

⁴⁶ On account of the appearance of bodies subjected to the addition of outward-facing pyramids, my suggested term for the procedure is 'pineapplling'. WENZEL JAMNITZER's *Perspectiva corporum regularium* (see note 31, p. 259) contains many pineappled solids.

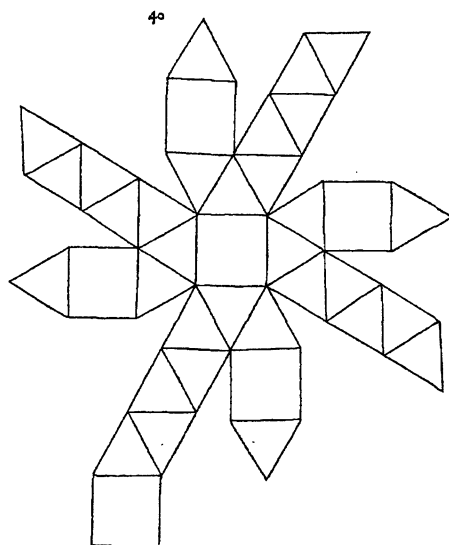


Figure 15

ALBRECHT DÜRER, *Underweysung der Messung mit dem Zirkel und Richtscheyt*, Nuremberg, 1525, folio Niiij verso, ed. STRAUSS (note 42), p. 340. Net for the solid now called the snub cube. Photo courtesy of the Trustees of the British Library.

the snub cube cannot in fact be produced by a simple truncation process like those described by PIERO DELLA FRANCESCA, even if we allow subsequent distortion as apparently practised (but not described) by PACIOLI. Thus in the case of the snub cube it seems likely that the construction of a symmetrical net may have been one step in DÜRER's discovery. Moreover, since in his preamble to the discussion of the Archimedean solids he gives instructions for making three-dimensional models from the nets, it seems probable that he tested his candidate solids by making such models.⁴⁷ Suggesting that DÜRER may have used this practical method of investigation is not meant to imply mathematical incompetence on his part, but merely the taking of care in checking results.⁴⁸ The other new solid in DÜRER's book, namely the truncated cuboctahedron, can, as its name implies, be produced by truncation, but we require the truncation to be followed by distortion. For this solid also, investigation starting with the construction of a symmetrical net seems to provide a plausible route to discovery.

⁴⁷ He recommends pasting two sheets of paper together and then using a sharp knife to cut through one sheet along the lines that are to be the edges of the solid, ed. STRAUSS (note 42), p. 328.

⁴⁸ KEPLER, who was a very good mathematician indeed, made actual models of geometrical solids he was investigating (see below).

What, however, of the six Archimedean solids that appear to have eluded DÜRER? If he could discover the snub cube, it is hard to see why he should not have also discovered the snub dodecahedron, whose net is very like that of the snub cube, but with pentagons in place of the squares. Moreover, DÜRER has omitted three solids shown in *De divina proportione*, though his omission was to be partly repaired, from his own manuscripts, in the posthumous edition of 1538. In the *Underweysung*, as in PACIOLI work, we may perhaps see the intervention of practical concerns. Like PACIOLI, DÜRER clearly believes himself to be concerned with an essentially unlimited number of solids having the properties he specifies. As we have seen, he describes such solids merely as being ‘many’.⁴⁹ There is accordingly no question of aiming for completeness, so he is free to choose examples as he pleases. What all his Archimedean solids have in common is that they do not include pentagons.⁵⁰ Pentagons can be drawn with straightedge and compasses (as is shown by EUCLID), and they are shown in DÜRER’s illustrations of the dodecahedron. However, they are rather awkward to draw accurately, and if a polyhedron includes a large number of faces small inaccuracies in individual faces can cause serious problems in assembling the solid.⁵¹ DÜRER was no doubt capable of the requisite degree of accuracy himself, but he may have decided to refrain from imposing the requirement on his readers (while the less practically minded editor of the later version preferred to include as many solids as possible). If DÜRER did take account of practicalities in this way, we are free to imagine that he in fact knew all the Archimedean solids, for judicious replacement of squares by pentagons in the nets he provides will give us all thirteen uniform polyhedra. KEPLER, who certainly knew DÜRER’s work, may have recognised this — which would help to explain why he claims no originality for his own work on the Archimedean solids. On the other hand, DANIELE BARBARO, who also certainly knew DÜRER’s work, not only omitted the snub cube from his catalogue of solids but (it seems) preferred to think in terms of truncation.

Daniele Barbaro

As a mathematician, DANIELE BARBARO was an amateur, in the good sense as well as (to some extent) in the bad one. He came from a Venetian patrician family — he was the nephew of the famous humanist ERMOLAO BARBARO (c. 1410–1471) — and served his city in the offices of Ambassador to the Court of St James’ and Patriarch Elect of Aquileia. He was the official representative of the Venetian Republic at the Council of Trent. In intellectual history, he is chiefly remembered for his excellent editions of VITRUVIUS’ *De architectura* (in Italian: Venice 1556, 1566; in Latin: Venice, 1566). His *Pratica della prospettiva*,

⁴⁹ See passage cited at note 45 see (p. 267).

⁵⁰ Or decagons derived from them.

⁵¹ As readers may guess, the present author writes from experience in the matter.

which is probably best regarded as an extension of his annotations to VITRUVIUS, mainly deals with what were seen at the time as Vitruvian mathematical themes — such as the use of perspective constructions for the design of stage scenery.⁵² However, it also takes account of modern work on related topics, including polyhedra, which are introduced as simple bodies whose structure can be explored by means of plans and perspective drawings. Two manuscripts of BARBARO's treatise are preserved in the Biblioteca Marciana in Venice.⁵³ The first is a draft (though some of it is, mercifully, in the hand of the copyist); the second is apparently the copy that went to the printer, and printed copies of the diagrams have been pasted into it.

As the title of the work implies, BARBARO's first concern is with perspective. In the earlier draft of the preface, he proposed to base his work on that of ALBRECHT DÜRER, but by the time he came to write the later draft he had decided instead to use PIERO DELLA FRANCESCA's *De prospectiva pingendi*, of which he had presumably acquired a manuscript copy (whether in its original Tuscan or in Latin translation is not certain).⁵⁴ Together with this change, we find a fairly drastic recasting of the structure of the work as a whole, and some considerable tidying up of the part that relates to polyhedra, partly by omitting some solids that BARBARO had presumably decided did not stand up to further scrutiny (in which KEPLER would have agreed with him — see below).

BARBARO's final treatment of perspective is entirely taken from PIERO, with acknowledgement. Indeed, BARBARO's work was of some historical importance in transmitting a simplified version of PIERO's chief results to later generations.⁵⁵ However, in the part of the work concerned with polyhedra, BARBARO acknowledges no intellectual debts. It none the less seems extremely likely that he did know PACIOLI's *De divina proportione*, whose second section is much concerned with Vitruvian architecture, though the subject is treated in an elementary manner that BARBARO might well have found infuriating. In any case, eight of the eleven Archimedean polyhedra that appear in *La Pratica della prospettiva* are to be found in the published version of PACIOLI's work: that is, they are either in PACIOLI's own part of the volume or in the printed version of PIERO's *Libellus* that follows it. The style of BARBARO's book is very similar to PACIOLI's. He relies heavily on visual presentation, usually providing only the briefest of

⁵² Perspective stage scenery is mentioned in the preface to *De architectura*, Book 7.

⁵³ DANIELE BARBARO, *Trattato della Prospettiva*, Classe IV, Codici XXXIX — XL, no 5446–5447.

