

Science and Philosophy in Classical Greece
Edited with a Preface by ALAN C. BOWEN
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Ratio and Proportion in Early Greek Mathematics¹

D. H. FOWLER

Let Aristotle introduce the subject. His *Topics* is a manual of syllogistic dialectic, a kind of formal debate between two people called here the 'questioner' and the 'answerer'. At *Top.* 158a31–159a2 Aristotle writes:

There are certain hypotheses upon which it is at once difficult to bring, and easy to stand up to, an argument. Such (e.g.) are those things which stand first and those which stand last in the order of nature. For the former require definition, while the latter have to be arrived at though many steps if one wishes to secure a continuous proof from first principles, or else all discussion about them wears the air of mere sophistry: for to prove anything is impossible unless one begins with the appropriate principles, and connects inference with inference till the last are reached. Now to define first principles is just what answerers do not care to do, nor do they pay any attention if the questioner makes a definition: and yet until it is clear what it is that is proposed, it is not easy to discuss it. This sort of thing happens particularly in the case of the first principles: for while the other propositions are shown through these, these cannot be shown through anything else: we are obliged to understand every item of that sort by a definition. The inferences, too, that lie too close to the first principle are hard to treat in argument... . The hardest, however, of all definitions to treat in argument are those that employ terms about which, the first place, it is uncertain whether they are used in one sense or several, and, further, whether

¹ I use the phrase 'early Greek mathematics' to denote the period up to and including the time of Archimedes.

they are used literally or metaphorically by the definer. For because of their obscurity, it is impossible to argue upon such terms; and because of the impossibility of saying whether this obscurity is due to their being used metaphorically, it is impossible to refute them. . . . It often happens that a difficulty is found in discussing or arguing a given position because the definition has not been correctly rendered. . . . In mathematics, too, some things would seem to be not easily proved for want of a definition, e.g. that the straight line, parallel to the side, which cuts a plane figure divides similarly (ὁμοίως) both the line and the area. But, once the definition is stated, the said property is immediately manifest: for the areas and the lines have the same ἀντανάγκησις and this is the definition of the same ratio. . . . But if the definitions of the principles are not laid down, it is difficult, and may be quite impossible, to apply them. There is a close resemblance between dialectical and geometrical processes.²

In brief: Define your terms! The mathematical proposition that Aristotle is describing, in his typically vague fashion, must be the following:

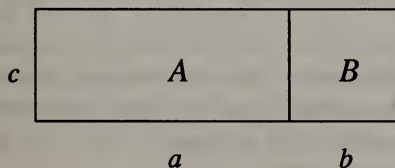


Figure 1

If a rectangle or parallelogram is divided by a line parallel to a pair of sides, as in Figure 1, then the ratio of the bases, $a:b$ is equal to the ratio of the areas, $A:B$.

I shall refer to this proposition hereafter as 'The *Topics* proposition': a similar result is proved by Euclid at *Elem.* vi prop. 1, where it forms the link between the study of the equal³ figures of books 1–4 and the similar figures of book 6 and later. It is no exaggeration to say that the *Elements*

² Most of this translation is taken from Ross 1908–1952 i; the mathematical example is adapted from Heath 1949, 80.

³ That is, equal in magnitude, according to the Euclidean conception of equality and inequality, set out in the Common Notions of book 1. This is sometimes described by mathematicians as 'cut and paste' equality.

hinges on this proposition,⁴ and so it is worth considering its proof in detail. This will be the subject of Section 1.

We start from Aristotle's 'first principles in need of precise definition':

What are 'ratio' (λόγος) and/or 'proportion' (ἀνάλογον)?

One difficulty with discussions about these words is that it is often uncertain, as Aristotle says, whether they are being used 'in one sense or in several... and literally or metaphorically'.⁵ They are often treated by ancient and modern writers as synonymous; the underlying concepts are expressed using a number of other apparently equivalent descriptions, and the words themselves have a very wide variety of non-mathematical connotations. Rather than introduce yet further words to identify the distinction I want to make and maintain, I shall hereafter use them with the following precisely differentiated meanings.

Ratio. Euclid states at *Elem.* v def. 3, that

Λόγος ἐστὶ δύο μεγεθῶν ὁμογενῶν ἢ κατὰ πηλικότητα ποιά σχέσις.

A ratio is a sort of relation in respect of size between two magnitudes of the same kind.

So, given these two homogeneous objects,⁶ *a* and then the ratio of *a* to *b* (abbreviated *a:b*) will be no more and no less than a description of just how this relation is conceived and expressed.

⁴ The role of the *Topics* proposition in the formal theories of proportion and the classification of incommensurables in the *Elements* is analysed in detail in Knorr 1975, ch. 8. However, his accompanying thesis depends heavily on the surprising difficulty of his anthyphairectic proof of *Elem.* v prop. 9 (that if $a:c :: b:c$; then $a = b$): see Knorr 1975, 338–340. But surely a practicing mathematician, faced with the difficulty that Knorr has uncovered, would proceed indirectly via the *alternando* property that he had just proved. For then $a:c :: b:c$ is equivalent to $a:b :: c:c$, whence the result follows immediately.

⁵ See, for example, the useful description in Mueller 1981, 138 (quoted in this very context in Berggren 1984, 400) of '[Euclid's] conception of definition as characterisations of independently understood notions'. Here we must draw out what these 'independently understood notions' might be and examine them against the historical background, such as we know it.

