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The story of the discovery of incommensurability, revisited

D.H. Fowler

pp.221-235 of *Trends in the Historiography of Science*, ed K. Gavroglu, J. Christianidis, & E. Nicolaidis, Boston: Kluwer, 1994, with some very minor variations

I take as my opening text the kind of thing my colleagues — certainly the mathematicians and often the historians of mathematics — might say about the beginnings of Greek mathematics. Something like this:

The early Pythagoreans based their theory of proportion on commensurable magnitudes (or on the rational numbers, or on common fractions m/n), but their discovery of the phenomenon of incommensurability (or the irrationality of $\sqrt{2}$) showed that this was inadequate. This provoked problems in the foundation of mathematics that were not resolved before the discovery of the proportion theory that we find in *Elements* V.

You must, at some time or another, have heard, or perhaps even have said, something like that. I've certainly said it; I even got into *Punch* for saying it!¹ But I shall try explain why I now disagree with everything in this line of interpretation. My space is limited so I will have to refer you for many of the crucial details in what follows to the thorough discussions I have cited or given in my book, *The Mathematics of Plato's Academy*; in fact, I hope this article will form the basis of the opening chapter of a sequel to this book.² I shall arrange my comments under various headings. First we have the matter of:

The nature of our evidence. Our evidence about Greek mathematics in general comes in very disparate forms, and almost all of it has been subject to an unknown amount of editing and interference. In particular our late sources — editions, compilations, and commentaries dating from the 2nd century AD onwards — are manifestly of very variable quality. So, in the first phase of my reconstruction, as represented by my book, I put it all to one side as far as possible, and ignore it.³ (This is very drastic, and my hope is to consider some of this later evidence in the sequel.) Moreover, some of the relevant evidence in early sources, in particular Euclid and Plato, comes in homogeneous slabs which often fit rather awkwardly in the various versions of the received interpretation. In some measure to redress the balance after my radical approach to the late texts, I endeavour to follow the principle that, if any piece of such an early slab enters the reconstruction in a significant way, then the proposal should also engage with

~~FOWLER, D.H.~~ 1994

¹. *Punch*, April 24, 1974. This magazine provided a humorous commentary on British life for 150 years until its closure in March 1992. In recent years, it ran a regular column 'Country Life' which explained itself as follows: "Not everything that happens in Britain gets into the national press. This feature presents some of the news that never made it." One reader sent in the following clipping: "The programme, which is about the development of number systems, will include an interview with David Fowler, of Warwick University, on the historical crises associated with the square root of two." This must ultimately have come from some Open University publicity about a TV programme I had helped make for their first History of Mathematics course, which had then been passed on by my university, picked up by the local newspaper, *The Leamington Spa Courier*, and submitted to *Punch* by a local reader. This programme was my first and reluctant venture into Greek mathematics, and I later disowned it, only permitting the Open University to continue broadcasting it if it also circulated a disclaimer by me to the students doing the course!

². Hereafter I shall refer to these as 'my book' and 'the sequel'.

³. At the outset, I must admit that one piece of evidence of late provenance plays a crucial role in the reconstruction, namely the material on 'side and diagonal' numbers and lines, found in Theon of Smyrna, Iamblichus, and Proclus. See the discussion in my book, pp. 100-4, which however does not deal with the material in Iamblichus (ed. Pistelli, 91.3 - 93.6).

DAVID H. FOWLER

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reconstruction in a significant way, then the proposal should also engage with the whole of its context. For example, any important use of any aspect of the curriculum in Plato's *Republic* VII should also connect with the whole of this curriculum, especially since Plato insists on its unity; any significant application of a proposition in *Elements* II should eventually involve all 14 of them; anything about *Elements* XIII, the book which contains the construction of regular solids and a lot more, should say something about Book X, the classification of incommensurables, and perhaps also about Books IV and II.

Please note: I am not saying early evidence is all good, late evidence is all bad. I am saying that our evidence is a hodge-potch that we cannot sort out until later in the project, but the best evidence is likely to be found in the coherent and obscure chunks of early provenance (this is analogous to the principle *difficilior lectio* of textual criticism), so let us start from this material, taken all in one piece. It is a methodological principle, not a simple value judgement of the evidence.

