

Science and Philosophy in Classical Greece
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On the Notion of a Mathematical Starting Point in Plato, Aristotle, and Euclid

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In this paper I wish to discuss the question of the status of the starting points of mathematics in the philosophies of Plato and Aristotle and in Euclid's *Elements*.¹ I will be mainly concerned with Aristotle since he has a good deal more to say on the question than Plato, and for Euclid we have only his practice to interpret. It is useful to have as a model for discussion some modern conception of mathematical starting points. I here present one briefly; it is designed to accommodate discussion of the ancients. We may begin with a division of starting points into terms, assertions, and rules; and a second division into logical and material. I give a rough illustration of each of the six resulting categories:

- primitive logical term: not ($\neg$)
- primitive material term: point
- primitive logical assertion
 - For any assertion P , $\neg$ both P and $\neg P$ (law of non-contradiction)
- primitive material assertion

If a and b are distinct points, there exists a third point c between a and b

¹ I use the term 'starting point' as a general term to cover a multitude of Greek expressions. I do not use the word 'principle', which scholars often use in the way I am using 'starting point', in order to avoid giving the impression that I am discussing the Greek word ἀρχή. In the *Elements* the starting points are the propositions Euclid labels definitions (ὅροι), postulates, and common notions. In the case of Plato I will be dealing primarily with the passage in the *Republic* in which Socrates talks about what he calls the hypotheses of the mathematicians. Aristotle uses a variety of terms in this connection, as will be seen in Section 2.

- primitive logical rule

If the assumption of P leads to a contradiction, then you may take $\neg P$ to be true (rule of *reductio*); if you have proved P and if P then Q you may say you have proved Q (*modus ponens*)

- primitive material rule

If you have proved A and B to be congruent to C , you may take A to be congruent to B .²

The distinction between logical and material starting points is usually made by specifying certain terms as logical, and making it a necessary condition for something to be a logical assertion or rule that it employ only logical terms. The distinction between logical and material has been argued to be arbitrary or at least unjustified; but these arguments are of no concern here. I shall simply assume that we have made a division about which we can all agree, and that the primitive logical terms include those of a standard formulation of the predicate calculus. I shall also assume that all theories use the same primitive logical terms, assertions, and rules, since to do otherwise needlessly complicates discussion. It will, however, be necessary to return briefly to the distinction between the logical and material in discussing Aristotle.

I shall also have nothing more to say about the notion of a primitive material rule until I discuss Euclid in the next section, since one of the simplifying features of standard specifications of the logical is to make material rules unnecessary. So, for example, standard specifications of the logical will allow one to infer that A is congruent to B from ' A is congruent to C ', ' B is congruent to C ', and the material assertion ' $\text{If } A \text{ and } B \text{ are congruent to } C, \text{ then } A \text{ is congruent to } B$ '; hence, the example of a material rule just given can be replaced by this assertion as a starting point. We can, therefore, assume that material rules are absent from a typical theory.

I shall also assume that our primitive logical rules and assertions and our primitive material assertions are minimal in the sense that we could not prove the same set of assertions using a proper subset of the starting points.

²The qualifying adjective, 'primitive', indicates that what is qualified is a starting point.

In standard accounts the only material rules will be ones permitting the transformation of given assertions into other assertions on the basis of material facts, e.g., that congruence is a transitive relation. Primitive material assertions are themselves a rule of this kind, since they permit one to make assertions on the basis of no previous ones. In the *Elements* the first three postulates are material rules of a kind quite unlike any in standard modern theories, since these postulates license the construction of new objects rather than the introduction of new assertions.

And I shall assume as well that the set of primitive material terms is minimal, though saying precisely what that means is too complicated to merit the time it would take.³ The rough idea is that the provable assertions of the theory do not include a 'definition' of any of the primitive material terms of the theory, i.e., an assertion enabling one to translate every assertion P of the theory into an assertion P' not containing a primitive term t and such that P is provable if and only if P' is. I make an analogous assumption for primitive logical terms, but only because it enables me to say the following briefly: I call a theory the starting points of which are minimal, minimal. Hereafter, unless I use the qualifier 'logical', terms should be understood to be material terms.

There are two further points I wish to make about the starting points of a theory. The first is that a standard theory may have no logical assertions, but it cannot get by without at least one logical rule. However, it is of some significance, at least historically, that there is a logical assertion corresponding to any rule, one which, it is tempting to say, is the assumption made by the person using the rule. One might think of the law of non-contradiction as an expression of the rule of *reductio*, and of the assertion 'If both P and if P then Q , then Q ' as an expression of *modus ponens*. However, the more important philosophical point is that even if it is possible to replace any particular logical rule with a logical proposition, no reasoning can proceed without some rules. In general, switching back and forth between rules and their propositional expression obfuscates issues, and it seems best to imagine that the distinction between the two is fixed for any particular theory.

The second point concerns definitions. In modern discussions definitions are not treated as starting points. They are simply abbreviations of complex expressions introduced to make complex assertions more intelligible to us, e.g., enabling us to say '28 is perfect' rather than '28 is the sum of all its factors less than it, including 1'. The only terms which are starting points are the primitive ones. However, Aristotle seems to think of definitions as starting points, and they are the most common kind of starting point in the *Elements*. Perhaps the simplest way to accommodate this discrepancy is to add to the starting points of a minimal theory, a set of defined terms and a set of definitions, where for simplicity one assumes that the definition of a defined term contains only primitives. Since I want to use the word 'definition' in my discussion of Greek authors I shall call defined terms non-primitive and definitions abbreviations. Ignoring logical terms, we can

³Throughout this introductory discussion I pass over formal complexities involved in the treatment of definitions because taking them into account would not affect the issues I treat.

say that the starting points of a minimal theory with abbreviations include the following:

- the material component—
 - primitive terms
 - non-primitive terms
 - primitive material assertions
 - abbreviations,
- the logical component (identical for all theories)—
 - primitive logical assertions (possibly empty)
 - primitive logical rules.

1. Euclid's *Elements*

Euclid's *Elements* may be divided as follows:

- a. books 1–4, plane geometry
- b. book 5, proportion theory
- c. book 6, plane geometry, presupposing proportion theory
- d. books 7–9, number theory
- e. book 10, plane geometry presupposing proportion theory and number theory
- f. books 11–13, solid geometry presupposing plane geometry, proportion theory, and (via book 10) number theory.

I have formulated this description to stress the sense in which the *Elements* builds on previously developed theories, even though it is clear that Euclid develops theories much further than his subsequent applications of them require.⁴ However, despite this building it is also the case that new theories are introduced in books 5, 7, and 11, that is to say, theories with previously unused primitive material terms and assertions. Obviously I make this point from a modern perspective; and I mean that if we were to represent the *Elements* as a formal theory corresponding as closely as possible to the original, we would be forced to introduce new primitives at those

⁴ The only case of possible building which I have not included is Euclid's alleged use of proportion theory in number theory. I have omitted this because I believe Euclid conceived book 5 as a geometric proportion theory, introduced a theory of proportions for numbers in book 7, and then took for granted correlations between the two theories in book 10. However, nothing I say here is altered by supposing that Euclid's number theory presupposes the proportion theory of book 5. For this and other claims about the *Elements*, see Mueller 1981.

points. However, if we look at the *Elements*, although we find at the beginning of book 1 definitions, postulates, and common notions—postulates corresponding loosely to primitive material assertions, common notions to either primitive material assertions or primitive logical assertions—at the beginning of the remaining books we find only definitions. I believe there are two related inferences we can draw from this: (1) Euclid did not believe that proportion theory, number theory, or solid geometry required its own postulates; (2) at the end of the fourth century there were no accepted presentations of these theories which included postulates, and probably no such presentations at all, presumably because no mathematician recognized the need for them. A further inference I draw is that the idea of such presentations of any mathematical theory was relatively new in Euclid's time, i.e., did not precede Plato's maturity. I believe the evidence suggests that Euclid himself is responsible for the postulates, but for the moment I will only say that, even if they are thought to predate, say, Plato's *Republic*, they should still be seen as the exception rather than the rule by Euclid's time.

The rule in the *Elements* and, I am suggesting, earlier in the history of Greek mathematics is a theory, the only explicit starting points of which are definitions. These definitions are, for the most part, either explications, which perhaps clarify the significance of a term to the reader but play no formal role in subsequent argument, or abbreviations in the modern manner. Examples of the former are 'A point is that which has no part' and 'A unit is that in virtue of which each thing is called one'; an example of the latter is 'An obtuse angle is an angle greater than a right angle.' Occasionally an assertion creeps its way into a definition as when Euclid adds to the definition of the diameter of a circle that the diameter bisects the circle; but these exceptions may, I think, be disregarded as indications of what Euclid thought he was doing; and, in any case, the assertions which do appear in the *Elements* after book 1 come nowhere close to overcoming the absence of postulates. In Euclid's practice the terms which are explicated play something like the role of primitive terms in modern theories; but, except in his practice, Euclid shows no sense of a distinction between abbreviations, which play or could play a role in argument, and explications which do not and hardly could. Moreover, the comparison with primitive terms is very limited, since in a modern presentation one expects all and only primitive terms to occur among the material terms of the primitive assertions in their unabbreviated form; whereas in book 1 Euclid does not include (even implicitly) all explicated terms in the postulates and common notions, and he uses a lot more terms than anyone with some notion of modern axiomatic method could possibly hope to characterize satisfactorily in five propositions. The impression one gets from reading the whole

Elements is that the fundamental operative notion of a material starting point is a definition. One defines the things one is going to reason about in order to make sure others understand what one is talking about; some of these definitions are formally usable abbreviations; the others serve only to help the reader grasp what is being talked about.⁵

We know that the question of the appropriate postulates and common notions for book 1 was a matter of much discussion in later antiquity, and that the manuscripts of the *Elements* were affected by that discussion. We have no way of being certain what Euclid's lists included, but the most plausible course would seem to be to follow Heiberg and Proclus, and accept the following:

Postulates

1. Let it be postulated to draw a straight line from any point to any point, and
2. to produce a limited straight line in a straight line,
3. to describe a circle with any center and distance,
4. that all right angles are equal to each other,
5. that, if one straight line falling on two straight lines makes the interior angles in the same direction less than two right angles, the two straight lines, if produced *ad infinitum*, meet one another in that direction in which the angles less than two right angles are.