⁶ I shall use this word 'object' as synonymous with 'magnitude' or μέγεθος. Euclid does not give a definition of the ratio of two μεγέθη, though he does introduce the idea, as will be noted below.

Proportion. We then read in *Elem.* v defs. 5 and 6 that

Ἐν τῷ αὐτῷ λόγῳ [= ἀνάλογον] μεγέθη λέγεται . . .

[Four] magnitudes are said to be in the same ratio (that is, proportional) if . . .

This means that proportionality is a condition that may or may not hold between four objects. Given a , b , c and d , we then answer with either 'They satisfy the condition and so are proportional' (abbreviated $a:b :: c:d$); or 'They do not satisfy the condition so they are not proportional'; or 'They fail to satisfy some homogeneity condition so the question of proportionality does not apply.' For example, it would be meaningless to ask, in the context of the *Topics*, whether $a:b :: A:5$.⁷

The *Topics* proposition has now two different formulations which I shall distinguish. Either a , b , A , and B are proportional, i.e., $a:b :: A:B$, or the ratio of a to b is equal to the ratio of A to B , i.e., $a:b = A:B$ (where we must now also describe the conditions under which two ratios are equal).

Having taken care of the definitions of the first principles, I shall go on to the 'inferences that lie too close to the first principle and which are therefore hard to treat in argument', and discuss how the proof of the *Topics* proposition depends on the underlying definition of ratio or proportion. Only when we are aware of the range of possible ways of giving sense to the concepts involved should we consider what might constitute a deductive proof of this result in a given historical context. These different proofs will then illustrate two other contrasts which I wish to emphasise and discuss: that of arithmetised versus non-arithmetised mathematics, and that of early versus later Greek mathematics.

⁷ There are several possible homogeneity conditions: we may have—(i) a , b , c , and d all homogeneous magnitudes; (ii) a and b homogeneous magnitudes, c and d homogeneous magnitudes, but a and c not homogeneous; (iii) a , b , c , and d all ἀριθμοί; (iv) a and b homogeneous magnitudes, c and d ἀριθμοί; or (v) a and b ἀριθμοί, c and d homogeneous magnitudes. Euclid is ambivalent about (i) and (ii) in *Elem.* v [see Mueller 1970]; (iii) is the topic of *Elem.* vii; and the absence of any link between the theories of *Elem.* v and vii—hence, the absence of any discussion of (iv) and (v)—leads to a notorious lacuna in the proof of *Elem.* x prop. 5.

1. Seven proofs of the *Topics* proposition

1.1 The naive 'proof'

Find a common measure d of a and b . Then the rectangle D [see Figure 2] will be a common measure of A and B , and as many times (say n) that d goes into a , D will go into A ; and Therefore,

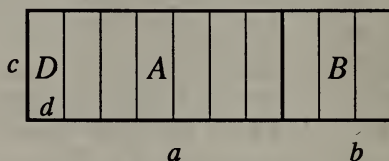


Figure 2

Comments. What does this tell us about ratio or proportion? What is the underlying definition? Neither of these ideas is mentioned explicitly and a generous interpretation of the 'proof' may suggest no more than an underlying definition that, for four ἀριθμοί, $m:n :: m':n'$ if $m = m'$ and $n = n'$; and a similar definition for four magnitudes $a:b :: A:B$ where $a = nd$, $A = nD$, $b = md$, and $B = mD$, for some magnitudes d and D .

What explicit evidence do we have for this proof, or some variation of it, as an argument of Greek mathematics of the fifth or fourth centuries BC? I know of none.

Notwithstanding these reservation, this kind of proof, often only implied, seems to dominate discussions of pre-Eudoxan ratio- and proportion-theory.

1.2 A geometrical definition and proof

Let $a, b, c, \dots$ denote lines, and $A, B, C, \dots$ plane regions of some suitably restricted kind; here, for example, rectangular regions will suffice. We suppose that these geometrical objects can be manipulated in the style of *Elements* i-iv. Define an operation, written here and later as a multiplication— $\otimes$ —(but any other word or abbreviation would serve equally well) as follows:

The product $a \otimes b$ of two lines is the rectangle with adjacent sides a and b , and the product $a \otimes B$ of a line and a plane region is the rectangular prism with base B and edge a .

And define

The objects x , y , z , and w are proportional if $x \otimes w$ and $y \otimes z$ make sense and are equal [see n3].

The proof of the *Topics* proposition then follows immediately [see Figure 3] since $a:b :: A:B$ means $a \otimes B = b \otimes A$, which is true since both are rectangular parallelepipeds with sides a , b , and c .

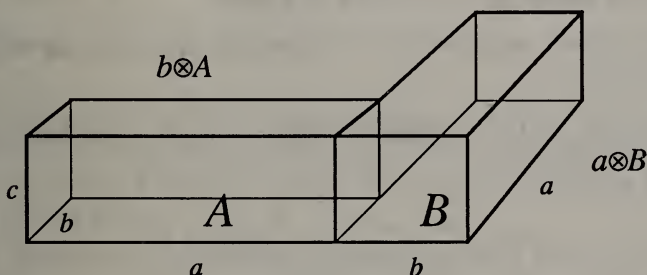


Figure 3

Comments. This proof is constructed from impeccably Euclidean ingredients [see *Elem.* ii def. 1 and vi prop. 16, where Euclid refers to ‘the rectangle contained (περιεχόμενον) by two lines’; and vii prop. 19, which gives a similar manipulation for four ἀριθμοί], and it is generally believed that this material dates from well before the proportion-theory of *Elements* v. The other basic results of Euclidean proportion-theory can be handled by an extension of this procedure. In other words, there is no difficulty in constructing a theory of proportionality from the basic techniques available, say, to Hippocrates and Theodorus. But I am not advancing this here as a proposal for a reconstruction of an early definition of proportionality; of this we have little or no evidence one way or the other, and all that we can say is that this kind of manipulation had become, by the time of Euclid’s *Elements*, a standard part of formal proportion-theory in geometry and ἀριθμητική. It is presented here in this form only to refute common assertions that fifth-century mathematicians did not have available the means to develop a theory of proportionality that would handle incommensurable magnitudes.

