

Science and Philosophy in Classical Greece
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What Euclid Meant: On the Use of Evidence in Studying Ancient Mathematics

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For most historians of mathematics the principal data are documents—records of past thoughts preserved in writing. It follows that the interpretation of documents is central to the methodology of historians and, hence, that discussions of the principles of interpretation can be brought to bear on efforts in this field.

As a specialist in mathematical history, I have found that my colleagues in the areas of literary studies tend to register surprise at the thought that mathematical texts are subject to *interpretation*, even as they take for granted that all literary texts require interpretation. Moreover, I would anticipate that associates in the disciplines of mathematics and the physical sciences would be surprised—perhaps appalled—at the suggestion that the understanding of technical documents could be illuminated through the insights of theorists of literary criticism. Somehow, the patent universality of mathematical discourse might be construed as precluding the relevance of critical principles whose objective is to offer guidance in the study of individuals in their special historical circumstances.¹

My project in the present essay is to explore this meeting ground between historical study and literary theory. My focus will be on the particular issue of the role of authorial meaning in the work of the critic. After a brief synopsis of some ideas from recent debates, I will discuss their bearing on three problems in the interpretation of Euclid's mathematics: his conceptions of ratio and proportion, his notion of fraction, and the aim over all of the *Elements*. From the outset I must emphatically disclaim any special expertise

¹ This ahistorical, Platonizing tendency of mathematicians and mathematical historians is noted and criticized in Unguru 1979.

in the wide-ranging field of hermeneutics. My approach here is entirely pragmatic: to select from the diversity of views those which I perceive can assist the historian in the effort to understand why disagreements arise in the examination of such problems and what is implied within the different options that one might espouse in their interpretation.

1. Authorial meaning in literary criticism

We all continually subscribe to the view that we can formulate our ideas in writing and successfully communicate them to others. After all, did the ancients not invent writing precisely for this end? But contemporary critics have come to recognize the difficulties in applying this common sense notion toward the interpretation of literature.² In the old *régime* one approached a text with the assumption that there was a *datum*, designated as the author's meaning or intention, which was the object of critical exegesis. Within the 'new criticism' of this century, however, profound doubts were expressed: one could multiply examples of how one and the same case had received diverse, incompatible accounts of its author's meaning; one could note the drastic consequences that the assumption of irony has for the interpretation of a text, yet the frequent difficulty of establishing an author's ironic intentions; and so on.³ By way of reaction, comes scepticism which emphasizes the problems of *access* and *relevance*: How can we presume to enter into the mind of an author? and Why should we even want to do this as part of our critical efforts?

A particularly trenchant statement of the sceptical position was put forward by W. K. Wimsatt, Jr. and M. C. Beardsley under the rubric of the 'intentional fallacy'.⁴ In the course of time, their essay has been invoked in support of positions far more extreme than theirs, so that the 'intentional fallacy' has come to signify for some the impossibility of any critical use of the concept of authorial meaning.⁵ In such exaggerated formulations, one maintains that the special, private circumstances of an author are beyond

² H. Parker [1984, 213-243] sketches some main currents in modern criticism, with particular emphasis on the teaching of American literature. He is decidedly more antithetical to the new criticism, even than Hirsch [see below].

³ See the synopsis of the arguments in Hirsch 1967, ch. 1.

⁴ Cf. Wimsatt and Beardsley 1946: their sceptical position on authorial meaning extends the views expressed in Richards 1929.

⁵ Hirsch [1967, 11-12] notes that the 'popular version' has severely exaggerated the claims that Wimsatt and Beardsley actually maintain in their essay. Cf. also H. Parker 1984, 214-215.

our ken and irrelevant to the critical task anyway; that one can consider only the public meanings attributable to the text. A text has no fixed meaning; the process of interpretation is dynamic, as readers respond to it in their individual ways. As critics, we are bound to our own historical circumstances. Instead of aiming to render a historically correct account of the text's original meaning, then, we should seek to articulate our own responses.⁶

Even those unsympathetic to this position admit to the positive effects its adoption has had on the critical disciplines and the teaching of literature in recent decades.⁷ Whereas it had been common earlier to glean literature as a source of historical, social or political information, for instance, one now could analyze literary products for themselves: a poem is a poem, and only incidentally, say, a record for the reconstruction of the author's biography.⁸ Nevertheless, the sceptical position effectively abandons the historical project: if no interpretation of a text is privileged, all are equivalent and it becomes meaningless to examine historical texts for their historical content.

While the sceptical vein represented by the intentional-fallacy argument has been intensified in some circles, others have proposed counter-arguments in defense of a more traditional literary methodology. A particularly thorough venture of the latter type is the hermeneutical study by E. D. Hirsch, Jr.⁹ A brief account can presume neither to do justice to the richness and subtlety of his discussion, nor to give due coverage to the rejoinders from advocates of other positions. I hope merely to provide here a synopsis of

⁶ I venture to note a certain parallel with contemporary developments in the philosophy of science: the older objectivist-positivist views now seem naive, as the subjective elements implicit in scientific theory and research have come to be recognized even by those who would still favor some form of scientific realism.

⁷ Cf. Hough 1966, 62 which maintains that for students of literature, Richards' theory was 'extremely fruitful' for providing 'a means of compelling close attention to the work itself and the processes involved in reading it, as a prophylactic against conventional and secondhand judgments. . . . But it is not the normal kind of reading.'

⁸ A pertinent example appears in the essay by Charles Kahn [see ch. 1, above]: that the older view of Hesiod's work had inclined toward an anthropological analysis, while recent efforts have attempted instead to grasp the impact and purposes of his poetry as *poetry*.

⁹ Hirsch's account is widely known among a diverse range of scholars in the humanities, and is highly respected even among those who do not adopt his relatively traditional position. Needless to say, the field of criticism over the past two decades has grown enormously, and one might incline to view such efforts as Hirsch's as outdated. My objective here, however, is not to show that one or the other critical theory is *correct* (whatever that could mean), but rather that Hirsch's views in particular can be *useful* for the practicing historian.

some key notions that will be of service in the subsequent discussion of ancient texts.

In Hirsch's view [1967, ch. 4], the author's meaning is not only a legitimate aim of criticism, it is the only possible aim: for it alone is shared by all interpreters of the given text.¹⁰ The critic's work, he maintains, is of two basic sorts: to give an account of the *meaning* of the text and of its *significance*. The latter embraces the major portion of criticism as such (indeed, the whole of it, in the sceptical view): the connections between the text and whatever else the critic chooses, the critic's personal response to the text, and so on. But a precondition of any discussion of a text's significance, Hirsch continues, must be an accurate grasp of its meaning. This entails two projects: to *understand* the text and to *explain* (or interpret) it. In explaining a text, the interpreter seeks to communicate its meaning to others. To this end, one typically resorts to paraphrases, recasting the text in terms calculated to be familiar to the audience.¹¹ One may well introduce elements entirely extraneous to the text itself, and it is a subtle demand on the interpreter to make sure the meaning of the text is not violated in the process.

To explain the text, the interpreter must already understand the meaning of the text, that is, the meaning intended by the author. The sceptics maintain, however, that this sort of meaning is inaccessible to us. But Hirsch [1967, ch. 5 and app. 1, sect. c] here introduces another distinction: if we insisted on *certainty* in our understanding of the author's meaning, the sceptics would be sustained. But the critic seeks not certainty, but *validity* of interpretation. Validity is a probabilistic notion; the interpreter engages in a heuristic process, refining and modifying tentative conceptions of the text's meaning, and so achieving interpretations of progressively increasing probability of being correct. In effect, the author's meaning is the limit of this heuristic process; without it, the process would have no object or criterion of accuracy. But how does one gauge the validity of one's account—that is, as being highly probable, or plausible, or merely possible? Hirsch cites four criteria: *legitimacy* (e.g., the account must

¹⁰ A useful synopsis of Hirsch's position on the verification of meaning appears in 1967, app. 1, esp. sect. c.