Common Notions

1. Things equal to the same thing are also equal to one another.
2. If equals are added to equals the wholes are equal.
3. If equals are subtracted from equals the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

The first thing I wish to point out is that the postulates include both assertions and rules. There corresponds to this division not only the distinction between theorems and problems, the latter being what we would call constructions, but also the distinction between the reasoning part of a proof (*ἀπόδειξις* in Proclus' terminology) and the construction (*κατασκευή*) which precedes it. Euclid's geometric reasoning is highly constructional in this way, and I see no reason to doubt that Greek geometry always was. However, even when this aspect of geometric reasoning is recognized,

⁵ For detailed discussion of some of the material in this paragraph, see von Fritz 1971, 393–414.

there is a tendency to focus on assertions and proofs rather than rules and constructions and, in particular, to speak of geometry as a matter of proving assertions from assumed assertions. This tendency may represent a philosophical bias, but, at least since Aristotle, accounts of reasoning have standardly focused on procedures by which assertions are transformed into other assertions and not on procedures by which constructions are built out of other constructions. The fact that Aristotle does not recognize primitive constructions as starting points of geometry suggests, although it hardly proves, that they did not occur among the presentations of geometry accessible to him. Since it also seems likely that the assertional postulates 4 and 5 are no earlier than the other three, there is some reason to think that nothing like Euclid's postulates was known to Aristotle [cf. Heath 1956, i 202].⁶ This is a point to which I return at the end of my discussion of Aristotle.

The common notions appear to be assertions relating to quantitative reasoning. Each could be transformed into a rule for such reasoning, common notion [5], for example, allowing one to go from ' a is part of b ' to ' b is greater than a '. Euclid's formulation of them as assertions is perhaps another reflection of the tendency to think of rules as founded on assertions. In any case, Euclid's list is quite inadequate to the quantitative reasoning he actually applies; and it is sufficiently inadequate to make me believe that Euclid had no desire to formulate a complete list, but settled for the most prominent principles he employs. A perhaps more interesting question is whether Euclid thought of the common notions as logical or material. From the point of view of standard predicate logic there is no question that the common notions are material; but I know of no fully satisfactory reason for denying argument about equality, addition, subtraction, coincidence, parts, and wholes, the status of logical reasoning. Here, as in the case of set theory, the division between logical and non-logical may be arbitrary. However, in the case of Euclid the issue may be refined by asking whether Euclid has a notion of rules of reasoning corresponding to our predicate calculus (or Aristotle's syllogistic). To be more specific: we know that Euclid follows such rules, and we know that he did not try to formulate the rules. Should we say that the absence of such an attempt reflects a lack of self-consciousness about these rules and, hence, an at least tacit belief that quantitative principles are the closest one comes to logical principles? Or should we suppose that Euclid acknowledged the use of logical principles,

⁶ The fourth postulate is much the most difficult to explain. Heath [1956, i 201] argues for an association with the fifth, but see Mueller 1981, 29–30.

but did not consider them to be his concern? The text does not allow us to decide between these alternatives, and to that extent supports the first.⁷

I conclude this section with a summary. The explicit starting points in the *Elements* are definitions, postulates, and common notions. Of these the definitions either correspond to abbreviations or they are what I have called explications. In thinking about starting points in the *Elements* and, hence, in thinking about them in Greek mathematics, we ought to think primarily about these definitions, even though explications play no official role in modern theories and abbreviations are starting points only by courtesy. In book 1 Euclid adds to the first definitions for plane geometry the postulates and common notions. The postulates correspond to primitive material rules and assertions. The common notions are general truths about quantities almost certainly intended to apply to numbers as well as geometricals. These assertions could be turned into rules without altering the character of the *Elements*. The questions whether they are logical or material starting points and whether they are the most general reasoning principles recognized by Euclid does not admit a clear answer. Certainly Euclid uses general logical principles, just as he uses primitive material and quantitative rules and assertions he has not made explicit. But using such principles does not constitute recognizing them. The following, then, is my list of acknowledged starting points in the *Elements*: explications, abbreviations, material rules, material assertions, quantitative assertions.

2. Aristotle

Aristotle's notion of mathematical starting points has been much discussed by historians of mathematics and historians of philosophy. In general the main passages which have to be looked at are well known, but no consensus on an overall reading of them seems to have emerged. In this section of my paper I will go through the passages in an order which facilitates what I think is their correct interpretation. For I believe that it is possible to find a relatively coherent and uniform view of mathematical starting points in Aristotle, and that standard accounts of the relationship between this view and Greek mathematical practice are not justified. Unfortunately, the content of the relevant passages overlaps and diverges in ways which necessitate discussing a variety of topics partially until, if all goes well, a total picture emerges.

⁷That is to say, the absence of a distinction in an author is *prima facie* (but only *prima facie*) evidence that the author did not make it.

When Aristotle sets out in *An. post.* i 7 to establish the impossibility of showing something by applying a proof in one genus to another, he announces that there are three things involved in proofs:

One is what is proved, the conclusion (this is a matter of belonging to some genus *per se*), one is the axioms (axioms are from which), and third is the subject genus, the properties and *per se* attributes of which are made clear by the proof.⁸ [*An. post.* 75a39–b2]

I will refer to this triad as the elements of a deductive science, and I will try to render plausible the view that these elements also represent Aristotle's basic conception of the starting points of a science. Aristotle offers other versions of the triad in other places.⁹ For example, in *An. post.* i 10 he writes,

Every demonstrative science concerns three things: the things it hypothesizes to be (these things constitute the genus of which it studies the *per se* properties), the so-called common axioms from which first things it proves, and third the attributes of which it assumes what each signifies. [*An. post.* 76b11–16]

⁸ My translations are not always literal. They are designed to facilitate my argument, but only by taking for granted what I think are relatively non-controversial interpretations.

⁹ In addition to the passages quoted in the text, the following one is generally thought to express the same doctrine.

Every demonstrative science investigates concerning some subject the *per se* attributes from the common opinions. Therefore, it belongs to the same science to investigate concerning the same genus the *per se* attributes from the same opinions. For that concerning which belongs to one science, that from which to one, whether it is the same or a different one, so that also either they investigate the attributes or one science composed of them does. [*Meta.* 997a19–25]

It seems clear that the common opinions in this last passage are the same as the common axioms: cf. Lee 1935, 113–114. But Aristotle's considered view seems to be that although there is a single genus belonging to each science, all sciences share the common axioms, which are themselves the domain of no single demonstrative science. Hence, Aristotle's view on the point raised in the last sentence quoted would be something like that one science investigates one genus from the common axioms, so that this science must necessarily investigate the attributes of that genus. Here he wants to leave open the possibility that there is a science of the common axioms (an issue raised in the preceding ἀπορία); but it is surprising that he goes so far in the direction of openness as to omit his own view, leaving only the possibility that two sciences or a composite science investigates the attributes.

After a brief excursus in which he points out that sometimes one or the other of these things is not explicitly hypothesized, he insists that by nature there are three things—the thing about which one shows, the things one shows, and the things from which one shows (περὶ ὃ τε δείκνυσαι καὶ ἃ δείκνυσαι καὶ ἐξ ὧν).¹⁰ In *Meta.* B 2 Aristotle writes,

If there is a demonstrative science of them, there will have to be some subject genus, and some of the principles will have to be properties, some axioms . . . ; for it is necessary for proof to be from some things, about some thing, and of some things. [*Meta.* 997a5–9]

On the basis of these and other passages it seems to me reasonable to say that for Aristotle the elements of a demonstrative science are the common axioms, the subject genus, and the properties associated with the genus. But there are several things to notice about Aristotle's characterization. First, the axioms are thought of as the premisses of scientific proof, the things from which one proves;¹¹ but the genus and the properties are apparently not thought of in this way. Secondly, Aristotle can speak of the genus in the singular or the plural, but presumably when he speaks of hypothesizing the existence of the genus he means hypothesizing the existence of things in the genus. However, it is important to see that even if Aristotle has in mind the hypothesis that, say, number or numbers exist, he does not seem to think of the hypothesis as a premiss of mathematical argument. In this sense, it does not matter much whether one speaks of hypothesizing the genus or hypothesizing its existence. Similarly, it seems to make no difference to Aristotle whether one speaks of the third element in demonstrative science as the conclusions or the properties shown in the conclusions to hold of subjects in the genus, but the latter formulation is somewhat more typical. Finally Aristotle speaks of assuming what the properties signify, and although this almost certainly relates to definitions, again the definitions are apparently not seen as premisses of argument. The picture one gets of a science then is that it proves properties of subjects in a genus from the common axioms. To do so it must take for granted the existence of the subjects and the significations of the properties. There

¹⁰ See also *An. post.* 77a27–29: 'I call common the things which all sciences use in the sense of proving from them, but not that about which they show things or the thing they show.'

