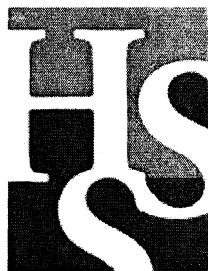


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Author(s): A. S. Saidan, Abu'l-Wafa', Muhammad ibn Muhammad ibn Yahya and al-Buzajani

Source: *Isis*, Vol. 65, No. 3 (Sep., 1974), pp. 367-375

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The Arithmetic of Abū'l-Wafā'

By A. S. Saidan*

Abū'l-Wafā', Muḥammad ibn Muḥammad ibn Yaḥyā, al-Būzajani (940–998) is known to the scholarly world for his contributions to astronomy and trigonometry.¹ What is here called the arithmetic of Abū'l-Wafā' refers to a work of his preserved for us in MSS Leiden, Or. 103, and Cairo, Riyāḍa 42 M. Medovoi is familiar with these two manuscripts and has written a study in Russian based mainly upon the former manuscript.² I have published the whole work in *Arabic Arithmetic*, using both manuscripts, with an introduction and commentaries based upon a comparison with other Arabic works of the same type.³ The purpose of this paper is to present an analysis of Abū'l-Wafā's work in English.

Two arithmetical systems co-existed in Islam: finger reckoning and Hindu arithmetic. The former, called in Arabic *ḥisāb al-yadd* (hand arithmetic), *ḥisāb al-'uqūd* (finger-joint arithmetic), or *ḥisāb al-Rūm Wa'l-'Arab* (arithmetic of the Byzantines and the Arabs), seems to be a survival of a Greco-Babylonian manipulative practice. Hindu arithmetic, called *al-ḥisāb al-Hindī* (Indian arithmetic), *ḥisāb al-takht* (board arithmetic), or *ḥisāb al-ghubār* (dust arithmetic), was imported from India with the numerals. Abū'l-Wafā's arithmetic is of the finger-reckoning type, which has the following important characteristics:

It has no numerals. Numbers are written in words, and calculations are performed mentally. To remember the results of intermediary steps, calculators bent their finger joints in conventional ways which enabled them to indicate whole numbers from 1 to

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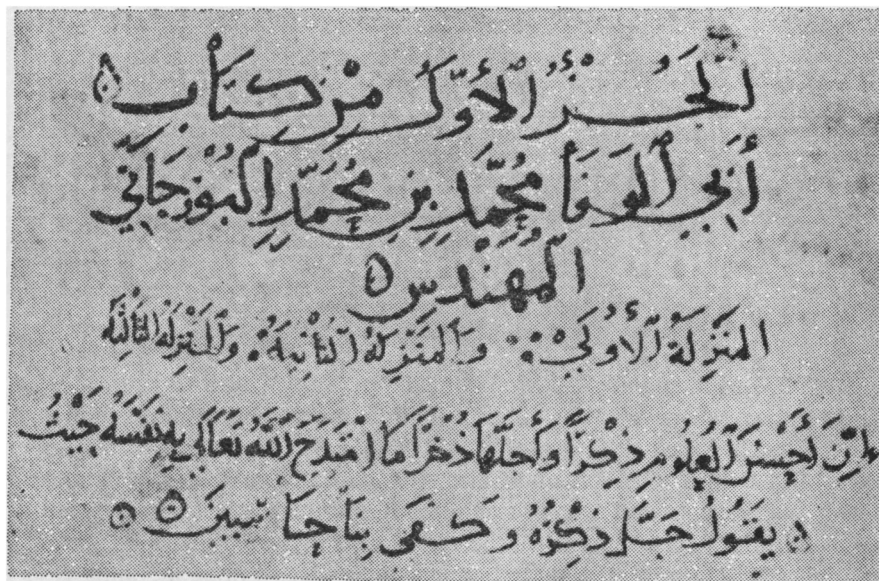
* Department of Mathematics, Jordanian University, Amman, Jordan.