This is a proportion-theoretic proof: the ratio $x:y$ is not defined.

1.3 An arithmetised interpretation

For a long time, and explicitly since Descartes, geometry has been 'arithmetised'. In this type of interpretation

The letters $a, b, \dots; A, B \dots$ denote, ambiguously, either geometrical objects that are manipulated geometrically, or 'numbers' (or 'numerical quantities') that are manipulated arithmetically.

The area of a rectangle, a number, is the product of the lengths of its base and height, e.g., $A = a \times c$ in Figure 1.

The ratio of two magnitudes is the quotient of the corresponding numbers, $x:y = x/y$.

These definitions, and the permitted manipulations of arithmetic, then yield the following proof:

$$a:b = \frac{a}{b} = \frac{a \times c}{b \times c} = \frac{A}{B} = A:B$$

Comments. This has nothing to do with early Greek mathematics. The first time that anything like this is found in Greek geometry is in the metrical geometry of Heron, in the first century AD. I shall discuss this issue further in Section 2.

What is a number? Answers to this question are, in fact, easy to supply and any of the different definitions of ratio to be given below may be considered as 'numbers' in some sense. But a really difficult question is, How can we describe, correctly and completely, arithmetic with these numbers? I believe that this question posed a profound and perplexing problem to arithmetised mathematics, though the deceptive ease with which post-Renaissance mathematicians were apparently able to manipulate decimal numbers, newly introduced in the West at the end of the sixteenth century, enabled them to set the problem to one side for two centuries. But no satisfactory answer to the question was known before Wednesday, November 24th, 1858, the day when Dedekind says he conceived his construction of the real numbers. In his *Stetigkeit und die irrationale Zahlen* [1872, see Dedekind 1901], Dedekind defines addition in detail and then goes on to say:

Just as addition is defined, so can the other operations of the so-called elementary arithmetic be defined, viz., the formation of differences, products, quotients, powers, roots, logarithms, and in this way we arrive at real proofs of theorems as, e.g., $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$, which to the best of my knowledge, have never been established before.⁸

This is a ratio-theoretic proof: ratio is first defined and then proportion is defined to be equality of ratio. This last step is often assumed to be a mere formality, but in fact it can be very subtle. For decimal numbers, for example, the statement ' $0.999 \dots = 1$ ' evokes the mathematical ingredients of Zeno's paradox of Achilles.

Interlude: the historical context

In *Elem.* v we find a theory of proportion, based on v def. 5, which is believed either to be due to Eudoxus or to be a development of Eudoxus' ideas, and which is dated, in conception at least, to around 350 BC, just before Plato's death.

The *Elements* is dominated by the sheer bulk of book 10 and the subtlety of its application in book 13: book 10 sets up a classification of certain kinds of mutually incommensurable lines, and book 13 applies this classification to the lines that arise in the construction of regular polygons and polyhedra. Although it contains few explicit references to the idea of ratio, this material is clearly connected with the idea of the ratio (*not* proportion) of these lines. See, for example, the terminology of the definitions of *Elem.* x, where new lines are described by their relation to an assigned line and distinguished according as this relation is either expressible ($\rho\eta\tau\acute{o}s$) or without ratio ($\acute{\alpha}\lambda\omicron\gamma\omicron s$). See also the description in the culminating *Elem.* xiii prop. 18:

The said sides, therefore, of the three figures, I mean the pyramid, the octahedron, and the cube, are to one another in expressible ratios ($\lambda\acute{o}\gamma\omicron\iota$ $\rho\eta\tau\acute{o}\iota$). But the remaining two, I mean the side of the icosahedron and the side of the dodecahedron, are not in expressible ratios either to one another or to the aforesaid sides; for they are $\acute{\alpha}\lambda\omicron\gamma\omicron\iota$, the one being minor, the other apotome.

⁸ Translation from Dedekind 1901, 22. This book also contains the translation of *Was sind unter was sollen die Zahlen* [1888], in which Dedekind returns to, repeats, and emphasises this view. He also has some very apposite remarks about the relation between his definition of the real numbers and *Elem.* v def. 5. See Section 1.7 for further comments on arithmetic.

This programme of *Elem.* x and xiii is attributed, on reasonably good authority, to Theaetetus. Plato's eponymous dialogue is an encomium to the dying Theaetetus who has just been carried from what is believed to be the siege of Corinth in 369 BC.

Hence, we have evidence of massive activity in the study of *ratios* of incommensurable lines, in which the expressible ratios such as the diagonal to side of a square play a prominent role, well before the development of the proportion-theory of *Elem.* v.