¹¹ I emphasize that for Aristotle the common principles are assertions (προτάσεις), things from which one proves, premisses, and not rules. Ross [1949, 531] places great stress on two passages, *An. post.* 76b9–11 and 88b1–3, in which Aristotle speaks of proving through (διδ) common things; but I agree with Barnes [1975a, 135] that no great significance should be attached to this preposition.

are many problems in this picture, including questions of its consistency with other things Aristotle says. I shall attempt to address these problems only after I have attempted to clarify the picture.

a. *The common axioms*

Since the mathematician also uses the common things but restricted to his own science, it also belongs to first philosophy to investigate the principles of mathematics. For that when equals are taken from equals the results are equal, is common to all quantities, but mathematics studies a certain part of the domain of the axiom in isolation, e.g., it studies lines or angles or numbers or some of the other quantities... [*Meta.* 1061b17–24]

Here Aristotle explicitly mentions as an axiom or common principle of mathematics what we know as common notion 3 of the *Elements*. However, in the parallel passage at the beginning of *Meta.* Γ 3 he only refers to ‘what are called axioms in mathematics’, without giving any examples, and he stresses the idea that these axioms are true of all things and are used by all reasoning. This characterization is obviously more appropriate to the context in which Aristotle is concerned with versions of the fundamental logical laws which we call non-contradiction and excluded middle, a point which is brought out clearly in the two statements of the ἀπορία which is being addressed in both passages:

Whether it belongs to the science [first philosophy] to consider only the first principles of substance or whether it also deals with the principles from which everyone proves, e.g., whether or not it is possible simultaneously to assert and deny one and the same thing, and the other things of this kind [*Meta.* 995b6–10]

and

It is an open question whether it belongs to one science or several to deal with the principles of proof. By principles of proof I mean the common opinions from which everyone shows things, e.g., that it is necessary to affirm or deny each thing, and that it is impossible for something simultaneously to be and not be, and all other such assertions. [*Meta.* 996b26–31]

One sees from these passages that Aristotle includes among the common principles of the special sciences clear instances of what we would call logical assertions and the common notions of Euclid’s *Elements*, which I have called quantitative assertions to avoid having to settle the issue of whether

they are logical or material. It is possible that Aristotle acknowledges some distinction between these two kinds of common assertion, since he rarely mentions both kinds together.¹² However, he talks about the two kinds in essentially the same way, so that the one passage [*An. post.* 77a26–31] in which he does mention the two together as common things can be taken as quite decisive evidence that Aristotle does not distinguish them.¹³

b. Problematic passages in the Posterior Analytics. In his translation of and commentary on the *Posterior Analytics*, Barnes [1975a, 136] provides a list of ‘various classifications of the elements of demonstrative science’ given by Aristotle, and says that ‘Aristotle himself makes no attempt to coordinate them.’ I believe that one can make reasonably good sense of all of these classifications in terms of the triad genus, properties, common axioms, and in this section I attempt to do so.¹⁴

At the start of *An. post.* i 10 Aristotle says,

I call the principles in each genus the things which cannot be shown to exist. Thus, what the first things [i.e., principles] and the things composed of them are is assumed; on the other hand it is necessary to assume that the principles exist, but to show that other things

¹²In *An. post.* i 10 76a41, 76b20–21 Aristotle mentions ‘equals from equals’ as a common thing, but never gives a logical example of an axiom in that chapter. At 88a36–b1, he cites the law of the excluded middle as a common principle.

¹³Theophrastus apparently did distinguish them. For Themistius [*In an. post.* 7.3–5] tells us that he defined axioms as certain opinions, some concerning things of the same category, e.g., ‘equals from equals’, some concerning absolutely everything, e.g., the law of the excluded middle.

¹⁴I do not discuss the last two paragraphs of *An. post.* i 10 (76b23–77a4). In the first of these Aristotle distinguishes propositions which must be believed, hypotheses, provable propositions which the teacher assumes without proof and the student accepts, and postulates which the teacher assumes and the student does not accept. This categorization does seem to me quite independent of all the others. (See on this passage von Fritz 1971, 365–366. I note that there are no other passages listed under ἀτήμα in Bonitz’s index which help to clarify the possible logical or scientific sense Aristotle attaches to the word ‘postulate’, although chapter 20 of the *Rhetorica ad Alexandrum* discusses rhetorical postulates.)

In the other paragraph Aristotle distinguishes ὅροι from hypotheses. I take hypotheses to be premisses in general (ὁσων ὄντων τῷ ἐκεῖνα εἶναι γίγνεται τὸ συμπέρασμα), but I am not certain whether a ὅρος is a definition or a term. (I am not inclined to think it is a *definiens*, although this cannot be ruled out. See, e.g., Mignucci 1975, *ad loc.* or Landor 1981.) Since the point Aristotle is making is that a ὅρος is not an assertion, taking ὅροι to be definitions would lend support to my overall interpretation; but I do not see any way to rule out the possibility that ὅροι here are terms and Aristotle’s point a relatively trivial one.

[i.e., the things composed of the first things] exist. For example we assume what monad or straight and triangle signify; and we assume that monad and magnitude exist, but we show that the others exist. [An. post. 76a31–36]¹⁵

Aristotle here contrasts the principles in a genus with the things composed of them, giving monad and magnitude as examples of principles, straight and triangle as examples of composites. The contrast between monad and magnitude represents the contrast between arithmetic and geometry, and it seems reasonable to assume that they represent the genera of the two sciences. Other passages suggest that magnitude is a stand-in for point, line, surface, and solid, although Aristotle usually uses as illustrations only the first one or two of these rather than all four. Similarly he sometimes speaks of number rather than monad as the genus of arithmetic, as he does in the passage from the same chapter quoted above. The fact that Aristotle here speaks of hypothesizing both that these principles exist and what they signify should be seen as an amplification of passages in which Aristotle only mentions the first kind of hypothesis in connection with the genus. For it seems obvious that one must know what the genus signifies as well as that it exists if one is to prove things about it; however, for Aristotle the hypothesis of existence is associated uniquely with the genus and, hence, is the most interesting one to mention in connection with the genus.

Triangle and straight are both geometric items, and I believe they should be placed in the class of what I have been calling properties. It does not seem to me to count heavily against this assumption either that Aristotle calls these things composites or that he speaks of proving their existence. For in the next passage in chapter 10 he speaks of proving the existence of the properties. In the passage Aristotle makes explicit another distinction he sometimes invokes in discussions of mathematical starting points, the distinction between two kinds of things used in demonstrative sciences, common ones, such as 'equals from equals', and special ones, the examples of which include both definitions ('A line is such and such, and the straight is such and such' [An. post. 76a40]) and

the things which are assumed to exist and concerning which the science investigates the *per se* features, e.g., monads in arithmetic, points and lines in geometry. For it is assumed that these things

¹⁵ This passage should be compared with the more obscure An. post. 87a38–40 for which Barnes' notes [1975a] are very helpful:

The science of one genus, i.e., things which are composed from the primary things and are parts or *per se pathē* of these things, is one science.

exist and that they are such and such. But what each of the *per se* properties of these things signifies is assumed, e.g., in the case of arithmetic, what odd and even and square and cube signify, and, in the case of geometry, what irrational and being broken and verging signify; but that these things exist is proved through the common things and from what has already been proved. [*An. post.* 76b3–11]

It is difficult to see that any sense can be attached to the notion of proving that these properties exist other than proving that they apply to their subjects [cf. Ferejohn 1982–1983, 394–395].¹⁶ Thus, if the composites mentioned at the beginning of chapter 10 are to be identified with these properties, proving their existence will also be proving that they apply to their subjects. Aristotle speaks of these properties as composites because he is thinking of them as defined in terms of the simple objects of the subject genus. In this sense the properties are not principles or starting points since they depend for their definition on other things, just as they exist only as belonging to other things.¹⁷ But they or their definitions are starting points in another way since they are taken for granted by the mathematician in his argumentation. What Aristotle has in mind by assuming existence is brought out in the next passage in chapter 10, the beginning of which was quoted early in this paper. In the part not quoted Aristotle mentions cases in which a science does not assume explicitly one of the three elements he has identified. He contrasts the necessary assumption of the existence of number, the genus of arithmetic, and assuming the existence of hot and cold, which it is not necessary to do because existence in this case is obvious. It seems to me relatively clear that the notion of existence involved here

¹⁶ Such a proof might be a construction; for example, Euclid's construction of an equilateral triangle in book 1 prop. 1 might constitute a proof of existence for Aristotle; but there is no textual reason for denying that Aristotle would think of the proof of the incommensurability of the side and diagonal of a square as a proof of the existence of incommensurability. I know of no good evidence for the frequently repeated suggestion that for Aristotle existence in mathematics was somehow connected with constructibility. Cf. Barnes 1975a, 92.

¹⁷ Cf. *Meta.* 1077b3–4 (where Aristotle says that *a* is prior in definition to *b* if the definition of *b* is composed out of the definition of *a*), 1035b4–14. Elsewhere Aristotle illustrates this priority in terms of point and line, and of line and triangle:

A belongs to *B* *per se* if *A* is in the definition of *B*; for example, line belongs *per se* to triangle and point to line since the substance of triangle and line are composed from line and point, which are present in the formula which says what they are. [*An. post.* 73a34–37]

In *Top.* 108b26–31 Aristotle mentions definers who treat the point as principle of the line and the monad as principle of number.

is neither technical nor profound. The 'assumption' that hot and cold exist is something we all make in our everyday conversation about the weather. When the arithmetician hypothesizes that there are numbers or units, he is only insisting that he is talking about something and asking that philosophical or ontological questions be put aside, just as, for the most part, such questions can be put aside when one talks about heat or cold. Only a philosophically trained (or mistrained) person would ask why he should believe there are such things as hot and cold. In the *Republic* Socrates indicates the kind of question which might be asked of the arithmetician:

What kind of number are you talking about in which the one is such as you demand, each equal to every other and not differing in the least and having in itself no part? [Plato, *Resp.* 526a2-4]

Socrates goes on to suggest that the arithmetician's response will bring out the intelligible non-sensible character of his objects, but Aristotle's position seems to be that the arithmetician will simply insist that he be granted that there are such things so he can proceed. My suggestion, then, is that when Aristotle speaks of hypothesizing the existence of the subject genus of a science, he has in mind a broad sense of existence precision about which serves no scientific purpose. The idea that a single science deals with a single genus is very important in Aristotle's doctrine that there cannot be a single universal science. But where did Aristotle get this idea? He does not offer any real argument for it. And it seems quite independent of syllogistic, which is a purely formal theory, although the idea of proving properties of a subject is undoubtedly related to the subject-predicate conception of an assertion (*πρότασις*) underlying Aristotle's syllogistic. Nor, I think, should the idea be connected with the doctrine of categories or highest genera of being. For although that doctrine might be taken to exclude the possibility of a transcategorical science, Aristotle's favorite example to illustrate the 'one science/one genus' doctrine is arithmetic and geometry, both of which presumably deal with species of the category quantity. In fact it seems likely that Aristotle takes restriction to a single genus as an observed fact about actual sciences, and that in this connection he uses the word 'genus' rather informally. All he is saying is that every demonstrative science deals with only one kind of thing.

In *An. post.* i 1 Aristotle describes the kinds of prior knowledge presupposed by learning:

In the case of some things it is necessary to assume in advance that they exist, and in the case of others it is necessary to apprehend what the thing said is; and in still others both are required. For example, one must assume that the law of the excluded middle exists, what

triangle signifies, and for the monad, both what it signifies and that it exists. For each of these is not equally obvious to us. [*An. post.* 71a12–17]

It is clear that we have to interpret the assertion that ‘the law of the excluded middle exists’ as the assertion that the law is true, but it should also be clear that the need for this interpretation in this passage does not by itself warrant interpreting ‘exist’ (εἶναι) as ‘is true’ in cases where the text does not require it. For in this passage Aristotle is trying to illustrate the trichotomy he applies in chapter 10 to the special starting points only (ὅτι ἔστι· τί σημαίνει· τί σημαίνει καὶ ὅτι ἔστι) in terms of the trichotomy of common axioms, properties, and subject genus. The result is perfectly defensible, but misleading in so far as it blurs distinctions made clearly elsewhere.