¹ The following are the major references for the study of Abū'l-Wafā' and Arabic arithmetic: George Sarton, *Introduction to the History of Science*, Vol. I (Baltimore: Published for the Carnegie Institution of Washington by the Williams & Wilkins Company, 1927). Carl Brockelmann, *Geschichte der arabischen Literatur*, 2 vols. + 3 supplements (Weimar: E. Felber, 1898–1902; Leipzig: C. F. Amelangs Verlag, 1901, 1909; Leiden: E. J. Brill, 1943–). Bernard Carra de Vaux, *Les penseurs de l'Islam*, Vol. II (Paris: P. Geuthner, 1921). E. S. Kennedy, *A Survey of Islamic Astronomical Tables* (Transactions of the American Philosophical Society, N.S. Vol. 46, Pt. 2) (Philadelphia: American Philosophical

Society, 1956). E. S. Kennedy, "The Arabic Heritage in the Exact Sciences," *Al-Abhath*, Dec. 1970, 23:327–344. Heinrich Suter, *Die Mathematiker und Astronomen der Araber und ihre Werke* (Leipzig: Teubner, 1900). A. P. Juschkewitsch, *Geschichte der Mathematik im Mittelalter*, trans. from the Russian by Viktor Ziegler (Leipzig: Teubner, 1964). A. H. Sabra, "'Ilm al-Ḥisāb," *Encyclopaedia of Islam* (2nd ed.; London: Luzac & Co., 1971), pp. 1138–1141.

² M. I. Medovoi, "Oh arifmeticheskoy traktate Aby'l-Wafy' (The arithmetical treatise by Abū'l-Wafā'), *Istoriko-Matematicheskies Issledovaniya*, 1960, 13: 253–324.

³ A. S. Saidan, *Arabic Arithmetic. The Arithmetic of Abū'l-Wafā'* (in Arabic), Part I (Amman: Jam'iyyat 'Ummāl al Maṭābi', 1971).



The initial page of the MS Leiden Or. 103.

9,999. This same device was repeated to indicate numbers from ten thousand onward.

It contains three sets of fractions. The first, here called the Arabic traditional system, supposedly arose from the fact that in Arabic there are only nine single names for fractions. These are the names of $1/2$, $1/3$, . . . , $1/10$. These fractions are called the *ru'ūs*, literally "heads," that is, principal fractions. In addition, there are the following subsets:

- (1) Composite fractions, which consist of two categories:
 - (i) $2/3$, $3/4$, . . . , $9/10$
 - (ii) Combinations of *ru'ūs*, like $1/2 + 1/7$, . . .
- (2) Fractions of fractions, like $1/2 \times 1/6$.
- (3) *Aṣamm*, that is, "deaf," fractions. These are fractions like $1/11$, $2/11$, which cannot be expressed exactly in terms of the other types.

Rules are stated, and always traditionally observed, to express every common fraction a/b in terms of *ru'ūs*, combinations thereof, and/or fractions of fractions. Thus $1/12$ is expressed as $1/2 \times 1/6$, which is said to be better than $1/3 \times 1/4$. Similarly, $5/12$ is expressed as $1/3 + 1/2 \times 1/6$, whereas $7/12$ is converted to $1/3 + 1/4$. Even the *aṣamm* fractions have to be approximated.

This practice was gradually overshadowed by the Indian concept of the common fraction a/b , but it survived till late in the Middle Ages. Arabic authors seem to have thought that it was of Arabic origin, but we have reasons to believe that it was a traditional practice coming down from the ancient Egyptian concept of the common fraction; Proclus in the fifth century expressed $23/25$ in the same way that Abū'l-Wafā' would have done.

The second set of fractions is that of the scale of sixty. This was mainly the concern of astronomers but found its way to problems of everyday life and served the purposes that percentages serve today.