¹¹ Hirsch's distinction of the interpretive and critical functions of textual commentary [1967, ch. 4, sect. b] might be used to suggest a position on the issue of geometrical algebra, currently debated among historians of ancient mathematics [see Unguru 1979]. To explain certain aspects of ancient geometry, it may become advisable, even necessary, to import notions from more recent fields, like algebra. This raises the possibility of anachronism, as is present in all analogical forms of exegesis. That risk becomes acceptable if the alternative is the learner's incomprehension.

assign to words of the text only those meanings which are possible for the author and his contemporaries); *correspondence* (each linguistic component of the text must be accounted for); *genre appropriateness* (where 'genre' embraces those conventions and expectations pertinent to the text which the author and his audience will share); and *coherence* (the interpretation must be plausible in the context of the whole of which it is part).

Hirsch [1967, 76–77, 237–238] observes that the criteria of genre and coherence sometimes lead into a hermeneutic circle. The broad notions of genre with which we initiate the examination of a text, for instance, have limited explanatory value; indeed, they function only as heuristic guides, as one refines one's conception of the text's meaning. Ultimately, knowing the intrinsic genre of the text is tantamount to understanding its meaning.¹² Similarly, in assessing the coherence of our interpretation, the whole against which we set our text will depend on our interpretation. Initially, when our view of its meaning is still open, the correlative ensemble of texts will be large; but as we sharpen our conception, the context will narrow. Testing an interpretation will involve showing that the author means precisely this in texts just like our text. As before, Hirsch obviates the problem of circularity by consideration of the heuristic element in interpretation. If we aspired to certainty, he argues, the reservations of the sceptics would be sustained, rendering the quest for author's meaning futile. But we do not demand certainty in most contexts of thought and action, and need not do so in hermeneutics either. Our aim ought to be to hit upon accounts of high probability; the process of *validation* of interpretations is precisely that of gauging the relative probabilities of competing interpretations. Thus, the uncertainties, the possibility of alternative views, the role of subjective factors—altogether familiar within a spectrum of human pursuits—are natural adjuncts of the interpretive enterprise. To propose these as fundamental objections against the viability of the search for author's meaning merely misconstrues what one's goals ought to be.

Thus, interpretation is not mechanical: it is a dynamic heuristic process, where one seeks to measure the validity of interpretations. This is enough, in Hirsch's view, to dispel the greatest difficulties raised by the sceptics. One can engage in an orderly quest for a valid interpretation of author's

¹² Hirsch [1967, 86] defines 'intrinsic genre' as 'that sense of the whole by means of which an interpreter can correctly understand any part in its determinacy'. By this he of course specializes the notion of genre, which in common usage denotes much broader categories of literary effort. Hirsch hereby captures the extremely close connection between grasping the meaning of a text and specifying its genre, but avoids the tautology whereby every text would constitute its own separate intrinsic genre.

meaning; author's meaning is the objective toward which this process is and must be directed. Hirsch observes that the sceptics admit this *de facto* by virtue of their participation in criticism: for inquiry would be senseless if all conceivable claims about a text had equivalent validity. Finally, understanding the author's meaning is the precondition for all the other inquiries in which critics engage, since it is the only feature of a text common to all potential critics.

This critical scheme provides a basis for examining the interpretation of texts from ancient mathematics. It also provides a cautionary note, by alerting the interpreter to the subtle difficulties that this activity poses, in particular, the hazards entailed in the interplay between the objective content of the texts and the subjective elements present in the experience of the interpreter. Mathematical texts are especially susceptible to being read in the context of the philosophical and mathematical predispositions of readers trained in the modern disciplines. Avoiding the misconstructions of authorial meaning that can result becomes, as we shall see in the following examples, the particular concern of the historian of mathematics.

2. The Euclidean concepts of ratio and proportion

As one would expect, the interpretation of ancient mathematical texts is strongly influenced by considerations grounded in modern mathematical theory, and these may introduce anachronizing tendencies. The discussion of the ancient convergence principles, specifically as they relate to the definitions given by Euclid at the beginning of his proportion theory [*Elem.* v], provides an interesting example. It is widely maintained that Euclid's definitions (in particular, def. 4) have the aim of excluding non-Archimedean magnitudes from the domain of geometry, a claim that is supported through consideration of subtle requirements of the Euclidean proofs. Similar observations are made for Archimedes' convergence theorems and his application of the so-called Archimedean axiom, with which the Euclidean definition is typically associated. But if one moves from the mathematics to the text, a different story emerges.¹³

Among the definitions prefacing the theory of proportions in *Elem.* v are the following:¹⁴

¹³ My account will be in substantial agreement with that presented in Mueller 1981, 138–145.

¹⁴ My translation from the text of Heiberg [1883, ii 2].

'Ratio' is of two homogeneous magnitudes the manner of relation [they have to each other] with respect to size.¹⁵ [def. 3]

'Having a ratio to each other' is predicated of magnitudes which when multiplied can exceed each other. [def. 4]

With reference to the latter definition, T. L. Heath offers this commentary:

De Morgan says that it amounts to saying that the magnitudes are of the same species. But this can hardly be all; the definition seems rather to be meant, on the one hand, to exclude the relation of a finite magnitude to a magnitude of the same kind which is either infinitely great or infinitely small, and, even more, to emphasize the fact that the term *ratio*, as defined in the preceding definition, ... includes the relation between any two *incommensurable* as well as between any two commensurable finite magnitudes of the same kind. [Heath 1956, ii 120: his emphasis]

By the phrases 'to be meant' and 'to emphasize', Heath clearly indicates his own intent to articulate the meaning Euclid himself had in mind. But it must seem remarkable that three such different meanings—homogeneity, the exclusion of non-finite (i.e., non-Archimedean) magnitudes, and the inclusion of incommensurables—could be covered in a single expression, and further, that Euclid could emphasize a claim about incommensurables without actually using the term.¹⁶ We thus confront a situation where, to use Hirsch's terminology, the effort to *understand* (or interpret) Euclid's text is separate from its *criticism*, that is, the elaboration of its mathematical implications.

Read in isolation, Definition 4 may indeed be construed as a condition intended to exclude non-Archimedean magnitudes. It would then be transcribed in the form that magnitudes A and B (for $A < B$) have a ratio

¹⁵ Contrast Heath 1956, ii 114 (emphasis his): 'A *ratio* is a sort of relation in respect of size between two magnitudes of the same kind.' Doubtless, Heath's rendition now would be considered standard. But in employing the indefinite article ('a sort of relation'), he appears to have lost a nuance of the Greek definite article (cf. my 'the manner of relation' for ἡ... ποῖα σχέσις). More important, Heath's version is vacuous, since in his rendering nothing is actually being defined (Mueller [1981, 126] calls this sense of the definition 'mathematically useless', but sets the onus of the difficulty on Euclid.) In my version, Euclid is specifying 'ratio' as a relation of *quantitative measures* of homogeneous figures. That is an essential and non-trivial condition, and would qualify as a definition on the supposition that the reader already grasps the notion of quantity or size (πληρότης).