In *An. post.* 72a7 Aristotle defines a principle as an immediate πρότασις, and goes on to describe πρότασεις as assertions and denials.¹⁸ He then says,

I call an immediate syllogistic principle which cannot be shown and which it is not necessary for a person to have to learn something a θέσις. But an axiom is something which a person must have if he or she is to learn anything whatsoever; for there are some things of this kind, and it is our custom to apply the term ‘axiom’ to them especially. One kind of θέσις is a hypothesis; it assumes one half of a contradiction, e.g., I mean, that something exists or does not exist; another kind, without this, is a definition. For a definition is a θέσις, since the arithmetician lays down that the monad is indivisible in quantity. But a definition is not a hypothesis; for what a monad is and that a monad exists is not the same. [*An. post.* 72a14–24]

All commentators point out that Aristotle rarely, if ever, uses the words θέσις and ‘hypothesis’ in the way explained here; and normally he does not refer to the learning situation in explaining axioms. But there seems to me no reason to doubt the text: the things which a person must acknowledge to be able to learn anything are the axioms, principles presupposed in all scientific argument,¹⁹ and θέσεις are the special principles for individual sciences. ‘Hypothesis’ appears to have the more general sense of assumption

¹⁸ The fact that Aristotle’s formulation very early in the *Posterior Analytics* is thoroughly propositional has had great influence on accounts of his doctrine of the starting points. For one attempt to minimize this passage, see Ferejohn 1982–1983, 382–383, n16.

¹⁹ Cf. *Meta.* 1005b5–23, where Aristotle describes the law of non-contradiction as something which one must know if one is to know anything.

at the beginning of this passage, but the contrast between hypotheses and definitions depends on treating them as existential assumptions. The transition from the general to the specific sense proceeds by the opaque phrase 'e.g., I mean, that something exists or does not exist', which is sometimes interpreted as 'i.e., I mean, that something is or is not the case'. This interpretation has the advantage of giving a clearer sense to the negative alternative, but makes Aristotle's opposition of definitions and hypotheses an apparent equivocation. It seems to me preferable to say that the negative alternative is included because of the general sense of 'hypothesis' introduced here, but that the concrete examples Aristotle has in mind are affirmations of the existence of the genus of a science.

I conclude that Aristotle's doctrine of the starting points of demonstrative science involves the division of starting points into common and special ones. The common ones or axioms include both quantitative and logical assertions, but Aristotle probably does not distinguish the two. The special ones are the subject genus and the properties of the genus. Aristotle frequently speaks of hypothesizing the existence of the genus or its members, and refers to definitions of the properties as well as of the genus and its members.²⁰ In this sense the special starting points can be thought of as propositional, but it is important to bear in mind that Aristotle thinks of the axioms as the only premisses used in demonstration. The hypothesis of the existence of the genus is the assumption that one is talking about something real in a science, and the definitions are simply determinations of the genus and the properties one is going to discuss. However, to say that the axioms are the only premisses of a science is not to say, at least for Aristotle, that all the theorems could be derived from them. For the axioms are too general to permit the derivation of specific truths. One needs to particularize them by bringing in a genus and its properties.²¹

²⁰ There is a close correlation between the elements of Aristotle's subject genus and the things whose definitions in the *Elements* I have called explications, and also between his properties and those whose definitions I have called abbreviations. However, I am not sure that Aristotle noticed this difference. And I certainly agree with Barnes [1975a, 134] that he did not distinguish primitive and defined terms.

²¹ See *An. post.* 88a36–b3, where Aristotle, in arguing that 'it is impossible for all syllogisms to have the same principles', considers the possibility that some of the common principles (of which he gives excluded middle as an example) might play the role of universal principles. Aristotle does not say that these principles are insufficient but only that the genera are different and that one proves with these genera through the common things. Shortly thereafter, in a very difficult passage, Aristotle considers the possibility that the primary immediate propositions are the principles and makes the curious remark that there is one for each genus, by which he perhaps means the definition of the genus [so Ross 1949, *ad loc.*].

That is to say, even if one could prove all the premisses of a science (the axioms) in a higher science, one would not thereby be able to prove all the theorems in the higher science.

c. *The mathematics known to Aristotle.* Aristotle sometimes mentions the common axioms in ways which would seem to insure that he is talking about a feature of mathematics known to his audience,²² and it seems safe to assume that mathematical texts known to him included 'equals from equals' and presumably at least the first three of Euclid's common notions. On the other hand, the absence of fundamental logical laws from Euclid's list of common notions suggests to me that Aristotle's inclusion of them among the axioms is a reflection of philosophical discussion in the Academy concerning the general principles of reasoning, discussion having no direct impact on Euclid. Philosophical discussion in the Academy and the alleged Platonic 'reform of mathematics' may also underlie the inclusion of the common notions in mathematical texts, but I can think of no considerations which weigh particularly heavily for or against this suggestion.

From a modern point of view the idea that the common axioms might be the only premisses of, say, geometric proof is incredible. There are at least two factors which may help explain why Aristotle adopted it.²³ The

Finally, Aristotle takes up the suggestion that although different principles are used in different proofs, they are all of a piece (*συγγένειος*). Aristotle responds by reasserting his doctrine that 'the principles of things differing in genus are different in genus'. 'For', he says, 'the principles are twofold, those from which and those concerning which; the former are common, the latter, e.g., number and magnitude, special.' Thus, even here, in the context of a discussion which relies heavily on the doctrine of the categorical syllogism, Aristotle on the whole treats the common principles as the only premisses and makes their non-universality turn on the fact that they are specialized through restriction to a genus.

²² See especially *Meta.* 1005a20, where Aristotle mentions 'the things called axioms in mathematics'. von Fritz [1971, 421–422] argues with considerable plausibility that Aristotle's analysis of the axioms as common as opposed to special starting points is Aristotle's own contribution and not a reflection of the mathematicians' understanding of their own practice.

²³ The fact that the common notions could not be employed reasonably in an Aristotelian syllogism does not strike me as particularly problematic in this connection, since Aristotle does not attend to issues of formalization in a rigorous way: see Mueller 1974, 48–55. The same relaxed standards are evident in his discussion of the way in which the laws of non-contradiction and excluded middle are used in demonstrations: cf. *An. post.* 77a10–25. He says that the latter is assumed in every *reductio* argument, but that the former is only used when the conclusion is in the form $P \ \& \ \neg \neg P$. This second claim is bizarre not only because it is hard to envision a scientist trying to prove such a proposition, but also because non-contradiction is used in any *reductio*. To argue for the claim

first is drawn from the *Elements*. As I have already mentioned, what we would call a Euclidean geometric proof is customarily divided into two parts called by Proclus the *κατασκευή* (construction) and the *ἀπόδειξις* (proof). Roughly, the *κατασκευή* depends on the postulates and previously established constructions, whereas the *ἀπόδειξις* depends on the common notions and previously proved theorems. Thus, there is at least the possibility of thinking that the only ultimate assumptions used in geometric proofs (i.e., the 'real' proofs, the *ἀπόδειξεις*) are the common notions. The supposition that Aristotle did think of proof this way would be strengthened if it could be rendered plausible that the geometry texts known to Aristotle included no postulates among their starting points. For if they contained only definitions and common notions, then Aristotle would have at least empirical grounds for thinking of the common axioms as the only substantive assumptions made by the mathematician. The argument from Aristotle's silence with regard to the Euclidean postulates seems to me quite strong in this case [cf. Heath 1921, i 336],²⁴ but a number of scholars²⁵ have argued that at least Euclid's first three postulates correspond to Aristotelian existence assumptions. I wish to argue briefly that the correspondences are at best very tenuous and probably non-existent.

I have argued elsewhere [Mueller 1981] that the first three postulates are not existence assertions at all, but licenses to carry out certain constructions. This position is, of course, quite compatible with the fact that they play a role analogous to the existence assumptions in modern formulations of geometry, as well as with the possibility that Aristotle thought of the postulates as existence assertions. However, Aristotle's description of scientific existence hypotheses corresponds neither to Euclid's postulates nor their modern analogues, but to the modern logical notion of a theory

Aristotle invokes features of the categorical syllogism, and it is true that any provable categorical proposition is provable without using non-contradiction.

²⁴ However, Heath [1949, 56] maintains that Euclid's first three postulates 'are equivalent to existence assumptions and therefore correspond to Aristotle's "hypotheses"'. I discuss briefly what seems to me the strongest evidence for a pre-Euclidean formulation of the postulates in two appendices.

²⁵ Most notably Lee 1935, 115–117. It is difficult to characterize von Fritz' position on the question of the correlation between Aristotelian existence assumptions and Euclid's construction postulates. He seems to concede all the difficulties but nevertheless insist on the correlation.

presupposing a domain or having an intended interpretation.²⁶ Euclid's postulates allow one to move from given objects of a certain kind (two points, a straight line, a point and a 'distance') to others (a straight line, a longer straight line, a circle). None of the objects constructed using Euclid's postulates is mentioned by Aristotle as an element of the genus; indeed, straight is mentioned as something whose signification we assume but whose existence we prove, and it seems to me reasonable to suppose that circle would fall into the same category, as triangle does. Moreover, Aristotle thinks there are existence hypotheses in arithmetic, but there is no trace of postulates of any kind ever being used in ancient number theory [cf. Kullmann 1981, 248-249].

For these reasons the attempt to correlate Aristotle's existence assumptions with Euclid's constructional postulates seems to me quite implausible. Moreover, the interpretation I have offered of these assumptions seems to me to correlate well with what Aristotle says about them and to cohere with Aristotle's general conception of reasoning and scientific knowledge. We cannot know whether or not the geometry of Aristotle's time included postulates in Euclid's manner; but Aristotle, who provides us with our best evidence for fourth-century mathematics, gives us no grounds for thinking it did.