It does not take much space to describe our positive and negative evidence concerning the early Greek mathematical idea of ratio (*not* proportion). Note first that *Elem.* v and vii describe theories of proportion for magnitudes and ἀριθμοί respectively, though some definitions in book 5—namely, definitions 3, 4, 9, 12, 13, 14, 16, and 17—refer to ratio. If we exclude material which, though it is expressed in terms of ratio is then immediately reformulated and used in terms of proportion, we have in the *Elem.* v def. 3 (quoted above), x and xiii (described above), vi def. 5, vi prop. 23, and vii prop. 5 (which refer to an operation of compounding ratios which then plays no further part in the *Elements*),⁹ and some definitions and propositions on the extreme and mean ratio, reciprocal ratios, and duplicate and triplicate ratios mainly to be found in *Elem.* vi. To this material in the *Elements*, we can add *Data* def. 2 ('A ratio is said to be given when we can make another equal to it') and the passage in Aristotle, *Top.* viii 3 with which we started and which may refer either to ratio or proportion. There are allusions to ratio in technical passages in Plato and Aristotle. That exhausts the positive surviving evidence. Our negative evidence is that we have no explicit sign whatsoever that early Greek mathematicians worked with any arithmetised conception of ratio: see Section 2 for further elaboration of this remark.

Let us now return to the *Topics* proposition.

1.4 Aristotle's proof

Aristotle summarises his proof thus:

τὴν γὰρ αὐτὴν ἀντανάρεσιν ἔχει τὰ χωρία καὶ αἱ γραμμαί· ἔστι δ' ὁρισμὸς τοῦ αὐτοῦ λόγου οὗτος.

For the areas and the lines have the same *antanairexis* and this is the definition of the same ratio.

⁹This is very clearly and thoroughly analysed in Mueller 1981, 88, 92–93, 135–136, 154, 162, 221, 225–226, and 229.

'Antanairesis' is an ordinary Greek word used to describe the subtraction of one thing from another; for example, we find it used interchangeably with ἀνθυφαίρεσις in commercial accounts on papyrus.¹⁰ The corresponding verb, with an adverb indicating a repetitive action, is used by Euclid in *Elem.* vii props. 1 and 2, x props. 2 and 3. Consider, for example, *Elem.* x prop. 2:

If, when the less of two unequal magnitudes is continually subtracted in turn (ἀνθυφαίρειν ἀεί) from the greater, that which is left never measures the one before it, the magnitudes will be incommensurable.

We can perform this operation of anthyphairensis on any pair of homogeneous objects. For example, given the ἀριθμοί (51,15), we subtract the smaller from the larger to get (36,15), then (21,15), then (6,15). The originally larger term is now smaller, so we now subtract it: (6,9), then (6,3). At this stage we see that the less or term 3 measures the term before it 6, and so [see *Elem.* vii prop. 2] the greatest common measure of 51 and 15 is 3. But also note, with Aristotle, that the relation in respect of size between 51 and 15, the 'anthyphairetic ratio', is characterised by this pattern: three subtractions, two subtractions, two subtractions, and no more.¹¹ If performed on two ἀριθμοί, the subtraction process will always terminate [see *Elem.* vii props. 1 and 2]; for two magnitudes, it may or may not terminate [see *Elem.* x props. 2 and 3].

Now consider the *Topics* proposition. We can characterise the relation of size, both between the two lines a and b and between the two areas A and B , by this subtraction process. But each subtraction of the line can be made to correspond to each subtraction of the rectangle standing on that line, and vice-versa [see Figure 4]. Hence, the pattern of the two subtraction processes will be the same. Moreover, since Aristotle says that this is the definition of the same ratio, the proposition is proved.

¹⁰ For examples taken from the same set of documents, the Zenon archive, see P. Lond. vii 1994.164, 176, 223 and 321 and vii 1995.333 (both dated 251 BC; here ἀνθυφαίρειν); and P. Cair. Zen. iii 59355.95 and 150 (243 BC; ἀνταναίρειν). Another equivalent, ἀνταφαίρειν, is found in Nicomachus, *Intro. arith.* i 13.11.

¹¹ One standard modern notation for this, used in my book and elsewhere, is to write $3:2 = [3, 2, 2]$. But the mathematical explorations can be carried through in natural language, without any symbolism, or using notations like this only as a convenient shorthand.

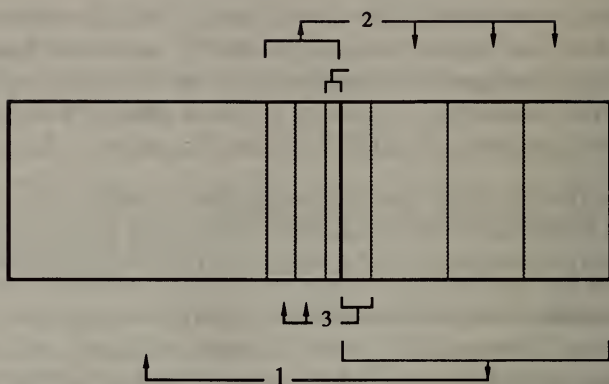
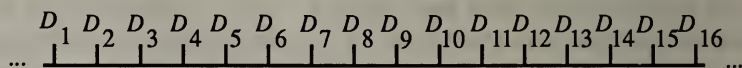


Figure 4

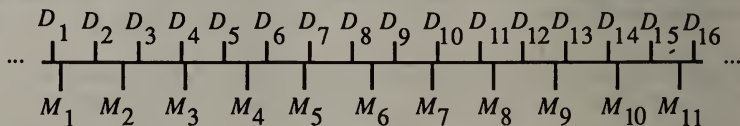
Note that this proof can be interpreted either in terms of ratio or of proportion: it is not entirely clear to which Aristotle refers.