¹⁶ The terms for 'commensurable' and 'incommensurable' first appear in the first definition of book 10.

if and only if there exists a finite integer m such that $mA > B$. Thus, for instance, for the indivisible element A of a finite figure B , there could be no ratio between A and B , since A taken any finite number of times could not be made to exceed B .¹⁷ The proofs of *Elem.* v prop. 8 and x prop. 1 both depend on such an assumption: that the smaller of two given magnitudes, when multiplied, will eventually become greater than the other.¹⁸ Since, furthermore, indivisibles were debated within early Greek natural philosophy and played a role in the heuristic analysis of figures by some precursors of Archimedes,¹⁹ one might accept that Euclid (or Eudoxus, the author of the source version of the theory) chose to exclude such cases from the formal theory of proportions of magnitudes. Indeed, already among ancient writers, the definition was read as a condition for convergence by the exclusion of non-finite magnitudes.²⁰

Nevertheless, this view of Euclid's principle runs into several difficulties. Most notably, Euclid's definition sets up a symmetrical relation between two given magnitudes, whereas the cited applications require only a property relating the smaller to the greater. These aspects of the question are well summarized by Mueller, so that I can omit their discussion here and turn at once to the presentation of an alternative view.²¹

Let us first take note of the definition of proportion on which Euclid's theory depends; it is announced immediately after the principle we have just considered:²²

¹⁷ The use of indivisible elements of figures, most familiar in the context of the work of B. Cavalieri (1598–1647), is characteristic of Archimedes' heuristic measurements in the *Method*: see Dijksterhuis 1956, ch. x, esp. 318–322. I am preparing a study of the Archimedean method of indivisibles and the evidence for precursors in the older Greek geometric tradition.

¹⁸ A statement of their critical assumption appears in n24 below. For an account of these propositions, see Mueller 1981, 139–142; van der Waerden 1954, 185–186, 188.

¹⁹ For a discussion of the evidence of pre-Archimedean uses of indivisibles, see Knorr 1982a, 135–142.

²⁰ In Hero, *Def.* no. 123, the Euclidean definition of ratio is observed not to apply to the class of points; that is, the comparison property of multiples is essential to finite homogeneous magnitudes: cf. Schöne and Heiberg 1903–1914, iv 78. By contrast, the scholiasts on Euclid's *Elements* call attention to the condition of homogeneity [cf. Heiberg and Stamatis 1969–1977, i 215–216: nos. 13, 15–17] or to the inclusion of irrationals [no. 14]. While these Heronian and Euclidean writers may draw on authentic critical traditions, their dating is obscure and their authority questionable in determining Euclid's actual purposes.

²¹ For additional details, see Mueller 1981, 138–145.

²² My translation is based on Heiberg's text [Heiberg and Stamatis 1969–1977, ii 1].

'Being in the same ratio'—a first [magnitude] to a second and a third to a fourth—is predicated whenever, in regard to the equimultiples of the first and third relative to the equimultiples of the second and fourth, according to any multiplication whatever, the former [equimultiples] alike exceed the latter, or alike equal [them], or alike fall short, taken in the same order. [def. 5]

That is, for magnitudes A, B, C, D , the ratio $A:B$ is the same as the ratio $C:D$ if, for arbitrary integers m, n , $mA > nB$ and $mC > nD$ obtain together, $mA = nB$ and $mC = nD$ obtain together, and $mA < nB$ and $mC < nD$ obtain together. Euclid's scheme for proportions thus turns on the formation and comparison of equimultiples of given magnitudes. Every application of the definition depends on the hypotheses, that $mA > nB$, that $mA = nB$, or that $mA < nB$.²³

The fourth definition, which has just preceded, may be read in the context of this statement of proportionality. It thus stipulates that magnitudes A, B will be said to have a ratio if the inequalities between their multiples can be satisfied; i.e., that there exist m, n such that $mA > nB$ and also m', n' such that $m'A < n'B$. This is the reading favored by Mueller, but does not appear to have been recognized by other commentators.²⁴ The condition, as now formulated, simply refers to the comparability of arbitrary multiples, just as the fifth definition and all the proofs dependent on it thereafter require.

The fourth definition does of course exclude non-Archimedean magnitudes. For it supplies a gap which the fifth definition by itself would suffer: if B and D were indivisible (that is, zero) magnitudes, for instance, then only the inequalities $mA > nB$ and $mC > nD$ would be possible; it might then be technically possible to prove the proportionality $A:B = C:D$, since the condition for the opposite inequalities would be vacuously true. But in view of the terms of its formulation, the definition is not easily seen to bear this latter consequence as its principal aim. Indeed, since it has been phrased in precise conformity to the demands of the definition of proportion in def. 5—whence one would locate the intention of def. 4 as the explicit

²³ One may note a degree of redundancy. The condition $mA = nB$ can be satisfied only if A, B are commensurable; in this case, the conditions on inequalities are unnecessary. On the other hand, if the inequalities alone are satisfied (this, of course, is the only possible situation for incommensurables), then this would suffice for establishing proportionality, even without reference to the case of equality.

²⁴ The usual transcription is that 'for some $m, mA > B$, where $B > A$ '. Cf. Heath 1956, ii 120; Dijksterhuis 1929–1930, ii 58; van der Waerden 1954, 186n; Frajese and Maccioni 1970, 298. This assumption, which indeed is made in *Elem.* v prop. 8 and x prop. 1, is taken by Mueller to be different from *Elem.* v def. 4 [see below].

precondition for def. 5—one may doubt that Euclid even recognized its implication for the elimination of non-finites. As all applications of the proportionality theorems (in books 6, 11–13) are in fact for cases of finite magnitudes, the difficulty does not there arise. On the other hand, the general theorems of book 5 do require a conditional restricting the domain to finites. One is free to judge as one likes how serious is Euclid's omission of an appropriate qualifying statement. It is clear, at any rate, that def. 4, as proposed in the *Elements*, does not expressly fill that role.

Evidence from Archimedes and the later commentators provides materials for sketching out the origins of these Euclidean definitions, by revealing the form of a technique of proportions alternative to what we now have in *Elem.* v. I have argued elsewhere [Knorr 1978b] that a certain technique evident in these sources can be assigned to the pre-Euclidean period, indeed, to Eudoxus himself. In the sketch I give here, I will refer to it as 'Eudoxan', with quotation marks to indicate the circumstantial nature of the attribution.

To prove a given theorem according to the 'Eudoxan' technique, one first takes up the commensurable case, usually a straightforward consequence of the assumption of a common measuring magnitude. To establish the incommensurable case, one adopts an indirect reasoning: if the ratios are unequal (say, $A:B$ is greater), one can construct a suitable magnitude B' commensurable with A , such that $A:B > A:B' > C:D$. (A constructing procedure is reported in an extant fragment and is comparable to the construction in *Elem.* xii prop. 16 [see Knorr 1978b, 187–188].) This condition is then shown to contradict geometric properties already shown for the commensurable case. In similar fashion, one shows that the contrary supposition (that $A:B$ is the lesser ratio) also leads to contradiction.

If we replace the ratio of commensurable magnitudes $A:B'$ with a ratio of integers equal to it, say $m:n$, the inequality just mentioned can be expressed in an equivalent form: that there exist integers m, n such that $mA > nB$ at the same time that $mC < nD$. This happens to be precisely the condition by which Euclid defines 'greater ratio' in *Elem.* v def. 7. Inverting the inequalities, we obtain the conditions for $A:B$ to be the lesser ratio (not, however, defined separately by Euclid). Equality of ratio would follow if there exist no integers for which either set of inequalities obtains; that is, if for all multiples, $mA > nB$ entails that $mC > nD$, while $mA < nB$ entails that $mC < nD$. In this way, the definition of 'same ratio' adopted in def. 5 can be deduced as the logical inverse of the definitions of 'greater ratio' and 'lesser ratio'.