Prima facie it would seem highly likely that the texts in mathematics known to Aristotle included definitions of the kind familiar to us from Euclid. Moreover, some passages in Aristotle suggest that definitions are the only starting points of a science. For example, at *An. post.* 90b24 (an aporematic passage) he calls definitions the principles of proofs, and at 99a22-23 he says that 'all sciences come about through definitions'.²⁷ Aristotle's recognition that these definitions do not or should not involve any existential implications, and that the source of these implications must lie elsewhere is a tribute to his powers of analysis. On the other hand, it is somewhat curious that he downplays the use of definitions as premisses. For in Euclid and in mathematical reasoning generally they do have this role; and Aristotle himself treats definitions as premisses in book 2 of the *Posterior Analytics*. However, the discussion of definitions in book 2 is notoriously problematic in itself and in relation to the account of starting

²⁶ I am not sure what von Fritz means when he says [1971, 393] that Greek mathematicians did not find it necessary to formulate the Aristotelian existential starting points explicitly. If he means that they were aware of these assumptions but did not formulate them, he is indulging in pure speculation. And if he means that their mathematics commits them to these assumptions, he is making a philosophical rather than a historical claim.

²⁷ Barnes [1975a, 109] lists other passages which he thinks express the same view.

points in book 1. Here I wish only to make a few suggestions which may help to clarify Aristotle's conception of starting points.

Aristotle's specific words for definition are ὁρισμός and ὅρος, but he frequently refers to definitions by using the expressions 'what it is' (τί ἐστι) and 'what it signifies' (τί σημαίνει). In book 2 Aristotle consistently uses the former expression until chapter 6 in which he raises objections to the view that one might be able to prove what something is. His second objection goes as follows:

How is it possible to show what something is? For it is necessary that a person who knows what a human or anything else is also know that humans exists; for no one knows what something is if it does not exist. But when I say 'unicorn' I may know what the expression or name signifies, but it is impossible to know what a unicorn is. [*An. post.* 92b4–8]

The text and interpretation of the next objection is disputed, but I need only a small and relatively clear part of it:

Therefore, there will be a proof that something exists, which is what sciences now provide. The geometer assumes what triangle signifies and shows that it exists. [*An. post.* 92b14–16]

The distinction Aristotle makes in the first of these passages is normally expressed as the difference between a real and a nominal definition. Apprehension of a real definition involves an apprehension of the existence of its subject, whereas a nominal definition only relates to words and, hence, bears no existential import. The second passage suggests that, at least in the case of properties, the mathematician uses nominal definitions. I believe that this is Aristotle's conception of all mathematical definitions, although this claim cannot be proved by arguing that Aristotle always uses the expression 'what something signifies' in connection with mathematical definitions. He does not, but it is striking how frequently he does. For example in *An. post.* i 1–10, there are seven occurrences of expressions related to 'what something signifies' in the vicinity of references to mathematics [71a14–16, 76a32–36, 76b6–12, 76b15–21]²⁸ and only two related to 'what something is' [72a23; i 10]; moreover, both of these occur in conjunction with expressions related to 'that something exists', so that speaking of what something is produces a more elegant coupling.

²⁸ The phrase 'what the thing said is' (τί τὸ λεγόμενον ἐστι) of *An. post.* 71a13 seems to me more likely to fall on the side of 'what something signifies' than on that of 'what something is'.

The suggestion I wish to make is that in the early chapters of *An. post.* i and in other discussions which clearly focus on mathematics, Aristotle thinks of definitions as nominal, and that the distinctive doctrines concerning definition in book 2 relate to real definition [cf. Gómez-Lobo 1981, Leszl 1980]. This assumption would explain why Aristotle speaks in book 1 of assuming or proving the existence of things the signification of which has been determined, but in book 2 he holds that knowing what something is entails knowing that it exists. It might also help to explain why Aristotle does not treat definitions as premisses in his descriptions of the three elements of deductive science, but does treat them as premisses in book 2. The idea would be that real definitions can play this role, but nominal ones cannot. Moreover, there seems to be some plausibility in the idea that nominal definitions are not assertions at all, e.g., that they are not really capable of truth and falsehood, and so could not play the role of genuine premisses in a science. In a sense the only truths assumed in mathematics are the common axioms since mathematical objects do not really exist and the definitions are purely nominal.

If my interpretation of Aristotle is correct, then the only real common ground between Aristotle's theory and Euclid's practice is the common notions. There is also a kind of commonality in the case of definitions, but we have no way of knowing whether Euclid understood definitions in the way Aristotle did. He may have, and he may also have believed that the different sciences in the *Elements* treat different genera assumed to exist. Neither belief is reflected in the way Euclid presents his starting points; he never asserts the existence of a genus, and he presents his definitions as if they were premisses of his arguments, or, at least, he uses them in that way.

To conclude my discussion of Aristotle I want to address what I take to be the most problematic aspect of my interpretation, my attempt to deny that for him the special starting points function as ultimate premisses of scientific proof. Clearly much of Aristotle's discussion of science is built around the idea of chains of deductive argument starting from unproved or immediate premisses frequently called principles (*ἀρχαί*). And, as I have mentioned, his discussion of definitions in *An. post.* ii does treat them as premisses in scientific arguments. The difficulties involved in harmonizing all major points made or apparently made by Aristotle in the *Posterior Analytics* are well known, but I doubt that it is worthwhile to try to defend my account by arguing that it is no worse off than other available ones. Instead I shall attempt to offer a way of explaining this inconsistency.

The logical picture of deductive science provides Aristotle with a strong argument that the observed situation in which mathematicians do not try

to prove their starting points is what must always be the case: deductive proof presupposes premisses which are not proved. These premisses are principles ($\alpha\rho\chi\alpha\iota$) for Aristotle, but obviously this fact does not entail that for him all principles are such premisses. Nor does it even follow that whenever Aristotle is discussing the question whether the principles can be proved, he is thinking of the principles as these premisses. That is to say, it is possible for Aristotle to ask whether it can be proved that number exists or that a number is a system of monads without his thinking of these statements as ultimate premisses. My proposal, then, is that we separate the purely logical notion of a principle as an ultimate premiss of proof from what might be called the analytic notion of a starting point, analytic because it is based on an analysis of mathematical practice. The two notions coincide in so far as one kind of starting point is an ultimate premiss and in so far as neither starting points nor ultimate premisses are provable, but they do not coincide completely since not all starting points are ultimate premisses. This lack of coincidence explains, I believe, the difficulty of mapping the analytic notion of a starting point onto the logical conception of an ultimate premiss.²⁹

3. Plato

Toward the end of book 6 of the *Republic* Socrates introduces Glaucon to what turns out to be a distinction between two kinds of reasoning, one exemplified in mathematics, the other in dialectic. He explains one feature of mathematical method in the following way:

I think you are aware that those who concern themselves with geometrical matters and calculations and such things hypothesize the even and the odd and figures and three kinds of angles and other things related to these in the case of each subject; they make these things hypotheses, as if they were known; they do not see fit to give any account of them either to themselves or others, as if they were evident to everyone; they begin from these things and proceed through the others until they reach by agreement that which they started out to investigate.

I know that perfectly well, he said. [Plato, *Resp.* 510c2-d4]

²⁹ The most interesting attempt at a mapping known to me is found in Hintikka 1972. However, the criticisms of it voiced by Ferejohn [1982–1983] and Frede [1974] seem to me very weighty. I note, however, that in his response to the latter, Hintikka [1974] virtually abandons the hope of understanding Aristotle's conception of starting points independently of his logic.

In terms of Aristotle's categorization of starting points in *An. post.* the examples of hypotheses mentioned by Socrates would most plausibly be interpreted as properties. But it is not clear what Socrates means by giving no account of these things. The most direct interpretation would be that the mathematician uses terms like 'odd', 'even', 'square', 'hexagon', 'right', 'obtuse', and 'acute' without defining them. I do not wish to rule out this interpretation, but, as I have indicated, it seems to me quite unlikely that these terms were used without definition in the mathematics known to Plato.³⁰ Hence, if Socrates' description is as obviously correct as Glaucon's answer suggests, Socrates may simply mean that the mathematician does not give justifications for his definitions. He or she says that an even number is one which is divisible into two equal parts and expects everyone to agree. In either case the important point is that Socrates focuses on what Aristotle calls properties, although there is no indication that Plato would draw any distinction between the underlying genus and these properties. That is to say, Socrates' list might have included 'point' or 'line' without affecting anything Plato says.

It is also important that Plato does not show any awareness of anything corresponding to either the axioms or the underlying genus mentioned by Aristotle or to Euclid's postulates, although Socrates does mention the active character of geometry [Plato, *Resp.* 527a6–b1]. In the *Meno* Socrates gives a relatively clear description of a mathematical hypothesis which is propositional but not a definition; however, the hypothesis is not intended to be a starting point in the sense I have been discussing, but a provisional assumption to which an unanswered question can be reduced [cf. Solmsen 1929, 104n1]. Of course the hypotheses of the *Republic* are also provisional in a way, but there is no indication that they are conceived propositionally except possibly in the sense that definitions are propositions. This point, of course, bears on the interpretation of Socrates' claim that dialectic can somehow do away with the hypothetical character of mathematics. The vocabulary Socrates uses in this passage makes it possible to interpret what he says as a matter of deducing hypothetical assertions from a single unhypothetical one. But this interpretation is certainly not necessary, and the fact that the mathematical hypotheses are properties or definitions and the unhypothetical starting point is the Good, makes it rather implausible. To be sure, we do not know what, if any, more precise picture underlies Socrates' account of dialectic in the *Republic*; but it seems to me that, from a more or less logical perspective, it is best to imagine the task of dialectic

³⁰ This is Solmsen's view [1929, 96–97]. He imagines that Socrates' description is a Platonic transformation of a mathematics based entirely on drawn figures. The interpretation I offer here is parallel to that of Sidgwick 1869.

with respect to mathematics as the rendering perspicuous of definitions through a systematic ordering of the concepts involved with some kind of non-deductive *Ableitung* of central concepts from highest ones [cf. Solmsen 1929, 101–103]. We do not know how such an *Ableitung* would work or even what good it would do, but we do no service to Plato and we read him inaccurately if we suppose that he believed in the existence of some transparent proposition from which all propositions could be deduced.