⁵⁴ BARBARO's inclusion of the passage in which PIERO uses the 'distance point method' of construction suggests his manuscript was in the vernacular. No corresponding passage is found in the known Latin manuscripts of *De prospectiva pingendi*. See J. V. FIELD *op. cit.*, in note 21, see (p. 252).

⁵⁵ The curious comment, in BARBARO's preface, that PIERO wrote for 'idiots' is presumably occasioned by the repetitious drawing instructions which form the bulk of the text in most of PIERO's propositions.

mathematical discussion. All the same, even in his account of the solids apparently derived from PACIOLI's book, we have evidence of BARBARO's having done some mathematical thinking of his own. For example, the cuboctahedron is described as obtained by truncation, to the mid-points of sides, from both the cube and the octahedron,⁵⁶ and the icosidodecahedron is similarly derived from the icosahedron and the dodecahedron.⁵⁷ As we have seen, PACIOLI had derived these solids merely from the cube and the dodecahedron.

The visual presentation of polyhedra in *La Pratica della prospettiva* includes many ground plans, together with perspective drawings of the solids concerned (the paired drawings are examples of how mathematical construction can be used — see Appendix), but for some of the more complicated solids we are merely provided with a net. Like any other form of representation, a net can give one an idea about the spatial structure of the solid that is more or less clear, depending on the skill of the draughtsman or the designer of the drawing. Unlike DÜRER, BARBARO did not make his own engravings, and the manuscripts show that his draughtsman simply made faithful copies of the drawings he was given. Thus in Chapter 16 BARBARO describes a solid (allegedly formed by cutting away solid angles of the cuboctahedron) whose net shows that it has vertices of two different kinds, one of which is surrounded by two equilateral triangles and two hexagons. Such a vertex would be flat, but neither author nor illustrator seems to have noticed, and a perspective view of the solid duly appears in the text. It would be possible to construct such a solid if some of its triangular and hexagonal faces were not regular, but the text does not mention such a possibility, and the illustrations appear to show all the polygonal faces as regular.⁵⁸ A similar solecism occurs in Chapter 19, where we have a solid whose net shows that some of its vertices are surrounded by three hexagons, again making flat vertices. Again, the silence in the accompanying text suggests that BARBARO had not noticed this fact, though the revisions to his earlier draft had eliminated other solids beset with the same kind of problem.⁵⁹

Near the end of his work on polyhedra, there are some slight indications that BARBARO may have been using the nets as a way of arriving at his results

⁵⁶ DANIELE BARBARO, *La Pratica della prospettiva*, Venice, 1568, 1569, Part 3, Chapter 8, pp. 58–60.

⁵⁷ DANIELE BARBARO, *La Pratica della prospettiva*, Venice, 1568, 1569, Part 3, Chapter 12, pp. 71–75.

⁵⁸ DANIELE BARBARO, *La Pratica della prospettiva*, Venice, 1568, 1569, Part 3, Chapter 16, pp. 90–93. A version of this solid in which the triangular faces are isosceles rather than equilateral is shown in perspective in WENTZEL JAMNITZER, *Perspectiva corporum regularium*, Nuremberg, 1568, plate B II middle left, and plate B IIII top right. Given their dates of publication it is just possible, but not very likely, that BARBARO knew JAMNITZER's work in its printed form. It is, of course, possible that he knew the work from a manuscript source. On JAMNITZER's book, see note 31 (p. 259).

⁵⁹ DANIELE BARBARO, *La Pratica della prospettiva*, Venice, 1568, 1569, Part 3, Chapter 19, p. 98; MS no 5447, f 53 verso.

as well as a way of presenting them. For instance, at the beginning of Chapter 25, a new solid is proposed as produced from the one described in Chapter 22 by 'changing' (*mutando*) some of the polygons that form its faces. In other passages, where the same verb is used, it certainly signifies a process of change by cutting off the corners of the polygon, but in this passage in Chapter 25 the context makes it clear that we are dealing with simple substitution of one polygon for another. Both of the solids concerned are shown only as nets.⁶⁰ In any case, in most of his work BARBARO appears to have been using the net only as a way of presenting his results. He arrives at this new polyhedra by cutting off solid angles from known solids. At least, this is what he claims to have done.

There are two recurring mathematical inadequacies in his accounts. The first is that he describes the truncations as either being made to the mid-points of faces or to points which divide the sides into three parts. Sometimes these three parts are (correctly) described as equal — as in the truncation of the octahedron to form a truncated octahedron.⁶¹ Sometimes we are not told where the points of trisection are to be — though we are correctly told how to choose the points on the sides so as to generate a regular octagon from the square faces of the cube when it is truncated.⁶² The second repeated inadequacy is that, like PACIOLI in the case of the rhombicuboctahedron, BARBARO makes no mention of the fact that his truncations sometimes produce faces that are not regular. In fact, it is not clear whether he has noticed this defect: his nets merely show regular faces. There is no mention of the process now known as 'distortion'. Just as PACIOLI apparently derived the rhombicuboctahedron (Figure 2, number 10 and Figure 10) by truncation of the cuboctahedron, BARBARO says that one truncates the icosidodecahedron to the mid-points of its sides to obtain the rhombicosidodecahedron (Figure 2, number 11). He shows the new solid as a net, a plan and a perspective drawing. There is no mention of the fact that the truncation he has described should have produced rectangular faces where the solid shown in the diagram has square ones.⁶³ This passing over of what a twentieth-century mathematician would regard as the problem of distortion is, of course, facilitated by BARBARO's representation of the solid in the form of a net — in which the possibility of making the faces square could easily be regarded as 'obvious'.

⁶⁰ DANIELE BARBARO, *La Pratica della prospettiva*, Venice, 1568, 1569, Part 3, Chapter 25, p. 104, Chapter 22, p. 101. Neither of the solids is of mathematical interest in the present context since both have vertices of more than one kind.

⁶¹ DANIELE BARBARO, *La Pratica della prospettiva*, Venice, 1568, 1569, Part 3, Chapter 11, pp. 61–63.

⁶² DANIELE BARBARO, *La Pratica della prospettiva*, Venice, 1568, 1569, Part 3, Chapter 9, p. 61. BARBARO uses the method described in note 18 (p. 252).

⁶³ DANIELE BARBARO, *La Pratica della prospettiva*, Venice, 1568, 1569, Part 3, Chapter 17, pp. 94–96.

As we have already mentioned, BARBARO includes nets for solids with more than one kind of vertex, though all faces are regular polygons.⁶⁴ Thus, despite the rejection of some solids from the earlier draft of his work, it is really not very clear what criteria he is using in deciding whether a solid is admissible or not. There is no evidence to suggest that he knew of PAPPUS' statement that ARCHIMEDES had discovered thirteen polyhedra whose faces were regular polygons of more than one kind. BARBARO's treatment of polyhedra in fact suggests that, like PACIOLI, he believes he is dealing with an unlimited number of solids.

There is nothing in the title of BARBARO's *La Pratica della prospettiva* to attract KEPLER's attention to the book, and it is not likely that any author he read would make a reference to it, since it is not concerned with the kind of optics that was of professional interest to an astronomer. BARBARO is addressing himself to a class of readers whose interest in mathematics is subsidiary to their main concerns. If KEPLER had come across the book by chance, he might have found it of some interest, but would certainly have recognised BARBARO's treatment of polyhedra with regular faces as being far from coherent. It is accordingly not difficult to explain why *La Pratica della prospettiva*, like *De divina proportione* and DÜRER's *Underweysung*, receives no mention in KEPLER's discussion of polyhedra in *Harmonices mundi* Book 2. In fact, KEPLER seems to have had little esteem for most of the geometers of his own and immediately preceding generations. Writing to a friend in 1606, he complains about the lack of good geometers:

Look around the nations. The Italians are in a dream (except for one Commandino and Giovanni Battista Benedetti, for Clavius is a German). The Netherlanders (*Belgæ*) indeed are engaged in commerce ...⁶⁵

One can well imagine that for KEPLER their lack of sustained mathematical argument, and their neglect of precise definitions of the type of solids with which they are concerned, would have classed PACIOLI and BARBARO among the dreamers. With them went PIERO DELLA FRANCESCA and — perhaps with more justification — LEONARDO DA VINCI.