Back now to my opening text. In fact, my principal objection to it is that it is founded entirely on late evidence and speculation, uncorroborated to a remarkable extent by any of our earlier sources, so it forms a very insubstantial base on which to start our reconstruction of Greek mathematics. I shall spell that out in more detail, and then propose an alternative starting point.

On Pythagoras and the early Pythagoreans, see, for example, W. Burkert, *Love and Science in Ancient Pythagoreanism*. I subscribe fully to his general conclusion, that the only kind of scientific activities and discoveries we can attribute with confidence to the early Pythagoreans are some remarkable findings in music theory and acoustics, the most remarkable being that consonance is associated with small integers. Most, if not all, of the mathematical stories have the ring of later legendising. In particular, consider:

The Pythagorean theory of proportion based on commensurable magnitudes. I know of no explicit evidence for this, early or late. From what I can work out, the thinking goes something like this: we, since medieval, times have learned at school about common fractions, that is m/n s, so that is what the Pythagoreans must have used, especially since we find these fractions later in Greek mathematics and Greek accounting. Occasionally this opinion gets expressed in print; see, for example, B. L. van der Waerden, *Science Awakening*, pp. 49–50 & 115–6. He writes:

It is probable that it was *calculation with fractions* which led to the setting up of *Elements* Book VII [which van der Waerden attributes to the early Pythagoreans]. Fractions do not occur within the official Greek mathematics before Archimedes but, in practice, commercial calculations had of course to use them....

Once again, I disagree with everything in this opinion. Fractions do occur in 'official mathematics', even, in a limited way, in the *Elements*: see the use of the word *meros*, plural *merê*, translated as 'part' and 'parts', especially in Book VII. For example, VII 37 & 38 talk of 'homonymous parts', of three & the third, four & the quarter, etc., and these *merê* are an ingredient of the way fractions are described in Greek. But I do not think there is any unambiguous evidence for *common* fractions, these m/n s, in any of our early texts including commercial calculations, and their apparent appearance in the late Byzantine copies which account for 99% or more of our evidence may be instead as scribal abbreviations. Our plentiful surviving explicit evidence is that Greek fractional practice was exactly the same as Egyptian practice: fractions were expressed as sums of the *merê*, as so-called unit fractions. The details of this argument are long, painful, and contentious because, while we have lots of different kinds of evidence, most of it is of the wrong sort or it comes from the wrong place or the wrong time. The details are given in Chapter 7 of my book and summarised in my article 'Logistic and fractions in early Greek mathematics'. And if you find the full conclusions of my overall thesis too much to swallow, we need only a much weaker version of it here, that the Greek mathematicians before Plato and Eudoxus used not common fractions, but Egyptian fractions. I'll return to fractions and arithmetic later.

Note that, here and elsewhere, 'Greek' as in 'Greek mathematics' simply means 'written in Greek'. Almost all of our evidence has been transmitted via Egypt, and then via the whole eastern Mediterranean, and that, of course, complicates the argument. Also, concerning fractions and division, I think that Diophantus needs a separate discussion.

The topic of incommensurability. At the end of my book, I review all of the evidence on this topic for the thousand year period up to Proclus and, there and here, I refer you also to another such review in W. R. Knorr, *The Evolution of the Euclidean Elements*, Chapter 2. Here are some opinions from these reviews.

- In the first surviving explicit mention of incommensurability,⁴ in Plato's *Theaetetus* (147a ff), the topic is handled confidently as a source of

interesting mathematical research. Incidentally, as to the date of Theaetetus' death, which is generally regarded as one fixed point, perhaps the only secure fixed point, in the shifting sands of the incommensurability issue, I also note that there is now one expert who dissents from the common view. H. Thesleff, specialist on pseudo-Pythagorean texts, writes: "I find it essential to note that historians of mathematics who take it for granted that Theaetetus was still alive in the 370s must be wrong. He made some important discoveries as a young man, and Plato and his friends were deeply impressed by this. But he is likely to have died in 390 BC."⁵

• The celebrated passage in Plato's *Laws* (817e ff) where Plato talks of "ignorance ... not worthy of human beings but pigs" is probably not referring to incommensurability in our sense here, but something else, very possibly the kind of techniques used in land measurement, where again things are not what mathematicians and historians of mathematics seem to assume they ought to be when, for instance, they parade stories from commentators about Egyptian land measurement as the origin of mathematics.