I suggest then that if we take what Plato says about mathematical hypotheses in the *Republic* at face value, then the mathematics, or at least the geometry,³¹ with which Plato was familiar contained as starting points at most, and probably at least, definitions. His view was that mathematicians proved things from these definitions or from undefined terms. I have already remarked that the conception of mathematics as resting on definitions alone is the dominant one in Euclid's *Elements* and that there are certain passages in Aristotle which suggest a similar conception. In any case Plato saw the hypothetical structure of mathematics as a shortcoming, which he thought could be overcome by justifying correct definitions of certain terms. Perfect justification would involve incorporation in a conceptual structure covering the whole of reality. This structure is sometimes called a universal science by modern scholars, but it is probably wrong to think of it as a universal deductive science. No doubt deduction and argument would be a part of it, but its upper level, that closer to the absolute starting point would involve derivation and justification in a much looser sense. We might then view the Platonic universal science as a two-tiered system with the following structure:

the ἀρχή of all
(‘dialectical *Ableitung*’)
the ‘hypotheses’ of the special sciences
the special sciences

Of course, one point of the notion of a universal science is precisely to deny the special sciences their special or isolated position. But to say that a discipline is part of a whole is not necessarily to deny that the part can be pursued on its own.

The *Timaeus* gives us some picture of how Plato conceived one special science, physics, which for him is, at least in part, subordinate to geometry. That dialogue also gives some hints that Plato might have espoused a development of geometry quite different from what we find in the *Elements*,

³¹ Plato's notion of arithmetic or logistic, as he usually calls it, is not as clear as it is frequently taken to be. But this is not an issue into which I can enter here.

but it seems to me reasonable to suppose that Plato also looked with favor on more straightforward deductive reasoning of the Euclidean kind. If we take this part of Platonic geometry as the relevant material for this paper, we may say that Plato views the science of geometry as using as starting points, or hypotheses as he calls them, only primitive terms or, more probably, definitions, where the definitions would presumably include both explications and abbreviations.

4. Justifying the starting points

For us a proof is primarily a means of justification. To prove P is to show that P is true in a way which justifies belief in P . However, one can also prove P as a way of teaching somebody that P is true. I will distinguish these two ways of using proof by speaking of proof as justification and proof as instruction. We, I think, tend to play down the notion of proof as instruction, particularly if the conception of proof is formal. We are willing to say a person has been taught and hence knows that the continuum hypothesis is independent of standard axioms of set theory if all he or she knows is classical set theory and that Paul Cohen was given a prize for the proof of independence. And we would certainly say that a person who knew the rudiments of Cohen's proof but not the details of forcing techniques knew Cohen's result. But Aristotle holds that we do not know anything provable unless we know its proof. Hence, if teaching is making known, teaching provable things has to be teaching their proofs; and teaching proofs is quite naturally identified with presenting them. Thus, for Aristotle, proof is both a means of justification and of instruction; proof serves to make known and to justify the theorems of a science.

However, for Aristotle, proof can do neither in the case of the starting points of a science, since he believes those starting points are unprovable. Sometimes he appears to defend this belief by turning it into the apparently less controversial doctrine that a science cannot prove its own starting points. But the more important form of the doctrine is that for certain sciences, including geometry and arithmetic, there is no higher science from which their starting points can be derived. In the case of the common axioms Aristotle appears to believe that they are not only unprovable, but that there is no way of making them known since he says [*An. post.* 72a16–17] that the axioms are a presupposition of learning anything. However, there are for him ways of making known the other kinds of starting points. Toward the end of his discussion of definition in *An. post.* ii, Aristotle concludes that definitions of the derived concepts are made known through their use in proofs, even though they are not themselves proved:

Some things have a cause different from themselves, and some do not. So it is clear that some definitions are immediate and principles, namely, the definitions of those things for which it is necessary to hypothesize or make evident in some other way both that they exist and what they are. The arithmetician does this, since he or she hypothesizes what a monad is and that it exists. Of things which have a middle and of which there is a different cause of the οὐσία it is possible, as we have said, to make what something is clear through proof without actually proving it. [*An. post.* 93b21–28]

Aristotle's best known discussion of making known the starting points is the last chapter of the *Posterior Analytics*, where he describes a process of induction and speaks of apprehension of the principles by νοῦς. The description suggests that induction and νοῦς relate first and foremost to the primary concepts or subject genus of a science.³² It is sometimes supposed that the topic of ii 19 is both the learning of starting points and the justification of our claim to have knowledge of them. The basis for this supposition is Aristotle's comparison of νοῦς and ἐπιστήμη as conditions of knowledge (ἀληθῆ ἀεί, *An. post.* 100b7–8) and contrast of them by saying that νοῦς is the more accurate of the two [cf. *Eth. Nic.* vi 6]. Thus, the impression arises that, although induction by itself does not justify our apprehension of the starting points, there supervenes as a result of it a self-justifying intuition of them, νοῦς.³³ I do not believe that it is possible to dismiss this interpretation entirely, but it does not seem to me to represent adequately all of Aristotle's thoughts on the justification of the starting points of the sciences [cf. Barnes 1975a *ad* ii 19; Burnyeat 1981, 130–133]. For there are clear indications in other treatises, notably the *Topics* and the *Metaphysics*, that he thinks it possible to provide justifications of a kind for them. I shall deal briefly with the three kinds of starting points in turn.

The only possible candidate for justification in the case of properties would seem to be justification of their definition and, in so far as a genus or its elements is also defined in the special sciences, the same notion of

³² Kahn [1981] stresses the fact that this chapter is most simply read as a description of concept formation. Traditionally it has been assumed that Aristotle must be giving an account of how primary propositions become known: see, e.g., Ross 1953, 58. Barnes [1975a, *ad loc.*] shows that one *could* read the text in terms of the apprehension of propositions.

³³ Cf. Ross 1953, 217: '[Induction is] the process whereby after experience of a certain number of particular instances the mind grasps a universal truth which then and afterwards is seen to be self-evident. Induction in this sense is the activity of "intuitive reason".'

justification would be relevant to the genus. Evans [1977, 50] argues that 'Plato conceived dialectic as essentially involving a search for definitions', but that Aristotle abandons this conception. Evans does not seem to me to do justice to a variety of passages in the Aristotelian corpus, including *Topics* vi-vii, and, in particular, to Aristotle's assertion that there can be a syllogism of the definition and essence [*Top.* 153a14-15]. This assertion is, to be sure, compatible with the position that there cannot be a deductive proof of a definition; but it equally does not mean just that there can be a valid deductive argument with a statement of a definition as conclusion. Aristotle seems to be saying that there is a kind of reasoning, usually called dialectical, which can be used to establish definitions. It goes without saying that this reasoning is not scientific because scientific reasoning is characterized by proceeding from starting points, including definitions. Equally, because the reasoning is dialectical, it can only be *ad hominem*, not absolute.

I have not found evidence that Aristotle thinks the existence of a genus can be justified by means of dialectical arguments. And I think it is reasonably clear that he does not think this. Two passages are particularly useful in this respect. The first occurs in *Phys.* ii 1 where Aristotle, after indicating what nature is, says,

To try to show that nature exists is laughable. For it is evident that many such things exist. But only a person unable to distinguish what is known through itself and what is not would show evident things through unevident ones. [*Phys.* 193a3-6]

In this case Aristotle is dealing with a starting point the existence of which he thinks is so obvious that anyone who asked to be convinced of its existence could be dismissed as stupid or merely contentious. The case of fundamental mathematical objects is not at all the same, but neither is Aristotle's attitude to the question of their existence at all as clear. At the end of *Meta.* M 1, before turning to this question, Aristotle says,

It is necessary that if mathematical objects exist, they either exist in sensibles, as some people say, or separate from sensibles—and some people do say they exist in this way—or, if they exist in neither way, either they do not exist or they exist in some other way. So that the issue for us will not concern their existence, but the manner of their existence. [*Meta.* 1076a32-37]

Here Aristotle seems to entertain the possibility that one might deny the existence of mathematical objects and then leave it out of consideration. Annas [1976, 136] calls the denial of existence absurd, and I suppose that

if one attenuates the notion of existence enough it is absurd: there must be some sense in which mathematical objects exist, e.g., as figments of the imagination, so the only task is to figure out the sense. In *Meta. M* 2 Aristotle rejects the two alternatives he mentions, and in *M* 3 he gives his account of the way in which mathematical objects do exist. This account, I suggest, provides the justification of the geometer's or arithmetician's postulating of a genus. It embeds those genera into an ontology, but neither the ontology in general nor the embedding of mathematical objects in it is a subject of scientific proof. We can, I think, be certain that Plato tried to embed mathematical objects into a general ontology, and, if we can believe Aristotle, the method of embedding was some kind of derivation from first principles. Aristotle has many detailed objections to the derivation, but his main procedural difference from Plato seems to me to be his insistence on the difference between scientific proof (in the sense of the *Posterior Analytics*) and other kinds of argument.

Aristotle's complex and obscure treatment of the common axioms in *Meta. Γ* has been the subject of much discussion which I cannot go into here. Instead I content myself with some general points. Aristotle deals only with the logical principles and not the quantitative ones; but I am inclined to think that he would suppose the quantitative principles could be dealt with in much the same way, although he might imagine a sequence in which the logical laws were established first and then the quantitative ones. The method is dialectical or, as Aristotle calls it at 1006a12, refutational (ἐλεγκτικῶς; at 1062a3 it is called πρὸς τόνδε or *ad hominem*). The passage in which Aristotle discusses the method is very important for my purposes. Aristotle first asserts that proof has to start from something unproven to avoid an infinite regress and that anything one might start from in trying to prove the law of non-contradiction would be more in need of proof than it. He continues,

But one can prove that the denial of the law of non-contradiction is impossible by refutation, if only the person who denies it says something. But if he or she says nothing, it is absurd to try to say something against a person who has nothing to say in so far as he or she has nothing to say. For such a person, in so far as he or she has nothing to say, is like a vegetable. I say that proof by refutation differs from proof because a person who proves might be thought to be taking as a starting point what is to be proved, but if another person provides the starting point, there will be refutation and not proof. The starting point in all such cases is not the demand that the person assert or deny some proposition, since one might take this to be a begging of the question, but that the person signify

something both to himself or herself and to someone else. For this is necessary if the person is to say anything at all. But for someone who will not signify anything there will be no such thing as speaking either to himself or herself or to another person. But if someone will give this much, there will be a proof. [Meta. 1006a11–24]

As I understand Aristotle's position, it is that the law of non-contradiction is an assertion P which cannot be proved in the strict sense because the premisses needed for such a proof would be more doubtful than P . But P can be derived from any premiss at all, so that all we need to refute anyone who denies P is the person's willingness to say something and mean it. Now from a modern point of view if P can be derived from any premiss at all it is provable, since, e.g., it can be derived from its own denial. The fact that the would-be prover has to provide this premiss is irrelevant, since as the argument proceeds the premiss is eliminated, and P is proved without assumption.³⁴

It seems to me that Aristotle has been misled here by a certain asymmetry in the way he treats dialectic and demonstrative science. At its center dialectic is for Aristotle a procedure of argumentation involving two people, a questioner (Q) and an answerer (A). Q's questions are designed to elicit assertions from A from which inferences are drawn until a proposition (possibly the denial of one of A's original assertions) is reached. Aristotle frequently abstracts from the human situation of dialectic to the extent of ignoring the possibility that inferences are incorrectly drawn, but not to the extent of thinking that the results of dialectic could be severed from their connection with the opinions of A. For Aristotle dialectic can serve to defend one person's opinion against another's objections or refute a person's opinions, but its success is strictly relevant to individuals. Plato, on the other hand, seems to have felt that prolonged and strenuous dialectical exercise could yield a profound insight, an insight transcending what we would call the strictly logical implications of dialectical exchange.