1.5 A proof using astronomical ratios

We consider an idealised model of astronomy, in which all motions are uniform and go on forever without change or the least deviation. So, for example, we tally each sunset, the beginning of each day, starting the record at some arbitrary point:



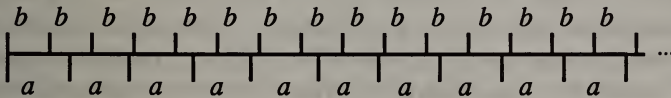
Then on the same tally, we can mark off some other uniform astronomical phenomenon like the conjunction of Sun and Moon which marks the succession of months. Suppose the period of this second motion is between one and two days long, else our tally will have to go on for a very long time before we see anything happening. We then will get a pattern like this:



$D, M, D, M, D, D, M, D, M, D, D, M, D, M, D, D, MD, \dots$

... 1 or 2, 1, 2, 1, 2, 1, 2, 1, 1, 2, 1, 1 or 2,...

Our fundamental insight is that these patterns also characterise the relation of size between the period of the two events: they define what may be called the astronomical ratio. Moreover, we can apply the same procedure to two geometrical objects. Take, for example, two lines a and b . We can now start the tally with a coincidence



a and $b, b, a, b, a, b, b, a, b, \dots$

Figure 5

In the configuration of the *Topics* proposition, the tallies of the bases, a and b , and the rectangles, A and B , will clearly give rise to the same pattern. So, again, with this underlying definition of astronomical ratios, the proposition is proved.

Again note that this is a ratio-theoretic formulation of the *Topics* proposition; indeed it is the definition of ratio that underlies the theory of proportion developed in *Elements* v. Also, in the geometrical context, when we can arrange for a coincidence with which to begin our tallying, there is no difficulty in identifying when two ratios are equal, and so converting this ratio-theoretic formulation and the proof into proportion-theory; we then get Euclid's proof of *Elem.* vi prop. 1. But we have no such liberty in the astronomical context, and different phase shifts between the two events will give rise to different patterns. How to recognise when ratios are equal in respect of size, and only differ in respect of phase, is then very far from obvious;¹² so one sees again that it is not always easy to pass from ratio-theory to the corresponding proportion-theory.

1.6 Variations on a theme

If we contemplate the last two proofs, we see that the pattern of many different addition or subtraction processes, performed on two homogeneous objects a and b , may be used to characterise the relation of size between a and b . For example, instead of using a process of alternating subtraction, we can always subtract from the first object, or from the second; instead of always undershooting, so subtracting with remainder, we can overshoot, and continue with the excess; at each step we can perform some specific predetermined scaling operation; and so on. The only general properties we use of the underlying objects is that we can compare any two to determine if they are equal or which is the greater, and that we can add any two or subtract the less from the greater.

Any such process, consistently applied, will generate a pattern that will then characterise the relation of size between the two original objects; and hence from each idea of ratio there will be a corresponding proof of the *Topics* proposition. Here are two examples of what I shall call decimal ratios and accountant's ratios.

Decimal ratios. Suppose that $a \geq b$. Compare a with b , $10b$, 10^2b , ... to locate that index k for which $10^k \leq a < 10^{k+1}b$. Then define

$$\begin{aligned} a &= n_k \times 10^k b + a_{k-1} \quad \text{where } 0 \leq a_{k-1} < 10^k b \\ a_{k-1} &= n_{k-1} \times 10^{k-1} b + a_{k-2} \quad \text{where } 0 \leq a_{k-2} < 10^{k-1} b \\ &\text{and so on.} \end{aligned}$$

¹² Such a procedure, found by E. C. Zeeman, is described in Zeeman 1986 and Series 1985.

The process will continue indefinitely; each n_j lies between 0 and 9, and if any remainder a_j is zero, then all subsequent terms $n_j, n_{j+1}, n_{j+2}, \dots$ will be zero. Also the procedure has been arranged so that this sequence is uniquely defined and cannot finish with an unending sequence of nines.

The pattern described by this sequence $n_k, n_{k-1}, \dots, n_0, n_{-1}, n_{-2}, \dots$ will then characterise the relationship of size between the pair a and b . For the first time in these illustrations, the second term b plays a privileged role, here as a 'unit' in a scale of measurement. Conventionally we write this particular ratio as a decimal number $a:b = n_k n_{k-1} \dots n_1 n_0 \cdot n_{-1} n_{-2} \dots$ (If $a \leq b$, we compare a with $b, \frac{b}{10}, \frac{b}{10^2}, \dots$ and get $0 \cdot 0 \dots 0 n_k n_{k-1} \dots$) Here again we have a proof of the *Topics* proposition, since the pattern of the subtraction process generated by a and b will again be the same as the pattern generated by A and B .

The importance of this algorithm for decimal subtraction is that there is an almost universal delusion, even among mathematicians, that we can easily perform arithmetical operations on these decimal ratios—that we can add, subtract, multiply, or divide decimal 'numbers'—and that it is obvious that this arithmetic satisfies the usual manipulations of arithmetic like

$$x \div y = (x \times z) \div (y \times z) = (x \div z) \div (y \div z)$$

for any three 'numbers' x, y and z . Also ':' and ' $\div$ ' are now treated as being virtually synonymous. This leads to the following version of the earlier arithmetised proof of the *Topics* proposition in Section 1.3.