• Aristotle's favourite mathematical illustration is "the incommensurability of the diagonal" as something all mathematicians know; but he never suggests that it is or ever was a disaster for any mathematical theory, even though, in closely related passages, especially in the *Metaphysics*, he is highly critical of the Pythagoreans. Curiously, Aristotle never specifies that he is talking of the diagonal of a square. Twice, both times in *Prior Analytics*, at 41a23 ff & 50a35 ff, he says something like: "from the assumption that the diagonal is commensurate, it follows that odd numbers are equal to evens", but Aristotle gives no more details of what he means by this. I find completely convincing Knorr's proposal (*Evolution*, pp. 228–232) that the so-called Pythagorean proof, revolving around this statement, was tacked on to the end of *Elements* X in two clumsy versions sometime after the time of Alexander of Aphrodisias in the 3rd century AD, in response to the needs of Aristotelian commentators, and that Alexander himself cobbled together the variant of this proof to be found in his commentary. However other natural mathematical explanations of the remark by Aristotle about odd and even numbers are possible, and our evidence for any interpretation of it so tenuous as to be unreliable as a basis for further reconstruction. I will come back again to Aristotle later.

• No surviving fragment or testimony of Eudoxus mentions incommensurability: the word-index to Lasserre, *Die Fragmente des*

Eudoxos, does not contain the words (*a*)*summetros*, (*ar*)*rhētos*, or *alogos*! To us, it may seem blindingly obvious that the principal achievement of the Eudoxan theory of *Elements* V must have been to accommodate ratios of incommensurable magnitudes, but no ancient source says that, and I intend to include a chapter on Eudoxus in the sequel that tells this story differently. For the moment, I'll have to leave it at that, and add that you'll already find most of the ingredients in my book, worked out in some detail.

• Proclus never quotes anything from Eudemus on incommensurability, though he cites Eudemus by name several times, and also writes about incommensurability several times. The one apparent exception to this, the passage in the catalogue of geometers where Eudemus appears to refer to Pythagoras, is almost certainly an interpolation; and the remark there that Pythagoras discovered the "doctrine of proportionals" is a modern editorial emendation of the text, where all of our manuscripts unanimously have "the doctrine of the *alogos*".

• A proper discussion of this word *alogos* would take us on a very long excursion into *Elements* Book X; instead, let us here just look very briefly at Euclid: As far as I am aware, the only time the topic of incommensurability appears in Euclid's works is in *Elements*, Books X & XIII, which form our only coherent slab of early evidence on the topic, and a very massive, very coherent slab it is, in bulk and content well more than a quarter of the *Elements*. Any discussion of incommensurability should deal with Book X; but if you have never looked at Book X, I can promise you that you will find it difficult, so I recommend you to try to get hold of a pamphlet by Christian Taisbak called *Coloured Quadrangles*. You may find that difficult too – I mean getting hold of Taisbak's pamphlet! – so I have given a version of it in Chapter 5 my book, and subsequently written an improved version of this: 'An invitation to read Book X of Euclid's *Elements*'. As to the role of Books X and XIII in my reconstruction of the incommensurability story, I must refer you to the rest of Chapter 5 of my book.

• The source of most of the stories about Pythagoras, Pythagoreanism, and incommensurability is the book *On the Pythagorean Life* by Iamblichus, so perhaps it is worth quoting the relevant passages in full:⁶

§18 (88) ... As for Hippasos, he was indeed a Pythagorean, but because he was the first to make public the sphere constructed from twelve pentagons he was lost at sea for his impiety: he got the reputation of having discovered it, but it all came from 'that man' – that is what they call Pythagoras: they do not use his name.