The third set arose from the local units of money, weights, and measures. Where 1 *dirham* = 6 *dānigs* = 48 *ḥabbas*, the *dānig* is used to denote $1/6$, and the *ḥabba* $1/48$.

However intricate it may look to us, finger reckoning was the system from which Arabic algebra, trigonometry, and higher mathematical notions were developed. It was already well established in Islam before the Hindu system made its first appearance in the Arab world. The Hindu system brought the numeral forms, the concept of the common fraction, and well-defined algorithms on the dust board. The amalgamation of the two systems resulted in the arithmetic handed over to the Latin arithmeticians of the Middle Ages. Arabic works on finger reckoning deal invariably with ratio, multiplication, and division in application to everyday problems. Finding the unknown quantity leads to algebra, and mensuration leads to root extraction and to trigonometry. Arabic works of Hindu arithmetic, on the other hand, start by introducing the nine numeral forms, the concept of place value, and the zero form, and then deal with addition, subtraction, multiplication, division, and roots. Business arithmetic, and therefore algebra and trigonometry, hardly found a place in early works of this type. They appear in later works where the two types are unified.

TEXT AND MANUSCRIPTS OF THE ARITHMETIC

Abū'l-Wafā's treatise is the earliest work on finger reckoning that has survived, and the most important. The author gave it the name *Fī mā Yaḥtāju ilai-hī al-Kuttāb wa'l 'Ummāl wa-Ghairuhum min 'Ilm al-Hisāb* (On what scribes, officials, and others need of the science of arithmetic). As the name implies, it was written for state officials, and therefore besides its value for historians of science it gives insight into tenth-century life in Islam from the administrative point of view.

The treatise is divided into seven parts called *manāzil*, literally "places." That is why the Cairo copy gives it the shorter name of *Kitāb al-Manāzil al-Sab'*, book of the seven places. Each part is in seven chapters.

The Leiden copy (112 folios), consists of the first three parts. The Cairo copy (230 folios), starts with part II, chapter 2, and goes to the end. It bears two sets of pagination: a recent one that extends from 1 to 230, and an old one that extends from 50 to 265 (the remaining fifteen folios which do not have this old pagination are in a different hand.) There are a few folios missing: part II, chapters 3 and 4; part IV, chapter 7; and part VII, chapters 1 and 2. The first folio is rather worn out. The book ends on folio 225 with the statement that copying was finished on Friday, 3 Ḥijja, 487 (Dec. 15, 1094). Folios 226–230 give problems of types discussed in the text and contain nothing of interest to us except that some numbers are denoted by Hindu numerals.

CONTENTS OF THE ARITHMETIC

The book starts with a long table of contents (pp. 65–69 in my printed edition, which will hereafter be noted in parentheses). The following summary will probably be more meaningful to the modern reader:

Part I, On Ratio (pp. 71–122).

Chapter 1 (pp. 71–74): definition of ratio, the four types of Arabic traditional fractions, rules to be observed in expressing ratios in the traditional way.

Chapter 2 (pp. 75–78): principal, composite, and fractions of fractions expressed as parts of 60.

Chapter 3 (pp. 79–80): expressing $n/60$ in the traditional way, n being a whole number.

$$\text{Example: } 25 = 15 + 10 \rightarrow 25/60 = 1/4 + 1/6$$

Chapter 4 (pp. 81–89): expressing $n/60$ in the traditional way, n being a whole number plus a principal or composite fraction.

$$\text{Example: } 11 \frac{1}{3} = 10 + 1 \frac{1}{3} \rightarrow (11 \frac{1}{3})/60 = 1/6 + 1/5 \times 1/9$$

Chapter 5 (pp. 90–112): expressing $n/60$ in the traditional way, n being a whole number plus a fraction of a fraction.