Three considerations indicate that Euclid's definition of 'greater ratio' is a vestige of an earlier form of the theory, as my proposal for the genesis of the Euclidean definitions suggests. First, the notion of a greater ratio is not developed as such; indeed, its definition is directly invoked only twice (namely, in *Elem.* v props. 8 and 13), in lemmas auxiliary to effecting proofs of proportionality in accordance with def. 5.²⁵ For instance, *Elem.* v prop. 8 establishes that $A > B$ implies that $A:C > B:C$, and similarly for the reverse inequalities; it is applied in v prop. 10, but the manner of appeal here is flawed, and an alternative proof founded directly on def. 7 would have been preferable.²⁶ Both theorems, in conjunction with v prop. 13, lead to the establishment of inequalities critical for the proofs of v prop. 16 and the following. For such uses, the concept of greater ratio is effectively superfluous, however, since it serves merely as an abbreviation for certain inequalities among multiples of given magnitudes. By contrast, in the 'Eudoxan' technique, manipulations of greater and lesser ratios are characteristic. This manner is imitated within the indirect arguments of book 12 (specifically, the theorems on the measurement of the circle, pyramid, cone, and sphere), where also the inequalities of v prop. 8 are conspicuous. Moreover, it is surprising that these same propositions do not exploit the definition of 'greater ratio', for that would have been natural and convenient. By here invoking the alternative assumptions of the sort characteristic of the 'Eudoxan' technique, Euclid appears to preserve marks of the older base of the theory of book 5.²⁷ Second, Euclid provides no construction for the integers whose existence is postulated in def. 7. This would doubtless cause uneasiness, were Euclid to have proposed theorems on inequalities of ratios corresponding to the proportion theorems of book 5. A construction lemma for the equivalent assumption does, however, appear among the texts associated with the alternative 'Eudoxan' technique [see Knorr 1978b, 187–188]. Third, as noted above, the convergence assumption alleged for the meaning of def. 4 is invoked only twice (namely, in *Elem.* v prop. 8 and x prop. 1), in the form that, given magnitudes A , B , where $A < B$, there exists a multiple m of B such that $mA > B$. But even here, one may question whether Euclid intends this

²⁵ For a criticism of the proofs, see Heath 1956, ii 152–153, 161–162; Mueller 1981, 130–131, 139.

²⁶ On the proof of *Elem.* v prop. 10, see Heath 1956, ii 156–157; Mueller 1981, 130.

²⁷ The basic similarity of technique between the proofs in book 12 and those in the alternative proportion theory is used to argue the Eudoxan provenance of the latter: cf. Knorr 1978b. For an account of Euclid's proof of the circle theorem [*Elem.* xii prop. 2] and the expected alternative method in the manner of the theory of book 5, see Knorr 1982a, 124–127; 1986a, 78–80.

assumption to be covered by his def. 4. For in general, Euclid manages cross-referencing via the literary device of verbal reminiscence; if he requires a previous theorem or postulate to justify the step in a proof, he will typically restate the terms of its enunciation in a paraphrase tailored to the present context.²⁸ It is remarkable, then, that in v prop. 8 and x prop. 1 the assumption of convergence is stated in terms quite different, indeed gratuitously different, from those in v def. 4.²⁹ I would thus infer that Euclid himself has formulated this assumption in the proofs, without perceiving—or, at the least, without wishing to mark—the connection with def. 4.

It thus seems far less clear than the usual view supposes, that Euclid's def. 4 was intended as a convergence assumption to exclude the case of non-finite magnitudes. The present account has shown how this definition is bound into the logical structure of the whole Euclidean theory of proportion. I think it possible that Euclid, in reworking the materials on proportion and convergence, as in v prop. 8 and x prop. 1, felt that the assumption he there had to make on the comparison of magnitudes was sufficiently obvious as not to require a special postulate. Certainly, there is no attempt to derive his assumption from def. 4, even though one could do that. Whatever textual affinities one can detect between the convergence assumption and the definition—and these are surprisingly few—can be accounted for through parallel developments from older sources.

Thus, far from being the definition's primary role, these applications in the convergence theorems are at best derivable consequences from it which Euclid appears not to have perceived. Significantly, when Archimedes formulates his own condition on convergence (Archimedean axiom), his terms

²⁸ Neuenschwander [cf. 1972–1973, 339–352] has discerned a pattern of close, often literal, recapitulation as Euclid's manner of cross-referencing in the planimetric books, especially prominent in book 2 and reasonably so in books 3–4. Comparable examples can be found in books 10, 12–13. van der Waerden [1979, 352–353] cites these insights as confirmation of a Pythagorean origin of books 2 and 4, a view with which Neuenschwander [1972–1973, 369–78] himself is in substantial, if not complete, agreement. It seems to me that the device of verbal cross-referencing is established; whether that owes to Euclid's editorial hand or is a feature of his sources, however, is a separate matter, by no means as clearly decided.

²⁹ In both of these propositions, the following formula is used: 'X when multiplied shall sometime be greater than Y; let it be multiplied, and let Z, a multiple of X, be greater than Y.'

indicate no linguistic affinity with Euclid's definition. For Archimedes asserts,

of unequal areas the excess by which the greater exceeds the lesser can, added itself to itself, exceed any preassigned finite area. [Heiberg 1910-1915, ii 264] ³⁰

Any effort to view the intent of Archimedes' postulate to be an explicit extension of Euclid's definition falters through the absence of verbal resonances between the texts. Indeed, the template for Archimedes' wording is readily detected in the applications of implicit convergence assumptions in the Eudoxan limiting theorems, e.g., statements of this sort as in *Elem.* xii prop. 2:

cutting the arcs in half . . . and doing this continually, we shall leave certain segments of the circle which shall be less than the excess by which circle EZHΘ exceeds the area *S*. [Heiberg and Stamatīs 1969-1977, iv 80-83]

The origins and meanings of Archimedes' postulate raise questions that go beyond the present context. But I would insist that the effort to analyze it ought to adhere to the same textual procedure that I have proposed for Euclid's definition. In both cases, the usual procedure, founded on recourse to considerations drawn from the modern mathematical field, attempts to conflate the meaning of the ancient texts with certain implications derivable from them. In general, this is a dubious interpretive procedure, and in the particular cases at issue here results in confusion instead of insight.

3. The ancient concept of fraction

Reading David Fowler's remarks on the ancient technique of ratios [see ch. 6, above], I was struck by the following remark:

[N]either the words nor the notation [employed for the expression of unit-fractional terms, or proper parts] contain those features that lead easily to our conception of our common fractions In fact, I do not believe we have any convincing evidence for anything corre-

³⁰ The principle is restated in essentially the same terms in the preface to Archimedes, *De lin. spir.* [Heiberg 1910-1915, ii 12], but quite differently in the fifth postulate of *De sphaer.* i [Heiberg 1910-1915, i 8]. For analyses, see Dijksterhuis 1956, 146-149; Knorr 1978b, 205-213.

sponding to our common fractions m/n in Greek scientific or everyday life.³¹

The thesis is provocative, challenging the basic intuition that anyone with the most elementary training in mathematics would today have, namely, that our own concept of fraction is essentially obvious and, hence an inevitable feature of any viable computational tradition. As before, we meet an interpretive issue centering on intention; for the most part, our texts present only calculations without conceptual elaborations, so that our attempts to formulate the ancient authors' *concept* amounts to our own view of the intention underlying their technical operations. Fowler is right to approach the question open to the possibility that the ancient and modern views *could* be different, especially in the light of certain special procedures of unit-fractions that mark the ancient practice in distinction from the modern. He is also right to insist that any conclusions be persuasively documented. Agreeing on these basic principles, we may proceed to scrutinize more closely his claim, that the Greek arithmetic tradition never evolved the conception of fraction we now take for granted. My chief interest here will be to locate the evidence bearing on the question and to grasp what the range of convincing interpretations could be.