§34 (246) ... The first man to reveal the nature of commensurability and incommensurability⁷ to those unworthy to share his teaching was so much detested, they say, that not only was he excluded from their common life and meals, but they built him a tomb as if their former companion had left human life behind. (247) Some say the supernatural power took revenge on those who published Pythagoras' teachings. The man who revealed the construction of the 'twenty-angled shape' was drowned at sea like a blasphemer. (He told how to make a dodecahedron, one of the 'five solid figures', into a sphere.) Some say this fate befell the man who told about irrationality and incommensurability.

This farrago of mutually inconsistent stories, which appear for the first time in a source of doubtful reliability and relevance dating from some nine centuries after the time of Pythagoras, is the main evidential base for much of what has been written about the discovery of incommensurability! I add the slightly perplexed comment that the most recent and authoritative study of the role of mathematics in neo-Pythagorism, D. J. O'Meara's book *Pythagoras Revived, Mathematics and Philosophy in Late Antiquity*, does not even seem to mention incommensurability.

The foundation crisis. Again I find Freudenthal, Knorr,⁸ and other writers very convincing when they argue that, far from being a period of crisis and confusion, the early fourth century was an extraordinary period of creativity, especially in Plato's circle; we have no historical evidence for any of the postulated difficulties of a 'foundation crisis'. But I want to go further and explore the possibility that the discovery was an incidental event in the early development of mathematics. So let us now look at the evidence concerning the effects of the discovery. This will involve tracking over some of the same material again.

The implications of the discovery of incommensurability. Just what precisely are the supposed problems raised by the discovery of incommensurability? As far as I know, no Greek text, early or late, tells us clearly of the mathematical difficulties raised by the phenomenon. Aristotle wrote of the innocent's surprise, and the way it then gives way to a more informed appreciation:

All men begin ... by wondering that the matter is so (as in the case of automatic marionettes or the solstices or the incommensurability of the diagonal; for it seems wonderful to all men who have not yet perceived the explanation that there is a thing which cannot be measured even by the smallest unit). But we must end in the contrary and, according to the proverb, the better state, as is the case in these instances when men learn the cause; for there is nothing which would surprise a geometer so much as if the diagonal turned out to be measurable (*Metaphysics* 983a 12–20).

Pappus' *Commentary on Book X of Euclid's Elements*, I.2, writes, much later, of how thereafter:

the soul ... wanders hither and thither on the sea of non-identity ... immersed in the storm of the coming-to-be and the passing-away, where there is no standard of measurement

from which passage I shall grasp, below, on the only substantial straw, the last four words: "there is no standard of measurement". There is the similar Scholium I to Book X, quoted in part in the Introduction to Book X in Heath's translation of the *Elements*, where you will find a long discussion that is remarkable for its lack of any hard evidence.⁹ And Proclus, in his *Commentary on the First Book of Euclid's Elements*, 60, writes:

The statement that every ratio is expressible (*phietos*) belongs to arithmetic only, and not to geometry, for geometry contains inexpressible ratios (*arrhetos logos*)

but this is not, very much not, the terminology of *Elements* X, which is built on a completely different meaning for expressible incommensurable lines and ratios.

So, prompted by Aristotle, let us try to "perceive the explanation" and "learn the cause" of incommensurability. But first, may I persist a bit longer with a variation on my question at the beginning of this section:

What precisely, to modern commentators, are the supposed difficulties revealed by incommensurability? A whole range of possibilities now seems to be on offer. With some overlap, there are the following:

- The Proclus objection: Incommensurable ratios cannot be expressed in (or by) numbers. Surely that is nonsense! We express them in numbers, and much of mathematics is, ultimately, numbers. Early Greek mathematicians could have expressed them in numbers. We have no explicit evidence that they did this, but I am arguing for a speculative interpretation in which they were expressed in numbers. My book is full of examples.