Chapter 6 (pp. 113–119):

Section 1: expressing a/b first as $x/60$, secondly in the traditional way.

$$\text{Example: } 7/15 = 28/60 = (12 + 10 + 6)/60 = 1/5 + 1/6 + 1/10$$

Section 2: approximating *ašamm* fractions.

$$\begin{aligned} \text{Example: } 3/17 &= (10 \frac{10}{17})/60 = 11/60 \text{ approximately} \\ &= (10 + 1)/60 \\ &+ 1/6 + 1/6 \times 1/10 \end{aligned}$$

For a better approximation, note that $10/17 = (35 \frac{3}{5})/60$
 $= 35/60$ approximately
 $= 1/3 + 1/4$

$$\begin{aligned} \therefore 3/17 &= (10 \frac{1}{3} + 1/4)/60 \\ &= (6 + 3 \frac{1}{3} + 1 - 1/4)/60 \\ &= 1/10 + 1/2 \times 1/9 + 1/6 \times 1/8 \end{aligned}$$

For a still better approximation the $3/5$ which was neglected above can be treated still further.

Chapter 7 (pp. 120–122): further examples.

Part II, On Multiplication and Division (pp. 123–201).

Chapter 1 (pp. 124–131): definitions and preliminary remarks. Multiplication and division are defined according to Euclid and Nicomachus. The operations are split into the following types, where H stands for whole numbers and F for fractions:

- (i) $H \times H$, $F \times F$, $H \times F$, $(H + F) \times H$, $(H + F) \times F$, and $(H + F) \times (H + F)$
- (ii) H/H , F/F , H/F , F/H , $(H + F)/H$, $H/(H + F)$, $(H + F)/F$, $F/(H + F)$, and $(H + F)/(H + F)$

Chapter 2 (pp. 132–159): multiplication and division of whole numbers.

1. Every number is composed of *marātib*, places, and *'uqūd*, literally, knots or finger knots. Thus three hundred and two is composed of three in the hundreds place and two in the units place; the three and two are the *'uqūd*. Places are units, tens, hundreds, repeated for thousands, repeated again for thousands of thousands and so on. *'Uqūd* may be one, two, . . . up to nine.

2. Multiplication of places by places. The rules here given lead to the formula $10^m \times 10^n = 10^{m+n}$

3. Division of places. Rules are here given leading to $10^m \div 10^n = 10^{m-n}$

4. Multiplication and division of the *'uqūd*. The multiplication table from 1×1 to 9×9 is assumed known.

5. The basic algorithm of multiplication appears from the following example, expressing in numerals what the author states in words:

46×28 : 2 places in each; $2 \times 2 = 4$, $\therefore$ 4 multiplications:

$40 \times 20 = 800$; $40 \times 8 = 320$; $6 \times 20 = 120$; $6 \times 8 = 48$

total 1288

Other devices are suggested, mainly for keeping order and avoiding oversight. The following is worth noting. The author suggests it to avoid Indian schemes, but it is rather Indian in disguise:

To multiply 489 by 67, go on as follows:

		seven	six
	nine	eight	four
	fifty-four	forty-eight	twenty-four
sixty-three	fifty-six	twenty-eight	

Add up, starting from the left, inserting the units and carrying the tens as units in the next place to the right: thus you start by putting down three and carrying six to the second place. Finally you obtain

three six seven two three

which means that the answer is 32763.

Further examples follow. For the zero the curved form $\sim$ is suggested.

6. Division is taken up gradually: division by numbers less than ten, division by tens, division by a number of two places, division by a number of many places. The schemes correspond generally to the schemes of multiplication.

Chapter 3 (pp. 160–173): the common denominator, addition, and subtraction of fractions.

When we think of a fraction we associate it with the picture of two numbers one above the other, the lower being the denominator. Finger reckoning did not have such a picture. Instead, the computer thought of two thirds and associated it with the (least) whole number whose two thirds made a whole number. What we call the denominator was called by him the *makhraj* (outcome), or the number from which the fraction comes out (as a whole number).

In this chapter Euclid's algorithm of the common factor is given and used to yield the common denominator. This is utilized in addition and subtraction of fractions.