First, a caveat: the discussion of conceptions is a tricky matter and perhaps better assigned to the philosopher than to the historian. Technical works, whether ancient or modern, devote little or no space to conceptual discussions; and it is too easy for us to state our own preferences in such specific terms that we can discern their presence or absence in older works, as we choose. What, after all, is the modern conception of fraction? In any relatively sophisticated modern account [see e.g., Waisman 1951] one will find an analysis which reduces all the properties of fractional numbers to relations (specifically, ordered pairs) of integers. Is it impossible, then, that Plato [*Resp.* 525e: cf. Lee 1955, 293] has some comparable objective in view when he insists that 'the unit is indivisible...' and that the experts will 'make you look absurd by multiplying it if you try to divide it...' Similarly, Euclid might be engaging in such a sophisticated project in the arithmetic books (7-9) of the *Elements*, where he may be viewed as handling fractional numbers under the guise of ratios of integers. In such cases, the restriction of the term ἀριθμός to whole numbers need not betoken a failure to grasp the concept of fraction.

³¹ A similar claim is made in D. H. Fowler 1983, 557. It is developed in D. H. Fowler 1987, ch. 7 (see 193, 226), where its validation depends critically on making a sharp break within the ancient arithmetical tradition around the 1st century AD (or earlier). In this way evidence from Hero and Diophantus is taken to attest only to the later phase of arithmetic technique.

In what follows, then, I recommend adopting a naive view of the concept of fraction. Our ancient writers may handle fractions as the quotients of division of integers, or alternatively, as ratios of integers. It will be enough, I propose, if the resultant entities in either case are treated as numerical terms, combinable with each other and with integers according to the rules of arithmetic. Notations or dictions need not disqualify an ancient effort *prima facie* as an acceptable manifestation of the general notion of fraction. Of particular interest for us is the ancients' remarkably persistent adherence to unit-fraction representations. Do these actually prevent the formulation of the general technique of fractions? As our evidence will derive from the full span of antiquity, we will also have to consider in what way late evidence bears on our views of the early (pre-Euclidean) period.

A rich store of fraction computations survive in the Akhmîm papyrus (in Greek, from 6th-century AD Egypt).³² It opens with an enormous spreadsheet, giving a systematic listing of the results of divisions of series of integers, first by 3, then 4, and so on up to 20, expressed as sums of unit-fractions. For instance, the entry for 12 among the 17ths is $2 \frac{1}{12} \frac{1}{17} \frac{1}{34} \frac{1}{51} \frac{1}{68}$.³³ This table is used repeatedly in the arithmetic problems which make up the rest of the papyrus. Typical of one kind of fraction computation, occurring in about 30 of the problems, is this (no. 8):

From $\frac{2}{3}$ subtract $\frac{3}{9} \frac{9}{99}$. What computation gives $\frac{3}{9} \frac{9}{99}$? the 11th of 5. $\frac{2}{3}$ of 11 is $7 \frac{1}{3}$. From $7 \frac{1}{3}$ subtract 5: remainder $2 \frac{1}{3}$. Of $\frac{1}{3}$ the 11th is $\frac{6}{33} \frac{6}{66}$. [Baillet 1892, 67]

The unit-fractions strike an alien note for the modern reader. But is it implausible to describe the writer's procedure as first raising terms to a common denominator, then subtracting, and finally simplifying the remainder? The subtrahend $\frac{3}{9} \frac{9}{99}$ is known from the table to be $\frac{5}{11}$; taking 11 as denominator, the terms become $7 \frac{1}{3}$ and 5, or a difference of $2 \frac{1}{3}$. The difference of the given fractions is thus the quotient of $2 \frac{1}{3}$ by 11.

We would prefer to simplify this to $\frac{7}{33}$ (raising terms by a factor of 3). But the ancient scribe is committed to answers in unit-fractional form, and so performs the division to get $\frac{6}{33} \frac{6}{66}$. Nevertheless, the unit-fractions play no computational role: the scribe first eliminates them by a suitable raising of terms before actually performing the required arithmetic operation, and only at the end is the result cast back into unit-fractional form.

³² The Greek text has been edited with French translation and commentary by Baillet [1892].

³³ For a survey of such tables, see Knorr 1982b. For other instances, see D. H. Fowler and Turner 1983; D. H. Fowler 1987, ch. 7. Our particular example of the division of 12 by 17 is cited also by D. H. Fowler on pages 115–116 of this volume.

The mathematical papyri relate to the humblest stratum of ancient mathematical instruction, and P. Akhmîm happens to be late in the tradition. From several centuries earlier a set of demotic (Egyptian) papyri contain problems of a similar kind. I cite one example to illustrate:

Subtract $\dot{3} \dot{1}5$ from $\frac{2}{3} \dot{2}1$: Take 5, 7 times: 35; its $\frac{2}{3} \dot{2}1$ is 25 and its $\dot{3} \dot{1}5$ is 14. Subtract 14 from 25: 11. Find 11 35ths: $\dot{4} \dot{2}8 \dot{3}5$.
 Answer: $\dot{4} \dot{2}8 \dot{3}5$.³⁴

Like the Greek scribe, the Egyptian works within the format of unit-fractions, while performing the arithmetic operations only after first clearing fractions, here by multiplying by 35. Presumably, that choice of multiplier (i.e., 35 or $5 \cdot 7$) has been hit upon through consultation of a table of 5ths and 7ths (since $\frac{2}{3} \dot{2}1$ is $\frac{5}{7}$ and $\dot{3} \dot{1}5$ is $\frac{2}{5}$).

For us, of course, the problem is complete with the value $\frac{11}{35}$. The demotic scribe, however, goes on to convert this result into unit form, by working out the division as $\dot{4} \dot{2}8 \dot{3}5$. In this he offers a clear precedent for the technique followed by the later Greek scribe of P. Akhmîm. Their procedural agreement, despite their chronological, cultural, and linguistic separation, indicates the stability of the elementary computational tradition.

From a higher stratum of Greek geometrical writing, the following examples may be noted: in *Dimen. circ. prop. 3*, Archimedes (3rd cent. BC) adjusts both terms of the ratio ($5924 \dot{2} \dot{4}$):780 by $\frac{4}{13}$ to obtain 1823:240; he next approximates an irrational square root as $1838 \frac{9}{11}$. His ultimate estimate for the ratio of the circumference and diameter of the circle (our π) is given as 'the triple and greater than 10 $\dot{7}1$ '. [Heiberg 1910-1915, i 242.17-18].³⁵ Admittedly, the manuscripts are late copies (10th century at the

³⁴ This is no. 60 of the demotic mathematical problems edited by R. Parker [1970]. Similar manipulations appear in nos. 56-59, 61.

³⁵ The result is also phrased as 'triple and greater by more than 10 $\dot{7}1$ ' [Heiberg, 1910-1915, i 242.19-21]. Yet another phrasing is adopted in the enunciation of prop. 3: 'the perimeter is triple of the diameter and moreover exceeds... by greater than ten seventy-firsts' [Heiberg 1910-1915, i 236.8-11]. In Knorr 1989a, pt. 3, ch. 4, I survey the ancient citations of this result. The oldest extant reference, it appears, is from Ptolemy, where the ratio is stated simply as greater than 'triple plus ten seventy-firsts' [Heiberg 1910-1915, i 513.4-5]. The wording in the enunciation of Archimedes' prop. 3 (as stated above) appears to have been framed by an editor following a model from Theon.

earliest); for the *Dimensio circuli* in particular, the text has undergone considerable alteration through editorial reworking and scribal carelessness.³⁶ Thus, it would surely be perilous to make claims about Archimedes' fractional notations on the basis of this textual evidence. But the character of Archimedes' computation, including the numerical values here cited, would not have been significantly tampered with by the later copyists and editors. Thus, the extant figures can be taken as reliable witness to the sequence of his computation. In particular, the ploy of attaching fractional increments (like $\frac{9}{11}$), however denoted, would be indispensable to his procedure.