⁶⁴ In *Harmonices mundi* Book 2, KEPLER introduces 'axioms' which are designed to exclude such solids: JOHANNES KEPLER, *Harmonices mundi libri V*, Linz, 1619, Book 2, sections 23, 24, p. 57; *Johannes Kepler gesammelte Werke*, ed. M. CASPAR et al, Munich, 1938-, 6, p. 78. (In what follows, the title of this edition will be abbreviated to KGW.) An English translation of this work has been in press since 1990: E. J. AITON, A. M. DUNCAN, J. V. FIELD, *Johannes Kepler. Five Books of the Harmony of the World* (translation, introduction and commentary), American Philosophical Society.

⁶⁵ JOHANNES KEPLER to SAMUEL HAFENREFFER, 16 November 1606, Letter 400, 1.8 ff, KGW 15, p. 359. The reference to commerce presumably indicates KEPLER's failure to recognise the historical importance of the developments to 'commercial arithmetic' introduced by such algebraists as SIMON STEVIN (1548–1620).

STEVIN's brief consideration of polyhedra in his *Problematum geometricorum . . . libri V* (Antwerp, [1583]) adds nothing of relevance to the present discussion. Nor do FOIX DE CANDALE's developments on the pseudo-Euclidean Books 14 and 15 in his edition of the *Elements* (Paris, 1566). KEPLER certainly knew CANDALE's work.

Art and Mathematics

PIERO and LEONARDO were two of the most accomplished artists of their time. Both accordingly bulk large in histories of Renaissance art. One purpose of the present essay is to suggest that they should also have a more substantial place than that usually allotted them in histories of Renaissance mathematics: PIERO apparently invented the use of truncation as a mathematical procedure, and it seems likely that LEONARDO introduced the notion of distortion (at least in the limited sense of applying a tidying-up process that made rectangular faces into square ones). In the fifteenth century there was, of course, nothing anomalous in an artist being taught as much mathematics as any other craftsman, particularly mathematics of the 'useful' kind that was taught in abacus schools. What these particular artists brought to their own further mathematical studies was something that fitted well with their craft concerns, namely a strong interest in symmetry. As is well known, EUCLID is markedly disinclined to employing the notion of symmetry in his proofs. We cannot tell whether he used it heuristically, but it would seem that reading his work is rather unlikely to suggest such a use to others. PIERO's work on polyhedra, though it is, as we have seen, very probably influenced by his reading of the *Elements*, is essentially unlike EUCLID's both in showing a concern with symmetry and in dealing directly with three-dimensional shapes rather than reducing the problem to a series of two-dimensional problems in different planes. For LEONARDO, we have too little detailed information to assess the extent of his mathematical expertise.⁶⁶ However, his sketches of mathematical solids show the same concern with three dimensions that we find in PIERO's writings. While PIERO is almost certainly the more competent mathematician, both he and LEONARDO are important for their concern with symmetry, and for their use of diagrams that are an aid to visualisation and can embody information beyond that to be derived from the words of the text that they illustrate. No doubt this increased power given to the visual seemed natural to a painter. To mathematicians, accustomed to the style found in editions of EUCLID, it was something new. A hint of its possible heuristic significance can be found in the work of DÜRER and BARBARO, who seem to have found 'nets' of polyhedra a useful intellectual tool.⁶⁷

As we have already noted, KEPLER may not have known PACIOLI's *De divina proportione*. However, the draughtsman responsible for the fold-out plate showing KEPLER's polyhedral explanation of the structure of the Solar System in the *Mysterium cosmographicum* (Tübingen, 1596, plate dated 1597) does seem to have known LEONARDO's pictures of skeletal versions of the five regular solids.

⁶⁶ Professor MARTIN KEMP has pointed out (private communication) that LEONARDO seems to make a habit of getting arithmetic wrong.

⁶⁷ DÜRER's mathematical work has received much more attention than that of his Italian predecessors, perhaps because of the known 'scientific' connections of his native city of Nuremberg.

In any case, KEPLER seems to have been quite good at drawing, if we may judge from his sketch for the frontispiece to the *Rudolphine Tables* (Ulm, 1627) and his earlier use of the *camera obscura* to make a topographical study (apparently just to see whether it could be done).⁶⁸ In the present context, however, KEPLER's technical facility is less relevant than his obvious interest in using diagrams and illustrations in his work. The elegant engraving showing the polyhedral theory in the *Mysterium cosmographicum* may well be regarded as essentially decorative and designed to help sell the book. However, the book also contains a fair number of other diagrams that are not strictly necessary, and as KEPLER gets older (and is thus in a better position to get his own way) it is noticeable that his works contain more and more illustrations. This is no doubt partly an effect of the development of fashion in book design, but we do know from his correspondence that KEPLER went to considerable trouble to provide illustrations of polyhedra for his *Harmonice mundi*. Models were made for him by his friend WILHELM SCHICKARD (1592–1635), Professor of Mathematics at Tübingen, and KEPLER himself made models in the course of his investigations.⁶⁹ Moreover, the work for which KEPLER is now best remembered, namely his calculation of the orbit of Mars (*Astronomia nova*, Heidelberg, 1609) required considerable powers of visual imagination: he needed to be able to see the three-dimensional configuration of the orbits of the Earth and Mars, in planes inclined to one another and intersecting in a line passing through the Sun, and then imagine the situation as each planet moved with variable speed round its orbit. As he was to remark in the second paragraph of his introduction to the *Astronomia nova*, he later found it hard to make sense of some of the diagrams he had supplied to help his readers. Readers must surely have found this remark discouraging as well as disarming, but it certainly suggests that the initial visualisation had been no trivial matter.⁷⁰

Although his visual imagination must surely have played a part in KEPLER's study of the Archimedean solids, his detailed account of them, in *Harmonice mundi* Book 2, is in very precise mathematical terms, considering fitting polygons together to form the faces of each solid. Like so much else in the *Harmonice mundi* this method links the work with *Timæus*, and KEPLER's concern — like PLATO's — continues to be with the faces as much as with the solid shapes they produce. In fact the purpose of *Harmonice mundi* Book 2, if

⁶⁸ See WALTHER GERLACH, 'Johannes Kepler — Life, Man and Works', in *Kepler: Four Hundred Years (Vistas in Astronomy, 18)*, eds A. BEER, P. BEER. Oxford, 1975, 73–95, where the sketch for the frontispiece appears as figure 3.8; for the camera obscura see HENRY WOTTON to FRANCIS BACON, Autumn 1620, in *Reliquiæ Wottonianæ*, London, 1651, 413–415, reprinted as Letter 892 in KGW 18, p. 42.

⁶⁹ See J. V. FIELD, 'Kepler's star polyhedra', *Vistas in Astronomy*, 23, 1979, 109–141.