Another method of addition and subtraction is given under the name of the method of scribes. This requires converting fractions to the scale of sixty and re-converting the sum or difference to the traditional Arabic form.

Chapter 4 (pp. 174–185): multiplication and division of fractions.

Here two kinds of fractions are discussed. Those we have so far encountered are absolute fractions. Two schemes for multiplying and dividing these are given: one scheme is much like what we now use, and the other resorts to the scale of sixty.

The other kind of fractions is a set of what are called *al-kusūr al-muntasibs* (related fractions). These arise from the units of currency in use. The arithmetic of these is much like that of duodecimals in the former British monetary system. What makes it more complicated is that the units of currency were never standardized in the Islamic world; they changed from time to time and from place to place.

Chapters 5, 6 (pp. 186–192): further multiplication and division of fractions.

Taking again H and F to denote whole numbers and fractions respectively, the types here discussed are:

$H \times F$, H/F , F/H , $(H + F) \times H$, $(H + F/(H, H))H + F$, $(H + F) \times F$, $(H + F)/F$, $F/(H + F)$, $(H + F) \times (H + F)$, and $(H + F)/(H + F)$

Chapter 7 (pp. 193–201): short methods of multiplication and division.

Here we meet one of the characteristic topics of finger reckoning that we hardly find in the Hindu schemes—shortcuts in multiplication and division. Finger reckoning faded away gradually, but these shortcuts continued to grow. The following are the rules given by Abū'l-Wafā':

1. Rules of multiplying and dividing by 5, $2\frac{1}{2}$, $33\frac{1}{3}$, ... and generally, $10^n/m$, n being any integer, and m a small integer like 2, 3, 4, ...

2. The above rules are utilized in multiplying by 15, $15 + 1$, ... Thus $15 \times n = n + 1/2n$ tens; $14n = (n + 1/2)$ tens $- n$.

3. $(10m + a)(10m + b) = (10m + a + b) \times 10m + ba$. Thus $59 \times 54 = (59 + 4)$ fifties $+ 9 \times 4$.

4. $(10mn + a)(10n + b) = (10mn + a + bn) \times 10m + ab$. Thus $64 \times 28 = (64 + 3 \times 8)$ twenties $+ 4 \times 8$.

5. $(10m + a)(10m + b) = (10m + a + b - 10) \times (m + 1)$ tens $+ (10 - a)(10 - b)$. Thus $59 \times 54 = (59 + 4 - 10)$ sixties $+ (10 - 9)(10 - 4)$. This rule is applied to the simple case of 5×3 . Here $10m + a + b - 10 = -2$, which is called a debt of 2. This is probably the earliest Arabic reference to negative quantities.

6. Multiplication and division of powers of 10. Thus $1200/16 = (12/16 = 3/4)$ of a hundred.

Part III, Mensuration (pp. 202–276).

Chapters 1, 2 (pp. 205–221): units of length, area, and volume; conversion.

Chapter 3 (pp. 222–233): mensuration of the circle.

The following are the rules here given. (d = diameter, r = radius, A = area, c = circumference, l = arc length, h = height of segment):

- | | | |
|-----------------------------------|------------------------------|----------------------------|
| 1. $A = r \times c/2$ | 2. $A = d^2 \times 11/14$ | 3. $c = (22/7)d$ |
| 4. $c = \sqrt{4A} \times 22/7$ | 5. $A = (c/2)^2 \times 7/22$ | 6. $A = c^2/11 \times 7/8$ |
| 7. $h = r - \sqrt{r^2 - (l/2)^2}$ | 8. $l/2 = \sqrt{h(d - n)}$ | 9. $d = h + (l/2)^2/h$ |