- Incommensurable ratios could not be fitted within the pre-Eudoxan style and scope of mathematics. That is false! The definition for lines that $a:b::c:d$ means rectangle (a,d) = rectangle (b,c) uses only believed-to-be early ingredients in a believed-to-be early way, and will handle most of what we need for the *Elements*, perhaps everything except for compounding, on which Euclid's own treatment is strange and unsatisfactory. But we have no evidence that this was an early definition.

- Incommensurability showed the inadequacy of the Pythagorean

doctrine that "all is number". Therefore, some add, it had to be concealed. Curiously, Aristotle, our principal early witness, from whom we learn of this "all is number", never advances this criticism, even though all of this material comes together in the *Metaphysics*. We find there lots of mentions of incommensurability, we find summaries and harsh criticisms of Pythagorean philosophy, but we don't find this objection. Zmnud has recently even proposed that this formulae "all is number" was Aristotle's invention.¹⁰ And, I would add, incommensurability need not show the inadequacy of the doctrine that "all is number", *pace* Proclus; indeed, in my reconstruction, the behaviour of the antyphraetic ratios of $\sqrt{n} : 1/m$ might even reinforce such a doctrine, and give a mathematical explanation of the 'expressible' lines that underlie *Elements* X and XIII.

• The discovery of incommensurability showed that the [Babylonian?] arithmetical basis of geometry was inadequate, so geometry had to be reformulated purely geometrically, for example, as in *Elements* II. We have no evidence of this supposed early, possibly Babylonian, arithmetical basis of early Greek mathematics, so this is pure speculation. Also, there are other explanations of the role of *Elements* II. I shall return to this issue later, when I discuss arithmetisation.

• Incommensurable ratios can only be approximated, while Greek mathematics aimed for precision. This assertion runs counter to our evidence: Archimedes is interested in approximation. Also Aristarchus and then, later, Hypsicles, Hero, Ptolemy, ... Moreover the fraction $1/7$ can only be approximated in sexagesimal arithmetic, and the fraction $1/3$ can only be approximated in the Greek system of land measurement which, by convention, only uses the parts $2'$, $4'$, $8'$, $16'$, ... So Greek mathematics was involved in approximation and the necessity of approximating simple numerical ratios was a well-known phenomenon.

• The discovery put into question the basic idea of ratio. I prefer to reformulate this, since no early text seems to express this concern: hints of it seem to surface first in later commentators like Iamblichus and Pappus, and then grow thereafter. So I shall change the question once again and ask instead:

Why do none of the early testimonies seem concerned with the manifest difficulties that incommensurability poses to our way of defining ratio? This emphasises that the difficulties may be *our* difficulties, not necessarily theirs; they may have had different ways of thinking about ratio. I finish by developing this theme a bit.

Incommensurability does present a problem to arithmetised geometry,

though ultimately I think it turns out that the problem was already lurking there even for commensurable manipulations. Let me try to explain. Arithmetised geometry is how we tend to think of geometry today: a line has a length, a number; a rectangle has an area, again a number which is equal to the product of the lengths of its sides; ratios are defined arithmetically, as quotients of numbers; and so on. So the geometry becomes translated into the arithmetical manipulation of numbers – addition, subtraction, multiplication, division, taking roots, etc. – and then this arithmetic is later abstracted into algebra. For example, the so-called Pythagoras' theorem becomes $a^2 + b^2 = c^2$, where the usual interpretation of this statement involves the lengths (i.e. numbers) a , b , and c of the sides of the triangle. It then seems that the numbers associated with things like $\sqrt{2}$ are rather complicated, so complicated that filling in all of these irrational numbers properly could not be done before the middle of the nineteenth century. Dedekind, the first, tells us that he succeeded on 24 November 1858; it was a Wednesday.

But read Dedekind carefully, and you will see that there are already serious problems with arithmetic. I go yet further than this, and argue that there need be no real problems in defining the set of numbers, for example, as decimal sequences, or sexagesimal sequences, or other such descriptions. However, a precise description of the arithmetic, even the arithmetic of the rational numbers when they are not being conceived as common functions, is intractable.¹¹ Hence the crucial role of my belief that early Greeks did not use common fractions, so they would not think of ratios in any way like our rational numbers, and so would not think that arithmetic was a natural and obvious basis on which to build their mathematics.