⁷⁰ As far as I know, though KEPLER enjoyed looking at landscape, he never expressed an interest in the visual arts. I am tempted to ascribe this silence to revulsion from the large and heavily erotic paintings by BARTHOLOMEUS SPRANGER (1546–1611) with which KEPLER's employer RUDOLF II adorned the walls of his palace.

one may judge it from the two sections at the end marked 'Conclusions', is to arrive at a hierarchy of polygons established according to the number of tessellations and polyhedra to which each contributes (more contributions imply a higher ranking). This method of constructing the ARCHIMEDEAN solids by fitting faces together round a vertex enables KEPLER to show that there are exactly thirteen such solids (for their defining properties see notes 1 and 2 above), and thus to show that his ranking of the polygons is based on an exhaustive catalogue of the figures in which they fit together.⁷¹

However, the names that KEPLER gives to the Archimedean solids in *Harmonices mundi* Book 2 do not reflect his method of constructing them from their faces. Instead he chooses names that refer to the way the solids may be produced by two forms of truncation. The first is by cutting into faces so that an n -gon becomes a $2n$ -gon, which gives him the solids shown in our Figure 1, with his numbers 1 to 5. He also uses the adjective 'truncated' (*truncum*) to describe the solids obtained by a similar process, followed by distortion, from the cuboctahedron and the icosidodecahedron (Figure 1, numbers 6 and 7), remarking in his text that these bodies are not actually obtained by truncation but are merely like those that would be so obtained. In naming the solids obtained by a second form of truncation, truncation to the mid-points of edges, KEPLER uses combined forms of the names of the two solids from which the new solid could be obtained. Thus we have the cuboctahedron and the icosidodecahedron (Figure 2, numbers 8 and 9). Truncated forms of these solids which require distortion to make their faces regular, acquire the prefix 'rhombi-' (see Figure 2, numbers 10 and 11). Apart from the brief comment about truncation that has been mentioned, KEPLER does not explain the names he gives the solids, but the names themselves are sufficient to imply a logical ordering of the solids that is quite different from that found in the course of their construction from their faces.⁷² Perhaps KEPLER intended to write this work up in a different form in his treatise on geometry, which remained unfinished, and survives only as a series of notes.⁷³ However, as things stand, the evidence he has left for his process of rediscovery is as tenuous as that left by his predecessors. KEPLER's strength, however, is in having rediscovered the complete set of Archimedean bodies, and having proved that he has done so. In this he shows himself the inheritor of the long tradition of rigorous geometry that goes back to the Ancients. As a mathematician, KEPLER could afford to do without the work on polyhedra published by LUCA PACIOLI, ALBRECHT DÜRER and DANIELE BARBARO. However, KEPLER's own mathematics and natural philosophy would have been different from what they in fact were if he had not been

⁷¹ See J. V. FIELD, *Kepler's Geometrical Cosmology* (note 26, p. 255).

⁷² Two solids which cannot be produced by truncation, the snub cube and the snub dodecahedron, see Figure 2 numbers 12 and 13, possibly derive their names from the fact that their solid angles have a blunted appearance comparable to that of a tip-tilted nose.

⁷³ See J. V. FIELD, 'Kepler's star polyhedra' (note 69, p. 275).

capable of something rather like the visual thinking embodied in their heavily illustrated and characteristically Renaissance treatises.

Appendix: Drawing polyhedra

Drawing the five regular polyhedra became a standard exercise in the perspective treatises of the sixteenth century. It has, however, no direct antecedent in PIERO DELLA FRANCESCA's *De prospectiva pingendi*, which is the source of almost all the other drawing problems found in such treatises. This minor puzzle hides a much more interesting historical fact, namely that when illustrating his own work on polyhedra PIERO himself did not always choose to draw the figures in correct perspective.⁷⁴

In fact, although PIERO does not propose the regular solids as perspective exercises, his drawings of them employ conventions that are notably more naturalistic than those commonly found in mathematical treatises of the fourteenth and fifteenth centuries, and indeed in many mathematical works of later date. The style of illustrations in these texts apparently derives from the Islamic manuscript tradition. For instance, parts are often turned so as to be seen in a way that shows their shape clearly. Figure A1 shows my redrawn versions of standard forms of pictures of various solids done using this kind of system. As illustrations to the relevant propositions in EUCLID, these pictures are entirely adequate, since they enable one to follow a proof by referring to points designated by given letters. Such diagrams are not, it seems, designed to be aids to visualisation in any other way.

We may see something of the variety of different styles PIERO adopted by examining the drawings in the manuscript of his *Trattato d'abaco* preserved in the Laurentian Library in Florence.⁷⁵ This manuscript is autograph, and one of its notable features is that the diagrams are drawn in the same brownish ink as that used for the text, thus suggesting that they may have been done by the same person — though since the figures are generally fitted in at the bottom of pages, they sometimes look as if they were inserted later. In any case, the figures are very neat, and rather small (diameters are generally a few cm). One or two of them show pencil markings that may represent changes of mind by the draughtsman, and could accordingly be taken as evidence for the figures being by PIERO himself. There are no known autograph drawings by PIERO, so comparisons could be made only with visible parts of

⁷⁴ Other essentially non-naturalistic drawing conventions are used in some diagrams in his perspective treatise, for instance those illustrating his proof of the correctness of his method of construction. See J. V. FIELD, 'Piero della Francesca as a practical mathematician', a paper presented to a conference in Arezzo in October 1992 and to be published in the Proceedings (ed. M. DALAI EMILIANI); and *idem*, *The invention of infinity: Mathematics and Art in the Renaissance*, Oxford University Press, 1997, esp. Chapter 5.

⁷⁵ PIERO DELLA FRANCESCA, *Trattato d'abaco*, Florence, Biblioteca Medicea-Laurenziana, Codice Ashburnhamiano 280 (359*).

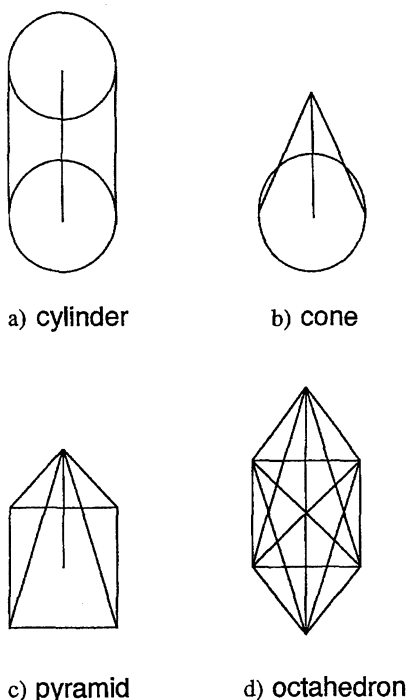


Figure A1

Diagrams of solids in the style found in most mathematical texts of the fourteenth and fifteenth centuries, as well as in many printed texts of later date. a) cylinder; b) cone; c) square pyramid; d) regular octahedron (found in this form in LUCA PACIOLI, *De divina proportione*, Venice, 1509, p. 9 verso). Drawings by J. V. FIELD.

underdrawings in his pictures. Such comparisons are most unlikely to be productive when we are concerned with drawings of polyhedra. In any case, since some of the bodies shown in the *Trattato* manuscript must surely have been unfamiliar, we have every reason to suppose that the drawings we have are at least relatively faithful copies after originals by PIERO.

Piero della Francesca's drawings of a cuboctahedron

Two propositions in the *Trattato d'abaco* concern the cuboctahedron. The first, which we have cited above, is to find the side of a cuboctahedron inscribed in a sphere of given diameter.⁷⁶ The second is to find the volume of the solid, but the proposition is so phrased that the sphere is again involved. The two propositions have essentially identical figures by way of illustration. Thus the

⁷⁶ See above at note 14 (p. 248).

use of a circle to indicate the sphere, and its being made to pass through all the vertices visible on the outer contour of the drawing of the body, cannot easily be dismissed merely as a copyist's misreading of PIERO's intentions.