10. A table of sines: If a semicircle is drawn with center M and diameter AMB , and points $C_1, C_2, \dots$ are taken so that the arcs $AC_1 = C_1C_2 = C_2C_3 = \dots =$ one unit, then, with $\pi = 22/7$, the semicircle is divided into 22 equal arcs. $\angle AMC_1 = 90^\circ/11 = m_1$, $\angle AMC_2 = 2m_1 = m_2$, $\angle AMC_3 = 3m_1 = m_3$, and so on. The author gives the lengths of the arcs $AC_1, AC_2, \dots$ and the lengths of their chords. Since chord $AC_r = 2r \sin 1/2 m_r$, the lengths of the chords gives sines of angles from 0° to 90° at intervals of $90^\circ/22$. Abū'l-Wafā' does not show how he calculates these lengths, but he says that he uses Ptolemy's theorem with approximations. His figures are correct to three decimal places. This table is henceforth used to find sines of given angles or arcs of given sines. The famous geometrical theorem attributed to Ptolemy is the one used in building up the sine tables in the *Almagest*. Al-Bīrūnī, however, and other Muslim

astronomers pointed out that this method of building up the table is laborious. Al-Bīrūnī wrote a treatise called *Istikhrāj al-awtār fī al-dāira* (extracting [lengths of] circular chords), in which he suggests easier methods. Abū'l-Wafā' prepared and used for astronomical purposes a more elaborate sine table in which the sine is given for each minute of the argument, to three sexagesimal places.⁴ But it is unlikely that he improved upon Ptolemy's method of calculating these chords.

For intermediary angles he uses linear interpolation.

11. He gives the formulas for the areas of circular sector and segment.

Chapters 4, 5 (pp. 234–261): mensuration of triangles, quadrilaterals, polygons, and composite figures.

About the triangle, practically everything we usually find in books of numerical geometry is given here, including the area in terms of the perimeter and the cosine rule. The same applies to quadrilaterals, except that the author makes no mention of the cyclic quadrilateral.

If a regular polygon has n sides of length l each, and the inscribed circle is of radius r , then the area of the polygon is $1/2nlr$. To find this r he uses his table of sines but adds a rule which he attributes to the Indians and criticizes as inaccurate and not built on deductive reasoning. The rule is

$$4r^2 = 2/9 l^2 \{n/2^{(n-1)} + 3\}$$

The composite figures he treats are:

1. rectilinear figures surmounted by circular segments, and
2. areas included between two intersecting circular arcs.

Chapter 6 (pp. 262–268): volumes.

The solids here considered are the cube, the cuboid, the prism, the cylinder, the cone, the frustum of the cylinder, the frustum of the cone, the sphere, and the spherical sector.

Chapter 7 (pp. 269–276): on finding distances.

A crude instrument is suggested for finding distances, heights, and depths where these cannot be found by direct measurement. The instrument consists of a rectangular board half an armlength wide and two or more armlengths long. All sides are graduated. To one corner a pointer is attached carrying two small hollow blocks. Let us say we want to know the width of a river we cannot cross. The board is fixed on a stand of a known height at one bank. The pointer is directed so that if you look through the blocks you see the other bank. By finding at which points along the perimeter the pointer meets the board, we can easily find the width of the river. The principle is obviously that of similar triangles.

Part IV, On Taxes (pp. 277–301).

Chapters 1, 2 (pp. 278–282): kinds of taxes and principles that govern their evaluation. The following are the main ideas:

1. Landowners pay to the state in money or in kind, at rates that differ, but payment involves three kinds of dues:

1. payment per unit of land, called the *tasq*.

⁴ Kennedy, *Survey of Islamic Astronomical Tables*, p. 145.