Let a, b, c, A , and B be as in Figure 1, and fix some standard line 1, the unit of length. This unit determines a standard square 1^2 , the unit of area. Denote the decimal ratios (or 'numbers') $a:1, b:1, c:1, A:1^2$, and $B:1^2, a', b', c', A'$, and B' , respectively. Then, by the supposed basic properties of arithmetic:

$$a:b = (a:1):(b:1) = a':b' \text{ and } A:B = (A:1^2):(B:1^2) = A':B'.$$

Use multiplication to define a numerical area, where

$$\text{area} = \text{base} \times \text{height}.$$

This turns out to be equivalent to the earlier numerical definition of area:

$$A' = a' \times c', \quad B' = b' \times c'.$$

Hence,

$$a:b = a':b' = (a' \times c'):(b' \times c') = A':B' = A:B$$

This 'proof' is ridiculous, though it would take a lengthy digression to analyse fully its mathematical and historical solecisms. As pointed out earlier, the flaw is that the proof depends on the underlying arithmetic, but this will not be properly set up before the development of the idea of the real numbers. Fortunately this digression is not necessary here; my concern is early Greek mathematics, and problems with and delusions about the role of decimal numbers in the arithmetisation of mathematics are not part of this. However, there is a very similar process that differs only by a change of base—sexagesimal ratios—that it is necessary to consider briefly, since sexagesimal arithmetic is found in Babylonia some 1500 years before the development of early Greek mathematics. Concerning this, I here make only the following two observations, and defer further comment to Section 2, below.

First, the problems with decimal and sexagesimal arithmetic arise with 'non-terminating' decimal numbers, those ratios in which an infinite number of the n_k are non-zero, and the consequent difficulties that arise from the possibility of a 'carry' through an arbitrarily long sequence of digits.¹³ Babylonian arithmetic shows a proper caution about this, since many (thought not all) of the manipulations that are found are restricted to the terminating or 'regular' sexagesimal numbers.

Second, our earliest trace of sexagesimal numbers in Greek mathematics are found around the second century BC, in the work of Hypsicles and Hipparchus. We have as of yet no explicit evidence of any influence of Babylonian arithmetical procedures on early Greek mathematics.¹⁴

Accountant's ratios. Let me illustrate this final definition by an example. The ratio 65:24 is more than twice, less than three-times; that is,

$$65 = 2 \times 24 + 17 \quad \text{or} \quad a = n_0 b + a_1 \quad \text{with} \quad a_1 < b.$$

We now describe 17:24. Since 17 goes once, not twice into 24, this ratio is more than half:

$$2 \times 17 = 24 + 10 \quad \text{or} \quad n_1 a_1 = b + a_2 \quad \text{with} \quad a_2 < a_1.$$

¹³ See Fowler 1985a and 1985b for illustrations of the difficulties with decimal arithmetic.

¹⁴ On this topic, see Berggren 1984, 397–398: 'If the event [of Babylónian influence on pre-Euclidean mathematics] cannot be located historically one must recognise the possibility that it may not have occurred.'

We continue by comparing the remainder 10 with 24; it is more than the third:

$$3 \times 10 = 24 + 6 \quad \text{or} \quad n_2 a_2 = b + a_3 \quad \text{with} \quad a_3 < a_2.$$

Finally

$$4 \times 6 = 24 \quad \text{or} \quad n_3 a_3 = b.$$

Hence, the process is described by the pattern 2, 2, 3, 4, and no more, a kind of pattern which again gives an immediate proof of the *Topics* proposition. Again, this is not a reciprocal subtraction process: after the first step, the current object is always subtracted from the second term up to the first overshoot, which then becomes the object for the next step.

The special interest of this process is that we again have an arithmetised interpretation. In our modern notation,

$$\frac{65}{24} = 2 + \frac{1}{2} + \frac{1}{2 \times 3} + \frac{1}{2 \times 3 \times 4}$$

or

$$\frac{a}{b} = n_0 + \frac{1}{n_1} + \frac{1}{n_1 \times n_2} + \frac{1}{n_1 \times n_2 \times n_3} + \dots$$

(This is, in fact, not unlike the way fractional quantities were expressed by Greek accountants and mathematicians, though our evidence also makes it quite clear that this particular algorithm was not used to generate the expressions we find them using, since their expressions do not exhibit the characteristic pattern $n_1, n_1 \times n_2, n_1 \times n_2 \times n_3 \dots$.) Again, although this suggests that there may again be an underlying arithmetised proof of the *Topics* proposition, the details of such a proof are far from obvious.

2. Arithmetic and ἀριθμητική

The previous section illustrated a series of distinctions between ratio and proportion, between complete proofs (seven, on my count) founded on explicit definitions and optimistic pseudo-proofs in which the crucial and difficult details were omitted (three, one in Section 1.1, two in Section 1.6), and between arithmetised and non-arithmetised mathematics. It is that last distinction that I now wish to consider.

In brief, my proposal is that early Greek mathematics and astronomy show no trace of influence of arithmetisation. The emphasis on 'early' in 'early Greek mathematics' is essential: after the amalgamation of Babylonian and Greek techniques, which seems to take place from the second century BC onwards, we do find examples of a Greek arithmetised mathematics in Heron and thereafter, and a Greek arithmetised astronomy in

Ptolemy and thereafter. Thus, there is a profound split between the aims and conceptions of early and later Greek¹⁵ mathematics, such as I conceive them: note, for example, how neither Theaetetus' programme of *Elements* x and its application in *Elements* xiii nor the sophisticated Eudoxan astronomy seem to have any place in these later arithmetised traditions. What is more, today's mathematics is profoundly arithmetised, founded as it is on the use and intuition of what has come to be known, comically, as 'the real numbers'.¹⁶ This again interferes with our understanding of early Greek mathematics; indeed one of the difficulties in the way of understanding my reconstruction is the problem of purging one's mind of this arithmetised way of thinking.