To pursue this theme further, early Greek geometry seems to me to be not arithmetised to a remarkable extent,¹² but later Greek mathematics is arithmetised. There are two different traditions in this later arithmetisation. The first is the astronomical one, using Babylonian sexagesimal numbers, which is not attested in Greece before Hypsicles and Hipparchus in the 2nd century BC. It may have been transmitted much earlier, but that is pure speculation, as was remarked earlier. The second is the Graeco-Egyptian unit fraction tradition which we find in our earliest testimonies like the Hibeh Papyrus, then later in the Heronian Corpus, and in Ptolemy alongside the sexagesimal astronomical calculations. The arithmetic of both poses theoretical problems, and both are spectacularly absent from the geometry of the *Elements*: they are incompatible with the

spirit of the proportion theory of Book V and remote from the spirit of the treatment of incommensurability of Book X. No wonder commentators from antiquity onwards, who seem to work in a vaguely arithmetised geometry, have a hard time, and no wonder there seems to be some confusion.

So let us try to purge our mind of this arithmetised geometry. (I did not find this easy and it took me some years spent with some good alternatives.) One problem we then face is defining ratio or proportion. (Note that this problem is not now directly concerned with incommensurability.) I have already pointed out that we have no real difficulty in fitting proportionality into the pre-Eudoxan style of mathematics. But our only evidence on how early Greek mathematicians actually handled ratio or proportion is the celebrated passage in Aristotle's *Topics* 158b29ff on *antiphrasis/antiphrasis* which suggests the use of the so-called Euclidean algorithm:

Given two numbers or two lines (or, with a bit of technique at our disposal, two more complicated geometrical objects), then see: how many times the second can be subtracted from the first; how many times the remainder can be subtracted from the second; how many times the next remainder can be subtracted from this remainder; etc. and this gives a string of numbers that characterise the relationship of size between the two things.

For example, the ratio of 60 to 26 will be twice, three-times, four-times exactly. Do the process for two numbers, and you will quickly see that it must stop after a finite number of steps, because if not "an infinite series of numbers will [arise], each of which is less than the other, which is impossible in numbers" (as formulated in *Elements* VII 31). Now do it with two lines. Early Greek geometers seem to take little notice of the philosopher's problems, and their lines can be chopped up indefinitely, so the possibility of the antiphrasis going or indefinitely presents itself. If you are a geometer, possessed of a bit of technique, and this question poses itself, you will soon find examples of it happening, as I shall illustrate below. Thus, as Aristotle might be saying, we could "learn the cause" and "perceive the explanation" of incommensurability in this way of thinking about the ratios of lines. But if, as Pappus might be saying, we involve geometry in "standards" of measurement", which I take here as a hint of

arithmetisation, then we encounter problems, as I have tried to explain.

Here then are some illustrations. First consider the problem of 'the diagonal and the side': draw any regular polygon, and evaluate the antiphrastic ratio of one of its diagonals and its side. The easiest and best known example is the pentagon, so I leave that for the reader, and will consider here the square. We easily see that the side S of a square goes once, but not twice, into its diagonal D (this follows from Socrates' comments at Plato's *Meno*, 82a-85d), and so the ratio (diagonal to side) is once, followed by the ratio (side to diagonal minus side). We are now faced with the evaluation of this ratio of the side to diagonal minus side, and some may feel that, like *Meno* at 84a, that we have made little progress. "It's no use Socrates, I just don't know". But, just as Socrates unblocks that impasse by conjuring a clever figure out of thin air, so I here draw Figure 1: Starting from the small diagonally placed square in the left-hand corner, with side s and diagonal d , we construct a larger square whose side S is $s+d$, and check that its diagonal D is $2s+d$.¹³

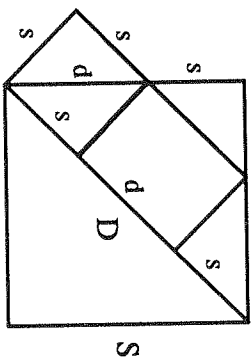


Fig. 1.