The first of the drawings in question is shown in our Figure 5 above. It has been chosen in preference to the second merely because in this first figure the vertices have been given letters (in a habitual mixture of upper and lower case forms). Figure A2 shows my own drawing of a cuboctahedron, seen with one triangular face perpendicular to the line of sight, and shown as if in plan, though (for the sake of clarity and in deference to the draughtsman's lack of skill) omitting indications of the parts that would be hidden if the solid were supposed to be made of some opaque material. A similar convention of opacity is adopted in PIERO's own figure of the cuboctahedron — though not in all his figures of other bodies. LEONARDO's illustrations for PACIOLI's *De divina proportion* show each body first as opaque (*solidus*) and then as a skeleton (*vacuus*). Our Figures 8, 9 and 10 above reproduce only the second type of picture, which seems to have fared better in its passage from watercolour drawing to woodcut. As can be seen by comparing our Figures 8 and 5, the orientation of the solid in LEONARDO's drawing is not the same as in PIERO's.

In Figure A2 one can see clearly that the six vertices of the solid that lie at the outer ends of the six edges radiating from the points I, K and P, the three corners of the innermost triangle, form the vertices of a regular hexagon, NMLRQS, and therefore lie on a circle. The vertices of the cuboctahedron in Figure A2 have been given letters corresponding to those in PIERO's figure (Figure 5). If we now imagine the cuboctahedron shown in Figure A2 as a solid figure, complete with its circumscribed circle NMLRQS, it is clear that by turning the body anticlockwise about the line LR (or any line parallel to LR) we can move the point K in space until we see it as lying at the mid-point of the arc LR, as shown in PIERO's figure (Figure 5). However, the circle NMLRQS now no longer appears as a circle, since it lies in a plane parallel to

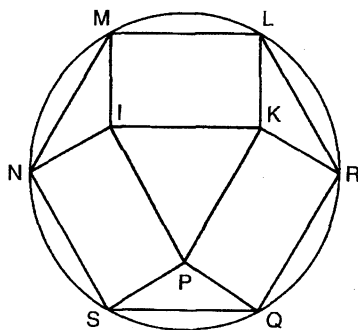


Figure A2

Diagrams of a cuboctahedron, in plan, with a circle running through six vertices. The lettering of the vertices corresponds to that in PIERO's figure (our Figure 5 above).

Drawing by J. V. FIELD.

that of the triangle IKP, which is no longer perpendicular to our line of sight. To be precise, the circle should be seen as an ellipse, but this is not a matter of any importance in the present context. In any case, the diagram in the manuscript is too small for checking the shape of the curve NMLRQS, but close inspection of the actual page concerned shows that the curve has in fact been drawn with compasses (a tiny hole appears in the position of the centre) and is thus intended as a circle. It accordingly appears that whatever drawing conventions are being followed in PIERO's diagram of the cuboctahedron, they are not those of mathematically correct (naturalistic) perspective.

Piero della Francesca's drawings of other polyhedra

PIERO does not adhere to one particular set of drawing conventions for all his illustrations of three-dimensional figures. The diagram after those of the cuboctahedron shows the PLATONIC regular dodecahedron, which has twelve pentagonal faces. As can be seen in Figure A3, the diagram shows the vertices of the solid as lying close to the surrounding circle but not all exactly on it. The circle has been drawn with compasses. Small indentations marking each vertex suggest that the drawing of the solid has been transferred from a preparatory study, probably one drawn in correct naturalistic perspective.

A different convention is to be found in the two diagrams of the truncated tetrahedron that illustrate the two propositions before those on the cuboctahedron. The two diagrams are shown in Figure 4. The figures are too small to measure with any useful degree of accuracy. They appear to give a naturalistic view of the solid, each of whose vertices is surrounded by two hexagons and a triangle. PIERO has explained that one may obtain this solid by cutting off the corners of a regular tetrahedron (that is, a regular triangular pyramid): each vertex removed leaves a triangle, and each face cut into leaves a hexagon.⁷⁷

Although points are given letters in PIERO's text, almost no lettering appears on the diagrams. My own copy of the diagrams, in Figure A4, uses letters starting at P because PIERO's ended at O. The copy was made by inscribing a regular octagon PQRSTUWX in a circle, then constructing the line YZ, equal in length to the side of the octagon, parallel to QR and WU passing through the centre of the circle and bisected by that centre. Joining up the points as shown then gives a figure which is nearly, but not quite, a plan of the visible part of the truncated tetrahedron. To make it an accurate plan, the lines XP, QR, ST, UW and YZ would all have to be equal to one another (since they all represent edges lying parallel to the ground plane) while the remaining lines should all be equal to one another but slightly shorter than the first set (since they all represent edges that are at an angle to the ground plane). The octagon

⁷⁷ PIERO ed ARRIGHI, p. 230; Cod. Ashb., p. 107 verso. Compare his account of the cuboctahedron, cited above, note 14 (p. 248). He does not explain how one ensures that the hexagons are regular.

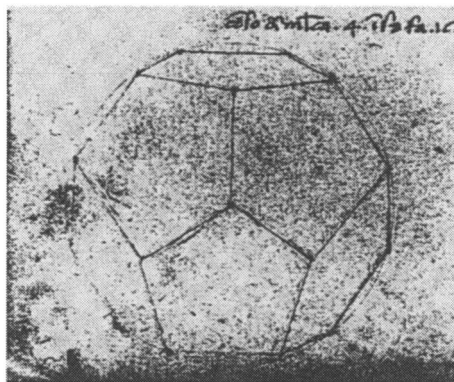


Figure A3

PIERO DELLA FRANCESCA (?), drawing of a PLATONIC dodecahedron (twelve pentagonal faces) with its circumsphere, diameter about 49 mm, From Piero DELLA FRANCESCA, *Trattato d'abaco*, Florence, Biblioteca Medicea-Laurenziana, Codice Ashburnhamiano 280 (359*), p. 110 verso.

PQRSTUWX should not be regular; it is, however, correct — if we are drawing a plan — to show P, Q, R, S, T, U, W, X as lying on a circle, though in space we actually have two equal circles, one being XPTS and the other QRUW. Neither of these circles is a great circle of the circumsphere of the solid, so if the figure is interpreted as a plan PIERO has not shown a circle with diameter equal to that given for the sphere (namely 6). It is, of course, possible that PIERO has in fact drawn the truncated tetrahedron in correct perspective (though without its circumsphere), but there are no signs that indicate the use of a preparatory study, and my personal impression is that PIERO's figure looks more like Figure A4 than like the corresponding view of the cardboard model of a truncated tetrahedron that I made in the course of my attempt to understand his figures.

Circumspheres

The conventions followed in PIERO's representations of the circumspheres of his solids are as variable as those followed for the solids themselves. It is interesting to make a comparison with the earlier method of representing the circumsphere found in the picture of the octahedron in Figure A1d. Here a circle through the vertices of the central square stands for the circumsphere, and has in fact got the correct radius (if the sides of the square represent the actual length of the edges of the solid).⁷⁸ PIERO's methods of representing the

⁷⁸ This is so in PACIOLI's version of the diagram; see LUCA PACIOLI, *De divina proportione*, Chapter 28, p. 9 *recto*. The circumcircle of the square, which lies in a plane of symmetry of the octahedron, will be a great circle of the circumsphere of the solid.