In this respect Aristotle's treatment of dialectic is quite unlike his treatment of scientific reasoning, which he seems to sever more or less completely from its human practitioners. In doing so he gives the appearance of a total and naive faith in the science of his day; but we may, if we like, suppose him to be adopting the position that, at least with respect to the mathematical

³⁴ Further evidence that Aristotle misses this point is provided by *An. post.* 77a31–35. There Aristotle argues that dialectic cannot prove any proposition because the dialectician argues on the basis of answers to questions and so could prove a proposition only if he could establish it from opposite assumptions. Aristotle assumes this cannot be done, but, of course, it can if the proposition is logically true.

sciences, his role is descriptive rather than prescriptive. However, it seems fairly clear that Aristotle thinks some of the features of science he takes for granted must be the way they are. In any case, for Aristotle science consists first and foremost, if not entirely, in the correct derivation of truths on the basis of the starting points. Moreover, he thinks of derivation as direct, since he believes that *reductio* arguments play no essential role in science [see *An. prior.* 62b38–40; *An. post.* i 26]. If we allow indirect proof based on the refutation of assumptions introduced by the prover, then there is no reason why the refutation Aristotle thinks possible could not be counted as an assumptionless proof.³⁵

It is frequently pointed out that the refutations Aristotle provides in *Meta.* Γ presuppose the law of non-contradiction. I do not think Aristotle would find this presupposition an objection to his procedure, for, as we have seen, a person who does not already know the law of non-contradiction is incapable of learning anything and, hence, in particular, of learning something by having it proved. Another way of putting this point is to say that Aristotle thinks the non-vegetable already ‘knows’ the law of non-contradiction and merely has to be shown that his pretence not to know it is indefensible.³⁶ From the formal point of view it seems to me best to say that Aristotle uses a rule corresponding to the law in order to prove its formulation as an assertion. This way of putting the matter makes clear how little Aristotle’s refutations actually accomplish, while also making clear that there is a sense in which he accomplishes what he sets out to do, i.e., to justify one of the common axioms of the sciences. The real shortcoming in Aristotle’s approach to the common principles is his failure to recognize explicitly that these principles also include rules, and that reasoning cannot justify rules of reasoning. But for purposes of my historical analysis the important point is that Aristotle thought the common principles were assertions which could be justified dialectically, and that, from our point of view, the justification, if it were possible, would constitute a proof. By insisting that the justification is not a proof Aristotle separates himself from Plato, but once again the separation would seem to be much more a matter of distinguishing what Plato does not than of refusing to

³⁵ In my discussion I have not made use of the interesting suggestion by Irwin [1977–1978] that the treatment of the law of non-contradiction in book Γ is a model for a broadened Aristotelian conception of science which treats a restricted class of dialectical arguments as scientific. As far as I can see, what I have said would not be much affected if my contrast between scientific and dialectical argument was transformed into one between a narrower and a broader kind of scientific argument.

³⁶ Cf. *Meta.* 1005b23–26 where Aristotle suggests that anyone who denies the law of non-contradiction does not believe what he says.

engage in a kind of reasoning about mathematical starting points which Plato enthusiastically espoused.

6. Summary and conclusion

I conclude by presenting the sketch I have offered in a more chronological sequence, leaving out certain alternative possibilities I have considered, and adding some small details.

1. The mathematics known to Plato at the time of the writing of the *Republic* probably acknowledged only definitions as starting points. Plato called these definitions or the things defined hypotheses and believed that the definitions and hence mathematical theories could be encompassed in a universal body of knowledge which justified the definitions and performed some kind of ontological derivation of all entities. The evidence suggests that Plato did not distinguish, at least clearly, between these justifications and derivations, on the one hand, and strict deduction of the kind associated with, say, Euclid, on the other.

2. The mathematics known to Aristotle included as starting points in addition to definitions at least the first three common notions of the *Elements*, perhaps called axioms or common axioms. We cannot know why or how these principles were added, but they may be associated with Academic reflection on reasoning and argument. Aristotle includes fundamental logical laws among the axioms, and tends to think of them as the only premisses used in mathematical demonstration. The other starting points of a science for Aristotle are unique to each science. He sometimes thinks of these special starting points as things: the underlying genus consisting of fundamental objects and the properties which are proved of these objects. But frequently he treats them as assertions, namely, the assertion of the existence of the fundamental things and the definitions of them and of the properties. But for Aristotle these starting points, even construed propositionally, function as presuppositions of argument rather than as premisses. This conception of the genus and its properties as starting points of the science is Aristotle's philosophical interpretation and not a pure description of the science of his day.

Like Plato, Aristotle thinks of the practice of science as the derivation of conclusions based on starting points not to be discussed in the science. But whereas Plato sees this practice as inadequate and to be superseded by a universal science, Aristotle sees it as inherent in the nature of science. Hence, he argues that no science can justify its own starting points because the only justification it can offer is a proof based on its starting points. Aristotle also argues that no higher science can prove the starting points

of sciences like geometry and arithmetic, but here he ultimately has to rely on his theory that every science must presuppose a genus. Aristotle usually makes this point by denying a universal science. Here he is arguing against Plato, but ultimately his disagreement comes to an insistence on the distinction between deduction and looser forms of reasoning. For Aristotle allows metaphysical or dialectical justifications of the starting points; and in the case of the common axioms the justification he envisages would, if it worked, amount to a proof.

3. In Euclid's *Elements* we find definitions, postulates, and common notions as starting points. The definitions predominate, and confirm one's sense that the introduction of postulates and common notions into Greek mathematics was relatively late. Indeed, it seems to me reasonable to think that the postulates are due to Euclid himself, and result from an analysis of the propositions and constructions needed to reach the major results of the end of book 1. The common notions are more problematic, but we can be virtually certain that their explicit formulation in mathematical texts predates Euclid. In any case, I see nothing in Euclid's starting points which would suggest to an unbiased reader influence from the work of Plato or Aristotle. If I had to choose between Plato and Aristotle in this regard, certainly I would choose Aristotle. But the greater plausibility of this choice is surely satisfactorily explained in terms of Aristotle's concern to describe the sciences as they are rather than in terms of his alleged influence on the way sciences turned out to be.

Appendix 1: On Speusippus and Menaechmus in Proclus

In his commentary on Euclid's *Elements* Proclus says some things about the fourth-century figures Menaechmus and Speusippus. What he says about the latter has been taken by some³⁷ as evidence that Euclid's constructional postulates were already known in the fourth century. I have nothing to add to the arguments which have already been given against this reading of the evidence,³⁸ but it is worthwhile to look at the relevant passages since they provide a good example of how cautiously the reports of Proclus on early mathematics and philosophy have to be treated. At Friedlein 1873, 77.7 Proclus introduces the now commonplace division of propositions (προτάσεις) into problems (προβλήματα) or constructions and

³⁷ Notably von Fritz [1971, 392]. von Fritz 1969, 94–95 offers a brief response to critics, which perhaps shows that Speusippus might have formulated the constructional postulates but does not make the possibility any more likely.

³⁸ Notably by Tarán [1981, 427–428].

theorems (θεωρήματα). In practice the distinction is quite clear, although formulating it in general terms is not entirely easy. Proclus says,

Problems include the generations of figures, the divisions of them into sections, subtractions from and additions to them, and in general the characters that result from such procedures [i.e., the objects constructed?], and theorems are concerned with showing the essential attributes of each [of the things constructed]. [Friedlein 1873, 77.8–11]

He then tells us that certain fourth-century thinkers, notably Plato's nephew and successor Speusippus, thought it right to call all these things theorems rather than problems on the grounds that theoretical sciences deal with eternal things in which there is no generation. Hence, it is better to say that constructed objects exist and that 'we look on our construction of them not as making but as understanding them, taking eternal things as if they were in a process of coming to be' [Friedlein 1873, 78.4–6].

This passage clearly suggests that Speusippus collapsed an already existing distinction of mathematical propositions into theorems and problems by insisting that all problems are really theorems. But there are a number of reasons for initial skepticism about this. One is that we have no independent evidence for the existence in the fourth century of the later distinction between theorems and problems. In the fourth century theorems are things contemplated, problems are things proposed for investigation. Second, the attempt to collapse the distinction seems misguided: to say that constructions are ways of apprehending eternal things is not to deny that there is a difference between constructing a square and proving the Pythagorean theorem, a distinction which is marked grammatically by Euclid, who formulates theorems as assertions, problems using the infinitive ('to construct a square on a given straight line', and so on). I suggest that if Speusippus wanted to substitute the word 'theorem' for the word 'problem', he simply wanted to get away from the conception of science as answering questions raised or carrying out tasks assigned (whether of constructing or proving) and over to the conception of it as apprehending eternal truths. He might have referred to constructions³⁹ to underline the inappropriateness of geometrical language (as Plato does in the *Republic*), but his doing so need not imply that the process of apprehending truths through proofs is any less misleading about the character of the world of theoretical science.