Early Greek mathematics does, of course, draw on the use and intuition of some kinds of numbers, as I shall describe briefly. Most fundamental are the ἀριθμοί, conceived in a very concrete sense which is best conveyed in English by the series

solo, duet, trio, quartet,

Moreover, the words usually occur with the definite article, which enhances further their concreteness. In formal mathematics the unit has a different status from the rest [see *Elem.* vii defs. 1–2], which means that sometimes this case has to be distinguished as, e.g., in *Elem.* vii prop. 2. The ἀριθμοί are also found in different grammatical forms, such as the repetition-numbers

once, twice, three-times, four-times,

The grammar of natural language describes the manipulations of the ἀριθμοί. For example, contrast 'four-times the duet gives the octet' with the abstract manipulations of abstract symbols, ' $4 \times 2 = 8$ ', that we tend to learn and use today. So there would not be the same temptation among early Greek mathematicians to extend the scope of these abstract manipulations and objects, for example, to extend $8 \div 4 = 2$ to the case of

¹⁵ 'Greek' means 'written in Greek'. Early Greek mathematics is geographically Greek: our evidence points to Ionia, Magna Graecia (Southern Italy and Sicily), mainland Greece, and then the Greek colony of Egypt. This *portmanteau* designation, 'Greek', later comes to encompass a vast collection of different traditions and influences.

¹⁶ See, for example, Mueller 1981, ch. 7 for recognition of the difficulty of incorporating *Elem.* x into arithmetised mathematics (e.g., 1981, 271: 'One would, of course, prefer an explanation that involved a clear mathematical goal intelligible to us in terms of our own notions of mathematics Unfortunately book 10 has never been explicated successfully in this way, nor does it appear amenable to explication of this sort').

$8 \div 3 = ?$, or to move to higher degrees of abstraction such as $a \times b = c$. The Greek formulation of division in formal mathematics tends rather to the use of manipulations such as 'the trio goes into the octet twice leaving a duet as remainder' [see *Elem.* vii props. 1–4], or the general verbal descriptions to be found in *Elem.* vii defs. 5–10. I shall refer to these Greek investigations of the ἀριθμοί by their Greek name, as ἀριθμητική, and thus distinguish between that which we find in early Greek mathematics and the later arithmetic which concerns more general and abstract kinds of numerical quantities.

Everyday Greek accounting does employ a system of describing fractional quantities, traces of which are also found in formal mathematics. This is based on the system of the μέρη,¹⁷ which are best conceived as the series:

the half, the third, the quarter, the fifth, . . .

and represented by a transcription of their Greek notation: ζ , γ , δ , ϵ , . . . , for example, as $\acute{2}$, $\acute{3}$, $\acute{4}$, $\acute{5}$, Here the definite article is an essential aid to understanding: neither the words nor the notation contain those features that lead easily to our conception of our common fractions, where we can pass almost imperceptibly from 'one fifth' and ' $1/5$ ' to 'two fifths' and ' $2/5$ ', and so on. Moreover, Greek fractional quantities are always expressed and always seem to be conceived as sums of different μέρη, in what is often called the 'Egyptian' system. In fact, I do not believe we have any convincing evidence for anything corresponding to our common fractions m/n in Greek scientific or everyday life. What is usually taken as the notation for common fractions¹⁸ seems rather to be an abbreviation employed extensively by Byzantine scribes but found in very few documents before then, in which the phrase τῶν m τὸ n ('of m the n^{th} ') is abbreviated as $\acute{n}/m$. But this phrase 'of m the n^{th} ', the standard phrase used to describe division, always seems to be conceived and is almost always immediately expressed as a sum of μέρη, for example

τῶν	ιβ	[τὸ $\acute{\iota}\zeta$]	$\zeta\acute{\iota}\beta\acute{\iota}\zeta\lambda\acute{\delta}\nu\acute{\alpha}\xi\eta$
of the	12	[the 17 th is]	$\acute{2}\acute{1}\acute{2}\acute{1}\acute{7}\acute{3}\acute{4}\acute{5}\acute{1}\acute{6}\acute{8}$

for what we now write as

$$\frac{12}{17} = \frac{1}{2} + \frac{1}{12} + \frac{1}{17} + \frac{1}{34} + \frac{1}{51} + \frac{1}{68}.$$

¹⁷ Greek has two words, μέρος and μόριον (plural: μέρη and μόρια), which appear to be perfectly synonymous.

¹⁸ For an influential description, see Heath 1956, i 42–45.

These expressions would have been looked up in division tables, of which many examples have now been published. The only manipulations of these expressions that are found are very restricted: 'of m the n^{th} ' is seen to be the same as 'of km the kn^{th} ', and 'of m the n^{th} ' and 'of p the n^{th} ' can be added or subtracted to give 'of $m \pm p$ the n^{th} ', where all of the expressions are, I repeat, still conceived as sums of $\mu\epsilon\rho\eta$. Nowhere, to my knowledge, do we get an example where two general expressions 'of m the n^{th} ' and 'of p the q^{th} ' are directly combined without going through some sequence of these basic manipulations.