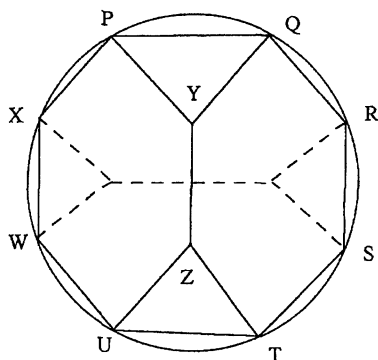


Figure A4

Drawing of a truncated tetrahedron, starting from a regular octagon inscribed in a circle. This drawing is not an exact plan of the solid, and the circle has not got the same radius as the circumsphere of the solid. Drawing by J. V. FIELD.

circumsphere for the regular dodecahedron and the regular tetrahedron (not illustrated in our Figures),⁷⁹ in which not all the vertices are shown on the surrounding circle, are at least close to naturalistic perspective. Indeed, they may even be mathematically correct. The convention used for the cuboctahedron is close to mathematically correct perspective but as we have seen it is certainly not exactly correct. For the truncated tetrahedron PIERO has chosen to imply the existence of a circumsphere by showing two circles which lie in it but which are not great circles. He chose to show up as much symmetry as possible rather than hold out for greater mathematical exactness. Rather similar decisions can be seen in some of his paintings, for instance the abandonment of correct mathematical perspective by the use of multiple viewing heights in his fresco of the *Resurrection*.⁸⁰

Circumspheres are not shown in the illustrations of polyhedra by LEONARDO DA VINCI in PACIOLI's *De divina proportione* or in the numerous illustrations in DANIELE BARBARO's *La Pratica della prospettiva*. They do, however, appear in DÜRER's *Underweysung de Messung mit dem Zirkel und Richtscheyt*, in his figures of the five regular solids.⁸¹ For the tetrahedron, octahedron, icosahedron and

⁷⁹ See PIERO ed ARRIGHI, p. 220; Cod, Ashb. 107 *recto*.

⁸⁰ This has often been remarked upon. For a discussion in a mathematical context, see J. V. FIELD, 'Mathematics and the craft of painting: Piero Della Francesca and perspective', in J. V. FIELD, F. A. J. L. JAMES (eds.), *Renaissance and Revolution: Humanists, Craftsmen and Natural Philosophers in Early Modern Europe*, Cambridge University Press, 1993, 73–95; and J. V. FIELD, 'A mathematician's art' in M. A. LAVIN (ed.), *Piero della Francesca and His Legacy*, Washington, D.C.: National Gallery of Art (Studies in the History of Art, no 48, Center for Advanced Study in the Visual Arts, Symposium Papers XXVIII), 1995, 177–197.

⁸¹ ALBRECHT DÜRER, *Underweysung de Messung mit dem Zirkel und Richtscheyt*, Nuremberg, 1525, Book 4, Figures 29 to 33, *ed. cit.* (note 42, p. 266), pp. 316–324.

cube the circumsphere is indicated by means of a circle passing through all the vertices on the outermost contour of the solid. An apparently similar convention is used to represent the circumsphere of the dodecahedron, but in this case it is clear that DÜRER, while drawing the solid as he has drawn the others, in 'parallel perspective',⁸² has forced vertices onto the circle in the same way that PIERO DELLA FRANCESCA did in his drawings of the cuboctahedron. That is, DÜRER also has sacrificed mathematically correct illusion in order to emphasise symmetry.

Drawing regular polyhedra in perspective

When fifteenth-century artists wanted to impart a sense of depth in two-dimensional works of art, they used perspective images of fairly simple geometrical shapes. Standard pictorial items included square-tiled pavements, square-coffered ceilings, houses (based on a cube), and hexagonal well-heads. All these examples can be found in PIERO DELLA FRANCESCA's treatise on perspective, *De prospectiva pingendi* (for modern editions see footnote 8 above), and in his paintings. They are also found in most subsequent treatises. The continued appearance of this kind of simple element in Renaissance pictures shows that such exercises in treatises were essentially practical. Since PIERO himself was plainly interested in the regular solids for their own sake, and does indeed seem to have made some perspective drawings of them, their omission from his treatise perhaps requires an explanation. An explanation is, in fact, very easily found: the regular solids, other than the ubiquitous cube (which PIERO does deal with), were of no general practical application in the making of pictures: images of polyhedra survive only rarely in paintings, and are otherwise confined to elaborate perspective exercises done in inlaid wood (*intarsia*). Such inlaid panels are used, for instance, to decorate cupboard doors with an illusion of their being open to display the contents of the cupboard, or to cover the backs of choir stalls with perspective townscapes.⁸³ PIERO presumably omitted polyhedra because he was addressing his text to the problems suitable for apprentice painters.⁸⁴ PIERO may, perhaps, have lent a mathematical hand to his friend the *intarsia* specialist LORENZO DA LENDINARA when he wrote his treatise on

⁸² That is, as if seen from an infinitely distant viewpoint, so that the uppermost and lowest faces of the solid both appear as straight lines.

⁸³ Notable examples of cupboard and window illusionism in *intarsia* are to be found in the Studiolo in the Ducal Palace at Urbino, dating from the mid 1450s. *Intarsie* of townscapes, probably dating from the late 1470s, are found in choirstalls in S. M. Gloriosa dei Frari, in Venice, and Sta Corona, Vicenza. See LUCIANO CHELES, *The Studiolo of Urbino: An iconographical study*, Wiesbaden, 1986; and A & J. F. TORMEY, 'Renaissance Intarsia: the Art of Geometry', in *Scientific American*, 247, 1982, 116-122.

⁸⁴ For a more detailed discussion of PIERO's perspective treatise see paper referred to in note 80 (p. 282).

perspective (now lost), which was presumably addressed to apprentices in the same specialism and might have contained some sections on such items for the display of virtuosity.⁸⁵

PIERO's perspective treatise ends with what are essentially two trick items: a goblet that seems to stand up from the table in which it is painted and a ring (the kind from which one suspends a lamp) that seems to hang down from the vault on which it is painted. The *trompe l'œil* element is similar to what we find in *intarsie*, but the objects concerned are more mundane. However, at the beginning of the last book of his treatise PIERO has described a completely general method which can be used to draw the perspective image of 'more difficult' bodies. This method is applied first to some simple examples, to show how it works, and then to such items as column bases, column capital, and human heads. The method involves making preliminary drawings showing the object from the front, from the side, and in plan, with some additional horizontal sections if these are needed to establish the shape. Figure A5 shows the set of drawings provided for the human head. It is obvious that PIERO could have produced corresponding sets of drawings that would have allowed him to construct the image of, say, a regular dodecahedron. In fact, the illustrations to his *Libellus de quinque corporibus regularibus* seem to have included such ambitious items as the perspective drawing of a regular icosahedron with a regular dodecahedron inside it, each vertex of the inner solid being at the centre of a face of the outer one. This diagram illustrates Case 10 of the third part of PIERO's work, and appears in Pacioli's printed version in 1509.⁸⁶ It would seem that PIERO omitted the polyhedra from his perspective treatise because he regarded them as a matter of interest to him as a mathematician rather than as an artist.

ALBRECHT DÜRER's attitude to polyhedra seems to have resembled PIERO's in this respect. In his *Underweysung der Messung* he deals with them in their own right, not as providing examples for exercises in perspective drawing. Indeed, he discusses them before he comes to perspective, so his illustrations of them are presented merely as illustrations. In fact, all polyhedra are shown as nets and only the five regular polyhedra are also shown in perspective. Moreover, the form of perspective used (so-called 'parallel perspective' — see above, note 82) is not entirely naturalistic. The seven Archimedean polyhedra in DÜRER's treatise, which must surely have been more unfamiliar than the five regular solids, are shown only in the form of nets. As we have seen, the reader is apparently advised to copy these nets and turn them into three-dimensional models for himself. DÜRER's reader is thus as it were presented with a printed do-it-yourself

⁸⁵ For PIERO's friendship with LORENZO see LUCA PACIOLI, *De divina proportione*, On architecture, Introduction, p. 23 *recto*, and R. LIGHTBOWN, *Piero della Francesca*, London: Abbeville Press, 1993, 74–75.