As far as notation is concerned, it is remarkable that in Eutocius' commentary on this proposition, where each of the square root values is checked by a fully worked out multiplication, the term 1838 $\frac{9}{11}$ is misconstrued as 1838 $\frac{1}{9} + \frac{1}{11}$ [Heiberg 1910–1915, iii 253].³⁷ This must indicate a notation for the fraction $\frac{9}{11}$ that closely resembled a common unit-fractional expression like 9 11. Thus, we can have reason to date the use of general fraction notations no later than the 6th century. While this may not seem to place them back very far, it does refute the supposition that such notations are artefacts of Byzantine scholarship. Moreover, it supports the view that the earlier traditions stemming from Archimedes, Hero, and Diophantus also had access to some form of general notation for fractions. Without insisting that the extant manuscripts preserve those notations with complete accuracy, we may assume that the computations with fractions in their works, which we take up next, were recorded in some comparable form.

From the *Metrica* of Hero of Alexandria (1st cent. BC) numerous computations with fractions may be cited. For instance, in *Metrica* iii 3 the

³⁶ My examination of the ancient versions of prop. 1 in Archimedes' *Dimen. circ.* indicates that the extant text is an adaptation from Theon's commentary on Ptolemy's *Almagest*: cf. Knorr 1986b; 1989a, pt. 3, ch. 1–3. This confirms widespread doubts about the extant text of this particular Archimedean work, although it proposes a far greater distance between Archimedes' original version and that extant than scholars have heretofore supposed. In a discussion of the numerical figures in the mss. of prop. 3, D. H. Fowler [1987, ch. 7.3a] emphasizes the textual corruptions and the great span separating the prototypes of the mss. from the time of Archimedes and cites these as factors complicating their use as indicators of the earlier notations.

³⁷ Heiberg gives the correct figure in the body of the passage (line 7), but the incorrect figure is the basis of the computation in the appended working out of the computation (col. 3, lines 11 *et seq.*). In a note on this faulty computation, D. H. Fowler [1987, 244n] surmises that Eutocius' procedure is 'fudged'. But, examining the passage more closely, I have argued [1989, 522–523] that Eutocius' own text, which sets out correct figures, must be founded on a correct computation; the supplementary work sheet, which purports to lead to the same answer via incorrect figures, would be due to a later editor.

division of $100 \frac{4}{5}$ by $10 \frac{22}{65}$ is stated to be $9 \frac{2}{4}$ [Schöne and Heiberg 1903–1914, iii 148].³⁸ The method of division is not explained; but since the result is exact, a measure of contrivance is indicated. Again, in *Metrica* iii 8 the result of multiplying $12 \frac{1}{14}$ by $5 \frac{5}{26}$ and dividing by $7 \frac{4}{7}$ is given as $8 \frac{1}{4}$, here also without explanation [Schöne and Heiberg 1903–1914, iii 158].³⁹ One suspects that a method of raising terms is employed in cases like these, comparable to the modern school procedure.

We possess several extensive collections of problems in metrics, containing series of exercises usable in conjunction with the study of Hero's *Metrica*.⁴⁰ Here examples proliferate in which fractions are manipulated in ways distinctly comparable to the familiar modern methods. These sometimes invoke unit-fractions but, as with the papyri, in a merely notational, not a computational, mode. It is important to note that, whatever notations or dictions are used for expressing fractions, the scribes freely manipulate these as numerical terms. For instance, even if the ancient equivalent of what we write as $5/26$ (as above) is expressed as 'of the 5 the 26th' or '5 26ths' or '5 26' or even '5 26 26' (where the duplication of 26 indicates the plural),⁴¹ and even if that term is there viewed only as the result of a division (namely, of 5 by 26), nevertheless, the quotient so found is treated as a numerical term, attachable to an integral term to produce a sum (the equivalent of $5 \frac{5}{26}$), which can then be multiplied or divided by similar terms.

The six books extant in Greek of Diophantus' *Arithmetica* comprise almost 200 arithmetic problems whose solutions are as often fractional numbers as they are integers.⁴² In book 2, for instance, in prop. 8 the solutions—for which Diophantus employs the term ἀριθμός, the standard Greek term

³⁸ The principal ms. (the Codex Constantinopolitanus, pal. vet. 1) dates from the 11th century. It is reproduced in photofacsimile in Bruins 1964, i.

³⁹ A modern procedure, exploiting cancellations, might run thus: $(\frac{169}{14}) \cdot (\frac{135}{26}) \div (\frac{53}{7}) = 13 \cdot 135 \div 212 = 8 \frac{5}{212}$; this could be reduced to very nearly $8 \frac{1}{4} \frac{1}{36}$, whence Hero's answer $8 \frac{1}{4}$.

⁴⁰ These are edited by Heiberg as the *Geometrica* and *Stereometrica* in Schöne and Heiberg 1903–1914, iv–v. Some examples of fraction computations from these works are discussed in Knorr 1982b.

⁴¹ This would be in accordance with a common tachygraphic convention in Byzantine texts. On ancient fraction notations in general, one may consult Heath 1921, i 41–44; D. H. Fowler and Turner 1983; D. H. Fowler 1987, ch. 7. Fowler frequently remarks on deficiencies in the older standard accounts, such as Heath's.

⁴² See Tannery's edition of Diophantus [1893–1895], and Heath 1910. The same prominence of fractional solutions is evident in those portions of the *Arith.* surviving only in Arabic: cf. Sesiano 1982, Rashed 1984.

for 'number' (that is, ordinarily, 'integer')—are $256/25$ and $144/25$. The solutions are fractional terms of this type in every problem thereafter in this book, through prop. 35, its last problem, save for props. 10, 19 and 23 (which result in mixed unit-fractional terms, like $72 \frac{1}{4}$), and 19 (which results in integral solutions).

It is clear that Diophantus' effort depends on a supple manipulation of fractions. This applies not only for explicit constant operands, but also for variables. For instance, in iv prop. 36 two indefinite fractional terms, $3p/(p-3)$ and $4p/(p-4)$ [lit.: '3p in the part ($\epsilon\nu \mu\omicron\rho\iota\omega$) $p-3$ ', and so on] are stated to have the product $12p^2/(p^2+12-7p)$ [Tannery 1893–1895, i 288.9–10].⁴³ In the same problem their sum is worked out to be $(7p^2-24p)/(p^2+12-7p)$, with the following explanation:

Whenever it is required to sum parts ($\mu\omicron\rho\iota\alpha$), for instance, $3p$ pt. $p-3$ and $4p$ pt. $p-4$, the number [*scil.* numerator] of the part shall be multiplied into the alternate parts [*scil.* denominator of the other], e.g., $3p$ into the parts [denominator] of the other, *scil.* $p-4$, and again the $4p$ into the parts [denominator] of the other, into $p-3$. In this way the addition has made $7p^2-24p$ of the part [denominator] which is the product of the parts, *scil.* $p^2+12-7p$.⁴⁴ [Tannery 1893–1895, i 288.1–9]

Doubtless, the account is cumbersome. But this is an accommodation to the learner, who is being asked to extend arithmetic operations beyond the simple manipulation of definite terms to the case of unknowns. The rule here is only stated, it is not proved. Moreover, it appears as an appendage to prove (that is, to check) the correctness of the solutions derived in the main text of the problem. One may well suppose that here, as is suspected elsewhere in the *Arithmetica*, the added section is due to a later editor, and that Diophantus himself could assume such operations without comment.