Most of our evidence comes from school or commercial texts, far removed from the kind of mathematics in which we are primarily interested here. Unfortunately the most pertinent mathematical text, Archimedes' *Measurement of a Circle*, survives only in a very late and corrupt version which shows clear signs of interference by scribes and commentators. Aristarchus' *On the Sizes and Distances of the Sun and Moon* has survived in a less corrupt state though, here again, as with almost all of our evidence, our only text is a Byzantine copy made in the ninth century AD. Nevertheless, my description above also fits the evidence that we find in both of these calculations.¹⁹

The arithmetic of these sums of $\mu\epsilon\rho\eta$ is very clumsy and does not show any promise of an interesting or useful mathematical theory. This might be interpreted as an explanation why early Greek mathematicians do not seem to bring to bear on their mathematics any intuitions about arithmetical manipulations with fractional quantities, if such an explanation is needed for us, today, to come to terms with what seems to be an uncomfortable feature of our evidence. My own preference is to state boldly and accept completely that such evidence we have of early Greek mathematics shows no influence of arithmetisation; and not, at this stage, to attempt to fabricate any further explanation.

3. *Envoi*

It may be difficult for someone brought up within the now universal and highly successful tradition of arithmetised mathematics to conceive that there are many ways of handling ratios other than as something that is, or is approximated by, some suitably formulated kind of numerical quantity, such as common fractions m/n for commensurable ratios, or some systematically organised collection of common fractions, like decimal or sexagesimal

¹⁹ A much more complete description of our evidence concerning Greek calculations appears in Fowler 1987.

fractions, for incommensurable ratios. I, at least, once found this difficult, so that the substantial explorations of non-arithmetised ratio theories set out in my book [Fowler 1987] were a liberating experience. Moreover, the exploration of these ideas revealed many remarkable and unexpected mathematical and historical insights. I do not wish to attempt to summarise this material here, so I will finish with one illustration and refer the reader to the book for more details.

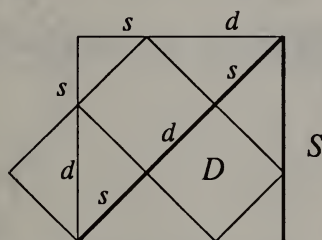


Figure 6

Let us work out the anthyphairetic ratio of the diagonal to side of some regular polygons. We start with the square. Let s and d denote the diagonal and side of some given square. The beginning of Socrates' encounter with the slaveboy at Plato, *Meno* 82a–85c brings out that $s < d < 2s$; hence, the first step of the anthyphaireses of $d:s$ will be one subtraction and not two, and the remaining steps will then be described by the ratio $s:(d-s)$. In the notation of n11 we can write this as $d:s = [1, s:(d-s)]$. The idea is general:

$$a_0:a_1 = [n_0, a_1:a_2] = [n_0, n_1, a_2:a_3] = \dots$$

We now need to evaluate $s:(d-s)$. Perhaps, like Meno's slaveboy, we also need the help of a diagram such as is given in Figure 6. Here we construct a new larger square whose side S is the side plus diagonal of the smaller oblique square in the left-hand corner:

$$S = s + d;$$

then, by filling in some lines in the figure, we see that the larger diagonal is equal to two small sides plus the small diagonal:

$$D = 2s + d.$$

Since the size, location, and orientation of our square are immaterial,

$$s:(d-s) = S:(D-S) = (s+d):s$$

which we can now evaluate as two subtractions, followed by the ratio $s:(d-s)$. We now go round and round: $s:(d-s)$ is therefore, twice, twice, followed by itself, so it is twice, twice, twice, twice, followed by itself, and so on. Hence, the anthyphairctic ratio of the diagonal to side of a square is once, twice, twice, twice, . . .

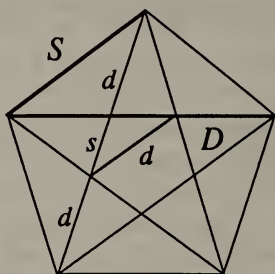


Figure 7

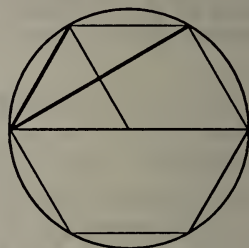


Figure 8

The reader is recommended to use a similar argument, applied to Figure 7, to evaluate the ratio of the diagonal to side of a pentagon. Now consider the ratio of a diagonal to the side of a hexagon [see Figure 8]. The longer diagonal is twice the side—that is a description of the anthyphairctic ratio—while the square on the shorter diagonal is three-times the square on side [for details, see Euclid, *Elem.* xiii prop. 12]. This ratio can be evaluated in the context of the following more general programme: Given a line and two ἀριθμοί n and m , we can use *Elem.* ii prop. 14 to construct squares equal to n -times and m -times the square on the given line. What can we say about the anthyphairctic ratio of their sides? The answer to this question plays a central role in my reconstruction: it involves heuristic explorations followed by a range of different proofs based on the figures of *Elements* ii; and it leads to a motivation for *Elements* x, and to a new description of the problems and motivations of early Greek mathematics.²⁰

²⁰ An outline of some of these interpretations may also be found in Fowler 1979 and 1980–1982.

An earlier version of this paper was presented at a colloquium, 'Logos et théorie des catastrophes: A partir de travail de R. Thom' (Centre culturel Cerisy la Salle, September 1982), and has subsequently circulated in duplicated form. I wish to thank the many people who have offered comments. Some of the topics discussed here are treated more fully in Fowler 1987.