(Note how the symbols, here and below, are pure shorthand for lines and contain no hint of arithmetisation.) With one eye on Figure 1, and with the insight that the ratios are unaffected by the scale or orientation of the figures involved, we take up our problem again, and see that the ratio (big side to big diagonal minus side) is the same as the ratio (little side plus diagonal to little side); and we can now evaluate this as twice, followed by the ratio (little side to little diagonal minus side) which is – scale and orientation aside – what we just started from. Hence the ratio (side to diagonal minus side) is twice, twice, twice, twice, continuing thus indefinitely, and so the ratio (diagonal to side of a square) is once, twice, twice, twice, ...

Let me evaluate this same ratio another way; or, to be more exact, let me give another proof that ratio (diagonal plus side to side) is twice, twice, twice, twice, ... Start with a square P and, by adding on a gnomon $Q+R+S = P$, as in Figure 2, construct a larger square of size $2P$, whose side will therefore be the diagonal of the original square; and then append a rectangle T equal to Q , as shown. Then ratio (diagonal plus side to side)

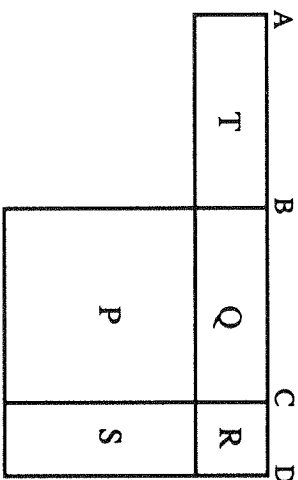


Fig. 2.

will be ratio (AD to BC), which we immediately see is twice, followed by ratio (BC to CD). The required proof will now follow if we can show that ratio (BC to CD) is equal to ratio (AD to BC); or equivalently, by a manipulation of geometrical proportions (see *Elements* VI 16; this manipulation was alluded to earlier), that rectangle (AD, CD) = square (BC), that is $T+Q+R = P$, but this underlies our construction, since $T = Q = S, Q.E.D.$

The case of the ratio of the longer diagonal to side of a hexagon gives rise to the ratio $\sqrt{3}$ to 1, where $\sqrt{3}$ denotes the side of the 3-fold square (which can be constructed, for example, using *Elements* II 14); and this example can be generalised to the investigation of the ratios like $\sqrt{n}$ to $\sqrt{m}$ – what I propose to call the problem of ‘the dimensions of squares’. We first use numerical techniques very close to those found in *Elements* VII to explore and conjecture what the answer might be. The second kind of proof above then deals with the simpler cases; a comparison of the mechanisms of Figure 2 and *Elements* II 11, gives us insight into the particular example of the ratios n -times, n -times, n -times, ...; and yet more heuristic exploration, followed by a generalisation of Figure 1 which

involves *Elements* II 13 & 14, gives a complete solution of this remarkable problem and a new interpretation of the whole of *Elements* II.

An analogous problem of ‘the dimension of cubes’ beckons, but proves to be of such redoubtable difficulty that in fact it remains unsolved today; Plato’s remarks at *Republic* 528b-c are still perfectly applicable! We can also explore related problems such as ‘the circumdiameter and side’ of polygons or polyhedra and see where they lead: the can provide explanations of the roles of *Elements* IV, X, and XIII. And the similar problem of ‘the perimeter and the diameter’ might be what lay behind Archimedes’ original calculation in his *Measurement of a Circle*.

Developing these ideas takes us into a different world. The ingredients are all early Greek, but they are fitted together to create a completely different picture. It is mathematically appealing, amazingly coherent (too coherent, in fact, for my comfort), and consonant with a quite extraordinary breadth of our evidence. And the topic with which I started, the simple discovery of incommensurability, plays no significant part in it, which is why my book contained no discussion of it beyond the bald and sceptical catalogue of our evidence at the end of its penultimate chapter.