³⁹ Proclus gives three examples of constructions (corresponding to *Elem.* i props. 1, 2, and 46) in his presentation of Speusippus' view. I see no more reason to suppose that these particular examples derive from Speusippus than any of Proclus' other examples which I discuss in this appendix.

This interpretation is confirmed by what Proclus says about Speusippus' alleged adversary Menaechmus. Menaechmus, he tells us, wanted to call all inquiries problems, but he distinguished two kinds of problems which might be proposed: one to provide what is sought, the other to see whether a thing has a certain property.⁴⁰ To suppose that Menaechmus abolished the distinction made by Proclus necessitates saying that he restored it as a dichotomy in the class of problems. The word 'inquiries', which Proclus has no motivation to supply, is a good indication that Menaechmus was speaking not about propositions, but about kinds of things into which one might inquire, i.e., problems in the standard dialectical sense.⁴¹ Menaechmus' division of problems may, indeed, be the origin of Proclus' (or even Euclid's) division of propositions into theorems and problems; but it is important to see that Menaechmus' relates to kinds of inquiries, not to mathematical texts like Euclid's *Elements*. Proclus' report on Menaechmus confirms what one would already expect, namely, that fourth-century geometers both proved theorems and carried out constructions; but it does not provide any evidence that the geometry textbooks of the fourth century marked the distinction in anything like the way Euclid does.

We are entitled to infer from this passage only that Speusippus called all geometrical knowledge theorems, i.e., matters of contemplation, for platonist reasons, and that Menaechmus made a distinction between two kinds of things into which a mathematician might inquire, i.e., between two kinds of problems. Sometime later Menaechmus' distinction was turned into one between two kinds of results (propositions) a mathematician might achieve, a construction (problem) and a theorem.

At Friedlein 1873, 178.1 Proclus turns to Euclid's postulates and axioms, which he considers to be kinds of principles (*ἀρχαί*). He suggests that the distinction between postulates and axioms parallels that between problems and theorems; but that principles must always be superior to the things after them in simplicity, unprovability, and self-evidence. He then cites Speusippus:

⁴⁰ Friedlein 1873, 78.10–13: ὅτε μὲν πορίσασθαι τὸ ζητούμενον, ὅτε δὲ περὶ ὁρισμένον λαβόντας ἰδεῖν ἢ τίς[ς] ἐστίν, ἢ πόλιν τι ἢ τί πέπονθεν, ἢ τίνας ἔχει πρὸς ἄλλο σχέσεις. I see no reason to accept Becker's suggestion [1959, 213] that περὶ ὁρισμένον should be πεπορισμένον and, therefore, no reason to accept his claim that the distinction made by Menaechmus was between solving a problem constructively and then investigating what has been constructed (for which reading one might expect τὸ πεπορισμένον).

⁴¹ Bowen [1983, 27n36], whose account of the Proclus passages discussed in this appendix differs considerably from mine, suggests that προβλήματα has another sense in the fourth century, namely geometrical demonstration or scientific deduction. The passages he lists do not seem to me to support the claim.

In general, says Speusippus, in the hunt for knowledge in which our mind is engaged, we put forward some things and prepare them for use in later inquiry without having made any elaborate excursion and our mind has a clearer contact with them than sight has with visible objects; but others it is unable to grasp immediately and therefore advances on them step by step and endeavors to capture them by their consequences. [Friedlein 1873, 179.14–22: trans. in Morrow 1970, *ad loc.*]

Proclus goes on to give examples to illustrate the difference between principle and subsequent result, and returns to the comparisons of postulates with problems and of axioms with theorems. He then says,

However, some people think it right to call all principles postulates, just as they call all things sought problems. Thus, Archimedes at the beginning of book 1 of *On Equilibria* says, 'We postulate that equal weights at equal distances are equally balanced.' But one might rather call this an axiom. Others call them all axioms, just as they call all things which need proof theorems. It would seem that these people have transferred words from special uses to common ones in accordance with the same analogy. [Friedlein 1873, 181.16–24]

It seems clear that Proclus is talking about Menaechmus and Speusippus; but it is striking that the example for calling all things postulates is drawn not from the fourth century but from Archimedes, who died at the end of the third and certainly did not call all principles postulates, but rather more or less completely disregarded the terminological distinctions Proclus thinks are important. We cannot exclude the possibility that Menaechmus used the word 'postulate' in something like the way suggested by Proclus, but the passage on problems suggests that at most he called anything taken for granted (or conceded) in a mathematical inquiry a 'postulate'.⁴² We have no reason to suppose that he distinguished kinds of postulates as he distinguished kinds of problems, nor that in calling them postulates he was reacting against a distinction between constructional and propositional 'principles'.

Proclus gives two pairs of examples to illustrate Speusippus' distinction between principles and things subsequent to them: Euclid's first postulate and his first proposition (the construction of an equilateral triangle); and his third postulate and the generation of a spiral by the motion of a point along the revolving radius of a circle. Tarán [1981, 427–428] has argued that

⁴² Possibly relevant to Menaechmus' discussion is the distinction Aristotle makes between a postulate and a hypothesis. See n14, above.

the examples are not Speusippus'. For my purposes it is sufficient to say that we cannot assume they are Speusippus' and, hence, cannot infer from this passage that Euclid's first and third postulates were already formulated as 'principles' in the mid-fourth century. We may, I suppose, accept that Speusippus called all principles axioms, but we have no very clear notion of why he would choose that word over other possibilities.⁴³ The last sentence of the last quotation suggests that Proclus had no information about the reason, but only assumes that the choice of 'axiom' as a name for principles is related to the choice of 'theorem' as a name for the things after the principles.

My conclusion is that the passages from Proclus which I have discussed tell us very little about fourth-century mathematics and philosophy of mathematics that we might not have guessed already. The most interesting information we get is perhaps that Menaechmus made a distinction between assertions to be proved and constructions to be carried out; for we have no explicit recognition of that distinction in Plato or Aristotle, although I think it must have been applicable to the mathematics they knew. We get the philological information about Speusippus' use of the word 'theorem' and Menaechmus' of 'problem' and perhaps about the former's use of 'axiom' and the latter's of 'postulate'. But none of this information seems to me to relate in any specific way to the content of fourth-century mathematics.

Appendix 2: Oenopides and Zenodotus

Proclus mentions [Friedlein 1873, 65.21–66.4] Oenopides of Chios in the so-called Eudemian summary of the history of geometry. It is natural to infer from this mention that Oenopides was active ca. 450 BC. Proclus also tells us [Friedlein 1873, 283.7–8] that Oenopides was the first to investigate the problem of dropping a perpendicular from a point to a straight line, a problem he thought useful for astronomy, and that [Friedlein 1873, 333.5–6] Oenopides was the first to discover [the solution to] the problem of copying an angle. In the second of these passages Proclus mentions Eudemus as source of information, so it is likely that Eudemus is Proclus' ultimate source for the first as well; there is also no reason to doubt that we are still dealing with the fifth-century Oenopides of Chios. Since the time of Heath [cf. 1921, i 175] it has been customary to say that Oenopides' innovation was to carry out the constructions in question with a ruler and

⁴³ It is interesting to recall Aristotle's conceptions of axioms as the common premisses of all sciences.

compass, since the dropping of a perpendicular could easily be solved using a draftsman's right angle. To this conjecture Szabó [1978, 275] has added another: Oenopides made conscious use of Euclid's first three postulates and is perhaps their originator. There is, however, a big difference between reducing certain constructions to others and laying down postulates as starting points. As for Heath's conjecture itself, it is very probable that Eudemus attributed to older geometers the solution of problems and proofs of theorems which he thought were presupposed by other knowledge ascribed to them [see, e.g., Dicks 1959, 302–303; Gigon 1945, 55; Wehrli 1969, 116]. Discussion of this point has largely focussed on Eudemus' ascription of certain results to Thales, but there is every reason to think he did the same sort of thing in the case of Oenopides.

Proclus mentions an Oenopides one other time in connection with a more philosophical matter:

Those around Zenodotus, who belonged to the succession of Oenopides and was a pupil of Andron, distinguished theorems from problems in the following way: a theorem inquires what property is predicated of its subject matter, a problem what is the case given that such and such is the case. [Friedlein 1873, 80.15–20]

This passage tells us all we know about Zenodotus, Andron, and the succession of Oenopides, so there is no real ground for von Fritz' assertion [1937, col. 2267] that Zenodotus was an 'Enkelschüler' of Oenopides of Chios.⁴⁴ The terms in which the distinction between theorem and problem is made are through and through Peripatetic,⁴⁵ suggesting a *floruit*

⁴⁴ And even if Zenodotus were the pupil of a pupil of Oenopides, there would be no more basis for inferring Oenopides' concerns from Zenodotus' than for inferring Socrates' from Aristotle's.

⁴⁵ As formulated by Proclus the distinction in question is almost certainly that between a categorical and a hypothetical assertion. There is no doubt that the description of a theorem is a description of a Peripatetic categorical assertion, the predication of a property of a subject. For evidence that a problem is being characterized as a hypothetical, consider Galen's description:

Another kind of proposition is that in which we do not maintain something about the way things are but about what is the case given that such and such is or what is the case given that such and such is not the case; we call such propositions hypothetical. [Galen, *Inst. log.* iii 1]

Cf. Aristotle's use of *τίνας ὄντος τὸ προκειμένου ἐστι* at *Top.* 111b17–18 with Alexander's comment *ad loc.*

If Proclus gives an accurate representation of Zenodotus' meaning, then Zenodotus presumably compared the givens of a problem with the antecedent of a conditional, the object constructed with the consequent. Thus, he might have

no earlier than the late fourth century for Zenodotus. Immediately after Zenodotus Proclus mentions the way Posidonius made the distinction (problems ask whether or not something exists, theorems what or what kind of thing something is) as if it were somehow derived from Zenodotus' (ὁθεν). This indication provides some support for a *terminus ad quem* of the 1st century BC, but in the absence of information about Andron or what is meant by the succession of Oenopides, the question of dating must be left open. Considerations of simplicity suggest that we identify this Oenopides with the fifth-century one mentioned by Proclus elsewhere, but this identification does not help to clarify the character of fifth- or fourth-century mathematics.

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read Euclid's first proposition as 'If there is a straight line, then an equilateral triangle can be constructed on it.'