⁸⁶ See LUCA PACIOLI, *De divina proportione*, Venice, 1509, 'Libellus in tres partes divisus ...' [PIERO DELLA FRANCESCA, *Libellus de quinque corporibus regularibus*], p. 17 *verso*.

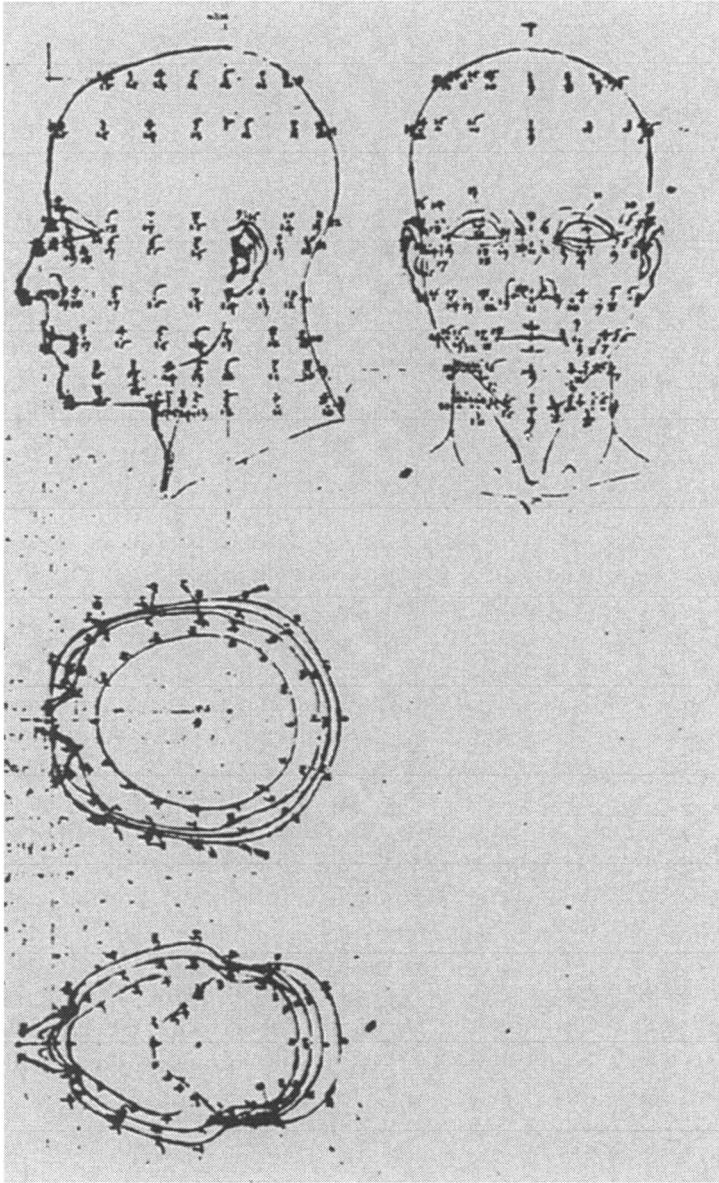


Figure A5

After PIERO DELLA FRANCESCA, Drawings of a human head. From *De prospectiva pingendi*, Book 3, Parma, Biblioteca Palatina, MS no 1576, folio 64 recto.

version of the manuscript plus models from PACIOLI adopted in the earlier versions of *De divina proportione*. DÜRER has not attempted to emulate the illusionistic substitutes for models that were drawn for PACIOLI by LEONARDO.

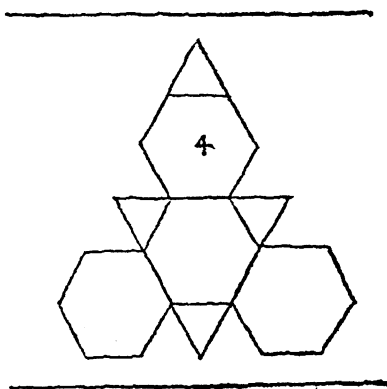
In later years, the rising level of mathematical education among the upper classes provided increasing numbers of *cognoscenti* capable of appreciating the characteristically Platonic of p. 244 mixture of aesthetic and mathematical pleasure that is offered by the drawings LEONARDO provided for PACIOLI. These drawings are, however, presented as illustrations. PACIOLI's text is, after all, not proposing to teach its readers how to draw. In contrast, DANIELE BARBARO's *La pratica della prospettiva* is making such a proposal, at least in principle. In practice, most of the text seems to be addressed to employers of craftsmen rather than to craftsmen themselves, but the relevant procedures are described in outline. Thus, while BARBARO's account of perspective follows PIERO DELLA FRANCESCA's very closely, it omits the repetitious drawing instructions of the original. There are several other omissions and elisions,⁸⁷ but only one substantial addition to PIERO's work: BARBARO includes polyhedra among the worked examples. This may be a relic of BARBARO's original intention of basing his work on DÜRER's account of perspective in the *Underweysung*, in which polyhedra appear not as perspective exercises but as solids 'which can be constructed with straightedge and compasses'.⁸⁸ BARBARO includes the polyhedra not as what PIERO called 'more difficult' bodies to be dealt with by the point-by-point method PIERO had described in his third and final book, but rather as bodies that can be constructed from ground plans.

This choice may be connected with BARBARO's concern with architecture, since in this period buildings were increasingly described by means of plans and sections as well as by perspective drawings and three-dimensional models. PIERO used ground plans and heights in his second book, where he dealt with prisms (the cube being a square prism). Problems of putting ground plans into perspective had been considered in the first book. To treat polyhedra by the method of PIERO's second book, BARBARO needs to find first ground plans and then the heights of the vertices of the solid above the ground plane. He does not explain in detail how the second task is accomplished, but series of verticals show that it has been, and the perspective picture of the solid is then constructed vertex by vertex, putting each at the appropriate height above the perspective version of the ground plan. Diagrams appear in pairs, showing the ground plan in its 'perfect' form (that is, its actual shape), with the perspective ('degraded') plan above it and the sets of verticals that establish the positions of the vertices (see Figure A6).

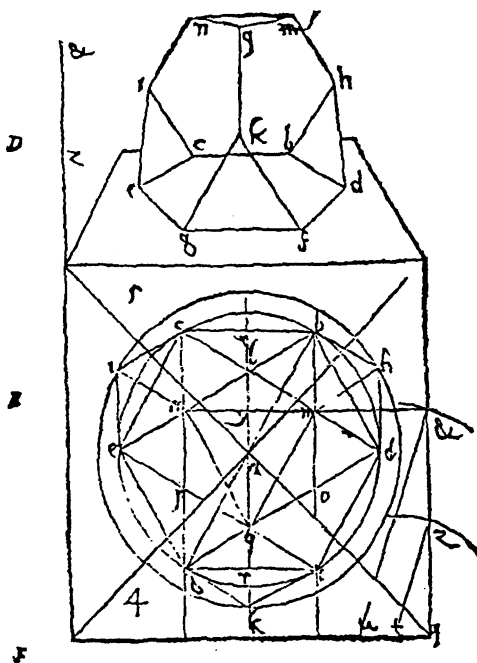
⁸⁷ See THOMAS FRANGENBERG, 'Piero in the Cinquecento' paper presented at a conference in Arezzo in October 1992 and to be published in the Proceedings (ed. M. DALAI EMILIANI) (in press 1993); and M. J. KEMP, 'Piero and the Idiots: The Early Fortuna of his Theories of Perspective', in M. A. LAVIN (ed.), *Piero della Francesca and His Legacy*, Washington, D.C.: National Gallery of Art (Studies in the History of Art, no 48, Center for Advanced Study in the Visual Arts, Symposium Papers XXVIII), 1995, 199–211.