I am constantly tinkering with the details of my book, so I shall finish with one final little addition to it, in the first of my dialogues, where my slaveboy discovers this antiphrastic definition of ratio under Socrates’ prompting. They start doing it on numbers (heaps of stones, in fact) and the slaveboy realises that the process must terminate. Then Socrates introduces the possibility of doing it with lines. At this point, at the end of B₃₆, on p. 28, add: ‘I wonder if it now can go on for ever’. That’s my slaveboy realising one of the causes and explanations of incommensurability, and it is just a passing remark on the way to discovering the much more remarkable things alluded to in S₃₇–S₄₃ and described above, alongside which this simple fact of incommensurability fades into insignificance.

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NOTES

¹ *Punch*, April 24, 1974. This magazine provided a humorous commentary on British life for 150 years until its closure in March 1992. In recent years, it ran a regular column ‘Country Life’ which explained itself as follows: ‘Not everything that happens in Britain gets into the national press. This feature presents some of the news that never made it.’ One reader sent

in the following clipping: "The programme, which is about the development of number systems, will include an interview with David Fowler, of Warwick University, on the historical crises associated with the square root of two." This must ultimately have come from some Open University publicity about a TV programme I had helped make for their first History of Mathematics course, which had been then been passed on by my university, picked up by the local newspaper, *The Leamington Spa Courier*, and submitted to *Punch* by a local reader. This programme was my first and reluctant venture into Greek mathematics, and I later disowned it, only permitting the Open University to continue broadcasting it if it also circulated a disclaimer by me to the students doing the course!

² Hereafter I shall refer to these as 'my book' and 'the sequel'.

³ At the outset, I must admit that one piece of evidence of late provenance plays a crucial role in the reconstruction, namely the material on 'side and diagonal' numbers and lines, found in Theon of Smyrna, Iamblichus, and Proclus. See the discussion in my book, pp. 100–4, which however does not deal with the material in Iamblichus (ed. Pistelli, 91.3–93.6).

⁴ H. Thesleff, *Platonic chronology*, on p. 18, n. 47.

⁵ I here leave to one side the notorious 'nuptial number' at *Republic* 546b, with its talk of the 'rational and irrational diameters of five'.

⁶ These quotations are taken from a new English translation, by G. Clark: *Iamblichus: On the Pythagorean Life*, Liverpool, 1989.

⁷ The translation has "symmetry and asymmetry", but surely (in)commensurability is the appropriate translation here of (α) *summetros*, as in the phrase "irrationality (αλογος) and incommensurability" at end of the passage. The later "twenty-angled shape" is *eikosagonon*, a non-standard name (perhaps found only here in Iamblichus) for *dodekaton*.

⁸ See Freudenthal, 'Y avait-il une crise de fondements...', and Knorr, *Evolution*, 306–12.

⁹ Contrast this with the sparse sections in Thomas, *Selections* i pp. 110–11 & 214–17, on 'The Irrational', where this and a passage from Aristotle are the only texts excerpted.

¹⁰ L. Y. Zihnd, "'All is number'?", "Basic doctrine" of Pythagoreanism reconsidered'.

¹¹ For a discussion and examples, see my '400 years of decimal fractions', '400.25 years of decimal fractions', and 'Dedekind's theorem: $\sqrt{2} \times \sqrt{3} = \sqrt{6}$ '.

¹² The only arithmetised passage I know, anywhere up to Archimedes and beyond, is in Plato's *Menon*, 82c ff, where Socrates says to the slaveboy: "Now if this side is two feet long, and this side the same, how many feet will the whole be?", and the passage continues in this arithmetised vein for the slaveboy's first two attempts. (I thank Wiburg Knorr for pointing this out to me in January 1991 in a train somewhere between Verona and Venice; I had used this passage for the introduction to my book without appreciating this feature!) The switch to geometry is then indicated by Socrates when he tells the slaveboy: "If you don't want to count it up, just show me on the diagram" (84a). And I think one can explain this singular exception by observing that the slaveboy is not a mathematician – that is the whole point of the episode. Note that an arithmetisation of aspects of everyday life occurs when a barter economy gives way to the use of money, which seems to have happened in Greece by the 7th century.

¹³ This my proposed interpretation of the figure being described in the texts on side and diameter lines; see note 3, above.

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