The basic fractional operations thus appear to be taken for granted, while only the more elaborate extensions need to be explained. Since Diophantus is reported to have written a treatise on fractions (the *Moriastica*), it is possible that this work provided the basis for these passages of the *Arithmetica*.⁴⁵ Although the date of Diophantus is uncertain, the basic expertise in dealing with fractions can hardly be considered a novel feature of

⁴³ Here I denote by 'p' Diophantus' symbol for the unknown number.

⁴⁴ Diophantus' term $\mu\omicron\rho\iota\omega$ (lit. part) here bears the sense of our 'denominator', while he uses the word $\alpha\rho\iota\theta\mu\acute{o}\varsigma$ (lit. number) where we would write 'numerator'.

⁴⁵ On lost Diophantine works, see Heath 1910, 3–4.

his mathematical work: one may note comparable cases in book 6 (props. 12–14, 19, 21–22).

The *Greek Anthology* contains dozens of epigrams involving arithmetic word problems, among them the famous epitaph of Diophantus [Tannery 1893–1895, ii 60–61].⁴⁶ Most of these are accompanied by scholia which work out the solution, some in considerable detail. In one common type of problem a whole is diminished by denominated parts leaving a given remainder. For instance, in no. 2, the provenance of the gold used for making a statue of Pallas is identified thus:

Charisios has given half the gold, Thespis an eighth and Solon a tenth part, and Themison an additional twentieth. The remaining nine talents plus his craftsmanship are the gift of Aristodikos. [Tannery 1893–1895, ii 44]

To find the whole weight of gold, the scholiast instructs us to find the least number having all of the stated parts, in accordance with the procedure in Euclid, *Elem.* vii prop. 39; the number is here 40; its $\frac{1}{2}$ $\frac{1}{8}$ $\frac{1}{10}$ $\frac{1}{20}$ (that is, $20 + 5 + 4 + 2 = 31$) when subtracted from 40 leave 9, as required, so our answer is 40. In the case of another remainder, say 6, he tells us to adjust the 40 in the ratio of 6 to 9, giving an answer of $26\frac{2}{3}$.

Dating the epigrams in the *Anthology* and their scholia is difficult: a definitive compilation was made early in the 10th century, from elements of diverse date and provenance.⁴⁷ Tannery holds open the possibility that the grammarian Metrodorus (assigned to the early 4th century AD) was responsible not only for the arithmetic epigrams, but also for the scholia. It is certainly clear that any writer with a knowledge of the arithmetic techniques of Euclid and Diophantus was able to produce such a commentary.

For our present purposes the central point is the scholiast's use of Euclid. The procedure for finding the least number with specified parts [*Elem.* vii prop. 39] is a modification of the finding of the least common multiple of specified integers [vii prop. 36]. Why do these propositions appear in the *Elements*? As they bring book 7 to a close, one cannot suppose that they were needed for the proofs of later propositions. But the examples from the *Anthology* reveal how the parts-procedure is neatly tailored for solving unit-fraction problems, just as the l.c.m.-procedure has an obvious function

⁴⁶ Epigram no. 13 sets out as an arithmetic problem the finding of Diophantus' age at death: in effect, his age less its 6th, 12th, 7th, and half parts is nine years. The scholiast works out the answer as 84 years.

⁴⁷ On the editing of the epigrams, see Tannery 1893–1894, ii x–xii.

within the addition and subtraction of general fractions. Further, Euclid's inclusion of the parts-procedure, an ostensibly superfluous variant of the l.c.m.-procedure, can now be perceived as motivated from computational practice. A view along these lines has been suggested by Itard [1961, 128], noting possible associations with methods employed in the Rhind Papyrus [cf. Mueller 1981, 80, 114–115; R. Parker 1970, 8–10].

I believe this survey represents the range of the evidence, although dozens of additional examples of similar kind could be cited. What claims does it support as to the presence or absence of a general conception of fraction? First, I question whether it is justified to impose a strict division between late evidence (e.g., Graeco-Roman and Byzantine) and early evidence (e.g., pre-Euclidean and Egyptian), when the computational techniques are so uniform. If the 'modern' techniques are clearest in Byzantine texts, this need not preclude their use much earlier. Indeed, the examples from Hero and Archimedes, for instance, recommend assigning them a very early provenance within the Greek tradition. Second, I consider it more reasonable that a sophisticated theory like that in Euclid *Elem.* vii developed on a sturdy base of practical technique, than that, conversely, the abstract theory should have arisen spontaneously and preceded its application by some long interval. The technical distinctions we can discern may as well be assignable to pedagogical, as to chronological factors. The quaint persistence of unit-fraction techniques, for instance, is most strongly marked in the papyri and the metrical collections, a genre of writing pitched to novices. Even here, however, the unit mode is a cover for a more general computational technique. Doubtless, it was left to the expertise of the teachers to compensate for the gaps in the papyrus textbooks in explicating the general procedure.

The pre-Greek tradition in Egypt is represented to us in only a few documents, like the Rhind Papyrus, dating from over a millennium before the Greeks.⁴⁸ Despite their antiquity, these documents appear to foreshadow the general fraction technique. From the Hellenistic period, the demotic papyri give evidence of knowing the more general method, as R. Parker [1970, 9–10] has observed; and they must surely have received this through the native tradition, rather than by borrowing from the Greeks. This would plausibly assign familiarity with the general technique of fractions to the Greeks in the pre-Euclidean period, through their exposure to earlier Egyptian methods.

⁴⁸ The Rhind Papyrus has been edited by Peet [1923] and Chace and Manning [1927]. For surveys, see van der Waerden 1954, ch. 1; Gillings 1972; Robins and Shute 1987. Some examples of its fraction computations are examined in Knorr 1982b.

BC/ Sometime in the late 3rd or early 2nd century AD the Greek mathematical tradition came into a windfall, by receiving the Mesopotamian astronomical and computational techniques.⁴⁹ The sexagesimal place-system provided a flexible instrument for elaborate computations far beyond the capacity of the elementary fraction methods. But as far as numerical conceptions of Greek arithmetic are concerned, as the evidence from Hero and the papyri reveal, this exposure made essentially no difference. For the Greeks never absorbed these new techniques into their elementary lessons on arithmetic. Indeed, teachers like Theon of Alexandria (4th cent. AD) found it useful to refer to the conventional notions of fractions in order to explain the manipulation of sexagesimals.⁵⁰ The very fact that the Greeks could adopt the sexagesimal system for the more advanced scientific uses would appear to indicate the prior existence of an adequate conception of fractions. That is, they were already prepared to translate their techniques of manipulating terms of the form '*n* of the *m*th parts' into the sexagesimal mode.

Operationally, the new procedures were radically different from the traditional ones known to the Greeks; to implement them, for instance, a whole new range of tabular auxiliaries had to be introduced, on the pattern of the ancient Babylonian computational system. But the introduction of sexagesimals need not have affected the underlying conceptual basis already established among Greek arithmeticians centuries before. Conversely, we should not assume that the persistence of unit-fractions in the popular arithmetics entailed any limitations in that underlying concept. In view of the silence of our sources on conceptual matters, we are compelled to treat the ancient concept of fractions as a sort of 'black box' associated with the technical procedures preserved in the texts. The evidence, in my view, would have to be far more extensive and explicit than it is to sustain

⁴⁹ For accounts of the Babylonian computational methods, see van der Waerden 1954, 37-45; Neugebauer 1957, 29-35.