⁸⁸ ALBRECHT DÜRER, *Underweysung der Messung mit dem Zirkel und Richtscheit* (Nuremberg, 1525), book 4, folio Miiij verso, DÜRER in fact shows most of his polyhedra only in the form of nets. See above.

A culo, & in esso sia descritta la superficie bcdefg, di sei lati eguali, sia poi tratta la linea bg,



& sopra'l centro z, fatto un'altro circulo di tanta circonferenza che la detta linea bg, formi in quello i punti di uno triangulo di lati eguali, che siano h, i, k, siano poi tratte le linee cd, eg, dg, bi, be, & ci, & si formi un'altra superficie di sei lati eguali i cui punti sono l, m, n, o, p, q, siano poi tratte le linee mn, mq, nq, & si formerà uno triangulo di lati eguali, i cui angoli saranno m, n, q. siano poi tratte le linee hm, hb, hd, ic, in, ic, & g, kq, kf, & serà descritto il perfetto i cui quattro esagoni sono bcdefg, bchlmn, hdmfqk, incqgk, i quattro trianguli mnq, hbd, icc, kfg. Le altezze, & i piani del detto corpo si fanno a questo modo. Sia presa la linea qk, & sia posto il punto r, dove ella è divisa da fg, & poi detta linea qrk, sia riportata sopra la linea inferiore del quadrato ne i punti t, u, sia poi partita egualmente la linea bc, nel punto z, & sia tratta la linea xq, la quale taglierà mn, in y, sia poi per lo spazio qy, e centro r, tratta la linea circolare, la quale taglierà la linea destra del quadrato nel punto z, & per lo spazio zr, & centro u, sia tratta la linea circolare, che taglierà la detta linea destra in &, per il che i punti &, z, l, sono i punti delle altezze, & perche tratta la linea &u, & zt, essa &u, è tanto, quanto è zr, & zt, è tanto quanto è yq, & perche nel corpo sodo. yq, è la prima altezza, ma non ad angolo giusto però zt, che è ad angolo giusto con &, esso punto z, è la prima altezza, & il secondo piano. Et perche nel sodo zr, è la seconda altezza, ma non ad angolo giusto, però &f, che è ad angolo giusto con su, esso punto &, è la seconda altezza, & terzo piano. Adunque se nel piano l, serà posta la superficie di sei lati bcdefg, & nel piano z, i punti hik, & nel piano &, il triangulo mnq, & tratte le linee, hb, hm, hd, ic, in, ic, kf, kq, kg, si formerà il detto corpo irregolare, tagliato dalla piramide, come si vede dalla figura 5. Ma perche essendo formato di superficie di sei, & di tre lati egli si può fermare con la triangulare, & con l'esagona, però se nel piano l, serà descritto il triangulo mnq, & nel piano z, i punti hik, & nel piano &, la superficie bcdefg, & tirate poi le linee (come s'è detto) nel perfetto egli si poserà con la basa triangulare. L'adombratione se intende chiaramente per la detta figura 5, nella quale uè il perfetto in pianta, & il d'agrada so dritto.



H

Descrizione

Figure A6

DANIELE BARBARO. *La Pratica della prospettiva*, Venice, 1569, part 3, p. 57, showing the truncated tetrahedron, in the form of a net, as well as in perspective and in plan. Photo courtesy of the Trustees of the British Library.

In the manuscripts of BARBARO's treatise, these complicated perspective diagrams show pin holes at important points. They have clearly been transferred from preliminary studies, but we cannot tell whether these were done by BARBARO himself — though his apparent impatience with the minutiae of construction to be found in PIERO's text suggest rather strongly that BARBARO would have employed a specialist draughtsman for the task. In any case, perspective illustrations cease after Chapter 16, and we are given only nets for some of the polyhedra which (since they are not in PACIOLI's work) we have reason to suppose were solids BARBARO had actually discovered for himself. This omission, which seems to devalue BARBARO's personal contribution, tends to confirm the impression that BARBARO believes he is concerned with an indefinitely large number of solids. If he had been working towards a fixed number he would surely have taken more care to chart his progress. The only solids which do have perspective illustrations later in the book are three of those formed by adding pyramids to the faces of previous solids. BARBARO has described these solids, and given nets for their triangular faces, in Chapters 26 to 33.⁸⁹ The three solids shown in perspective are those in which pyramids have been added to the pentagonal faces of the dodecahedron, to the triangular faces of the icosahedron and to the triangular faces (but not the pentagonal ones) of the icosidodecahedron.⁹⁰ No indications are given of how these illustrations are drawn. Ever since Chapter 26, BARBARO's work has been slowly drifting away from providing examples of perspective construction and towards becoming a study of polyhedra.

As KEPLER was to show, in *Harmonices mundi*, Book 2, Proposition 26, the addition of suitably proportioned pyramids to the faces of the regular dodecahedron and the regular icosahedron can form two new regular polyhedra whose star pentagon faces are partly hidden inside the solid (see Figure 3 above).⁹¹ Neither BARBARO nor his illustrator seems to have noticed this fact. The pyramids added to the dodecahedron are indeed isosceles and rather tall — though those shown in the corresponding net are equilateral,⁹² as are those in the corresponding diagram by LEONARDO in PACIOLI's work. However, the edges of BARBARO's pyramids are not shown as aligned with those of the underlying dodecahedron. There is accordingly no doubt that BARBARO has not discovered the solid KEPLER derived from the dodecahedron. His diagram is, all the same, rather decorative, and it almost certainly provided the inspiration for a piece of inlaid marble paving in St Mark's church, Venice, which shows

⁸⁹ DANIELE BARBARO, *La Pratica della prospettiva*, Venice 1568, 1569, Part 3, pp. 105–110.

⁹⁰ DANIELE BARBARO, *La Pratica della prospettiva*, Venice 1568, 1569, Part 3, Chapter 34, pp. 111–113.

⁹¹ See the paper referred to in note 69 (p. 275).

⁹² DANIELE BARBARO, *La Pratica della prospettiva*, Venice 1568, 1569, Part 3, Chapter 28, p. 106.

a closely similar solid.⁹³ The pyramids added to the icosahedron shown in the second of BARBARO's short final series of perspective diagrams are clearly equilateral (like those shown in the corresponding net) and their edges do not align with those of the underlying icosahedron. The resultant solid in fact looks decidedly unlike the one KEPLER derived from the icosahedron, and it has not (as far as I know) provided material for unlikely claims of priority.

BARBARO's treatise on perspective seems to have been used as a source by many later writers. Thus, following his example, and that provided by LEONARDO's spectacular illustrations to *De divina proportione*, regular polyhedra came to provide standard exercises of skill in perspective drawing. Later treatises did not, however, take up BARBARO's account of the different construction method described in PIERO's third book. This construction for 'difficult bodies' is not discussed, and almost all perspective treatises end with descriptions of sighting devices. While apparently exalting a mathematical exercise the treatises have in fact told the reader how one could avoid it. Curiously enough, it seems that readers either did not notice or did not object. Perhaps it is not only in the twentieth century that unfamiliar mathematics can mesmerise people out of exercising their common sense.

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⁹³ The panel, which lies just inside the left door in the main façade (as seen from inside the porch), has sometimes been described as showing KEPLER's new solid (which is to ignore the non-alignment of edges) and has also been ascribed to PAOLO UCCELLO (1397–1475). None of the authors concerned cites any evidence to sustain this ascription. UCCELLO, who was famous for his love of perspective construction, merely seems to have been considered a suitable designer for an undatable item, though the foliate border of the panel surely cannot be fifteenth-century.