- ii. payment to the surveyor, called the *ayīn*.
 - iii. payment to the accountant, called the *rawāj*.
 - 2. The Euclidean notion of proportion applies to all.
- Chapters 3–7* (pp. 283–301): problems of tax calculations.
- The following six units are involved:

1 *dirham* = 6 *dānigs* = 60 'ushairs of currency

1 *jarib* = 10 *gafīzes* = 100 'ushairs of capacity

Manipulations involving these units require special attention. We should remember that each of these has a twofold significance. Thus a *dānig* is a unit of money and may also denote one sixth. Similarly the *gafīz* is a unit of capacity and may also denote one tenth. Thus $a \text{ dānigs} \times b \text{ gafīzes} = a/6 \times b \text{ gafīzes}$ if the required result is expected to give some capacity. It is $b/10 \times a \text{ dānigs}$ if the result is that of money. The nature of the problem decides which result should be taken.

Part V, On Exchange and Shares (pp. 302–329).

Chapter 1 (pp. 303–306): kinds of *kurrs* and conversion from one kind to another. The *kurr*, a unit of capacity for measuring crops, varied in value from place to place. Four such values are given with tables of conversion.

Chapters 2, 3, 4 (pp. 307–315): kinds of crops, their standard values, exchange.

Crops are divided according to price into four major kinds. Under barley, for example, are placed all crops whose price is more or less equal to that of barley. The relative values are given and the problem of exchange arises. The problem becomes worth skillful consideration when the crops to be exchanged are given in different *kurrs*. A conversion table is supplied for this purpose.

Chapters 5, 6, 7 (pp. 316–329): problems related to harvest.

- 1. Distribution of shares; the share of the state and that of the tax collector.
- 2. Assessment of total price.
- 3. Harvest valued by different units of capacity.

Part VI, Miscellaneous Topics (pp. 320–346).

Chapter 1 (pp. 331–336): conversion of payment in kind to cash.

Chapter 2 (pp. 337–340): exchange of money units.

Chapter 3 (p. 341): payment in kind or cash with respect to weight.

Chapter 4 (pp. 342, 343): pay and annuities of soldiers.

Chapter 5 (pp. 344, 345): fodder for animals as entitlement of owners in civil service.

Chapter 6 (p. 346): impost dues; (not complete).

Chapter 7: problems related to mail and official journeys (missing from the manuscript).

Part VII, Further Business Topics

Chapter 1 (missing): units of weight.

Chapter 2 (missing, except for a few lines, pp. 346, 347): conversion of weights and related topics.

Chapter 3 (pp. 348–353): assessment of wages.

Chapter 4 (pp. 354–357): wages paid in kind.

Chapter 5 (p. 358): wages for mortaring and similar labor.

Chapter 6 (pp. 359–360): wages for bricklaying.

Chapter 7 (pp. 361–367): five problems, on fractions and streams filling a well.

TOPICS NOT TREATED BY ABU'L-WAFA'

Abū'l-Wafā's arithmetic is lengthy compared to the other Arabic books on finger reckoning that have come down to us. Yet we miss in it some important topics that we find in the others.

Algebra: In connection with proportion all other works consider the unknown and usually present in a seemingly traditional way the types of equations that may be encountered. Abū'l-Wafā' ignores these. In one case at least he obtains a quadratic equation and gives its positive root without telling how he does it. He states, however (p. 132), that he has written a commentary to the algebra of al-Khwārizmī. This is probably why he overlooks algebra in this work: he has dealt with it elsewhere.

Extraction of roots: In all Arabic works of arithmetic this is included as a basic operation. Abū'l-Wafā' gives the roots encountered in connection with mensuration but does not tell how he obtains them. Is this because he had discussed this in some other work? I am inclined to give a negative answer. Finger reckoners must have had mental ways for obtaining roots, exactly or approximately, but the practice of presenting these as a branch of arithmetic came probably with Hindu arithmetic, which Abū'l-Wafā' did not master.

Casting out nines: All other works, whether of finger reckoning or of Hindu arithmetic, use casting out nines for checking results. Later ones use casting out eights or elevens as well. Some medieval Arabic commentators state that casting out nines is an Indian practice. The odds are, therefore, that this is a practice which Abū'l-Wafā' did not know of. There are places in this work where ignorance of the Indian practice is shown and others where it is deprecated.