⁵⁰ On sexagesimal computation in general, see Theon, *In Ptol. ad i 10* [Rome 1936, 452-62]. In particular, Theon refers to the familiar Greek tradition, as represented by Diophantus, to explicate the operations on sexagesimal parts; for instance, Theon notes [Rome 1936, 453]:

In the case of (fractional) parts of the unit... in the manner in which, according to Diophantus, the species are altered in the multiplications of the parts of the unit, for the reciprocal first power (ἀριθμοστόν), [e.g.] the third, multiplied into itself makes the reciprocal second power (δυναμοστόν), [i.e.] the 9th and alters the species, in the same way also here the parts of the unit alter the species.

Cf. Tannery 1893-1895, i 8: 'ἀριθμοστόν into δυναμοστόν makes δυναμοστόν'. Theon goes on to elaborate case after case of sexagesimal parts (e.g., minutes times minutes make seconds, minutes times seconds make thirds, and so on).

the hypothesis of a significant separation between the ancient and modern concepts.

4. The aim of Euclid's *Elements*

In the preceding sections we have taken up difficulties in the interpretation of Euclid's theories of proportion and number. Besides clarifying the meanings of particular passages, we have seen ways to exploit Euclid as a source for understanding aspects of the pre-Euclidean development of these theories. But the primary task of a literary analysis of a writing must be to explicate the meanings and objectives of the work as a whole. We undertake such an analysis for the *Elements* in the present section.

What sort of work did Euclid intend the *Elements* to be? Hirsch's account of genre will be useful here. The notion of genre, he proposes [1964, ch. 3], serves as focus for the external or public determinants of a text, complementary to the internal or private meanings of the author. Upon first encountering a text, we begin to construct its meaning on the basis of provisional views we have as to its genre, or type classification. Our idea of the genre embraces the expectations we have about the text, as we pursue our examination of it. At first our genre idea is quite broad; but it is progressively narrowed, being refined, modified, sometimes discarded and replaced by an entirely new genre idea, in the course of our reading. When we ultimately possess an adequate reading of the text, our idea of its genre (or, intrinsic genre) will be closely accommodated to this.⁵¹ What is notable in this account is the reciprocity between the meanings attributed to a text and the ideas one has of its genre. The genre delimits at each moment the possible meanings one can assign to the text; but the genre idea is always provisional, subject to revision as it comes into conflict with new details and implications of the text.⁵²

As we approach the particular case of the *Elements*, we will find that views of its aim and meaning are accompanied by implicit views of its genre. Conversely, establishing a credible view of the genre will assist one's inquiry into Euclid's aims.

The question of Euclid's objectives was already raised in antiquity. His commentator, Proclus (5th cent. AD), observes [Friedlein 1873, 70–71] that the aim (σκοπός) of the *Elements* can be viewed in two respects, relative to its research findings and to its instructional uses. In the former regard,

⁵¹ For Hirsch's definition of 'intrinsic genre', see n12.

⁵² Conversely, Hirsch [1967, 71–77] offers interesting examples where misconceptions of the genre can lead to persistent misconstruals of a text.

Proclus sees the culmination and goal of the work in its elucidation of the five cosmic figures (that is, the five regular polyhedra, elsewhere styled the 'Platonic figures').⁵³ But from the pedagogical viewpoint, he continues, Euclid aims to provide an introduction (στοιχείωσις) toward perfecting (τελείωσις) the learner's understanding (διάνοια) of the whole of geometry.

Proclus' first surmise, which is probably his own insight,⁵⁴ may be dismissed as the expected emanation of his own Neoplatonism [cf. Heath 1956, i 115]. To be sure, it is remarkable that so much of the *Elements* comes to bear on the constructions of the five solids in book 13—including results from the plane geometry of books 1–4 and 6, the solid geometry of book 11, and the theory of irrational lines of book 10. Even so, much of the contents of these books is unrelated to the solid constructions, while the substantial portion of the *Elements* contained in its other books would be wholly left out of account—the proportion theory of book 5, the number theory of books 7–9, and the exhaustion theorems of book 12. In Proclus' defense, one might observe that his position is hardly less credible than modern theories of Euclid's Platonism, based almost entirely on the definitions prefacing books 1 and 7.⁵⁵

But in his second suggestion, about Euclid's pedagogical aims, Proclus takes the essence of the work to be its elaboration of complex theorems out of the simplest and most fundamental starting points. In this he foreshadows certain modern views, which emphasize the axiomatic architecture of the *Elements*. A conspicuous feature of the *Elements* is its deductive structure.⁵⁶ As a prototype of modern foundational efforts like Hilbert's *Foundations of Geometry*, it lends itself to a mathematical analysis of its axiomatic technique—e.g., what changes must be made in Euclid's scheme of axioms to make it sufficient for demonstrating the propositions in each

⁵³ Proclus [Friedlein 1873, 23] also mentions the figures in the context of remarks on Plato's *Timaeus*. Referring to the same solids, Pappus once calls them 'the five figures of (παρά) the most divine Plato' [Hultsch 1876–1878, 352.11–12]. As a scholiast to the *Elements* observes [Heiberg and Stamatis 1969–1977, v.2 291.1–9], the Platonic association is due only to his inclusion of the solids in the *Timaeus*, for their discovery, he claims, goes back to the Pythagoreans and to Theaetetus.

⁵⁴ Proclus qualifies his statement with 'I would say.' Heath [1956, i 33–34] notes, however, that such passages need not indicate Proclus' originality.

⁵⁵ The most extensive effort to link Euclid with specifically Pythagorean and Eleatic precedents is Szabó 1969. I criticize some aspects of the argument in Knorr 1981b.

⁵⁶ This is the focus of the analysis by Mueller 1981.

of the major branches of Euclidean geometry?⁵⁷ There has also been an extensive discussion of the related philosophical issue of Euclid's conception of first principles: What is it about Euclid's axioms and postulates that suits them to be the starting points for his deductive system?, in particular, How does Euclid's conception compare with Aristotle's prescriptions about the starting points of formal expositions of scientific knowledge?⁵⁸

Implied in these views are two complementary notions of the genre of Euclid's *Elements*. In looking to the five solids as the focal point, that is, in defining the work in terms of its subject, Proclus conceives it as the exposition of a technical field of research. But when he, like the modern scholars mentioned, emphasizes its deductive system, the work moves into the category of philosophical exposition—the actualization, as it were, of some implied program of axiomatics. To these forms of technical exposition and philosophical exposition, we can add a third, also recognized by Proclus, the introductory textbook. In the following discussion, we will explore how the decision relating to genre affects our view of the goals Euclid pursues and his success at attaining them.

Proclus amplifies his second point on the aim of the *Elements* by considering the meanings of the term 'elements' (στοιχεία). The characterization he borrows from Menaechmus, Euclid's 4th-century precursor, is of particular interest, since it is one Euclid himself would have known. Menaechmus, as Proclus [Friedlein 1873, 72–73] reports, recognized two senses for the elements of a geometric system: in the relative sense, any proposition can be termed an element in the context of any other proposition whose demonstration assumes it; in the absolute sense, the elements are those simpler propositions (e.g., axioms) on which whole bodies of theorems depend. Proclus observes that under the former sense many propositions could qualify as elements, whereas under the latter sense the term will apply only to a specific, relatively small set of propositions.⁵⁹ One may note in Menaechmus' first sense of the term the reciprocal character of elements: two propositions can be elements of each other, if the proof of the one can be effected by assuming the other and conversely; this indicates a certain

⁵⁷ In his brief comparison of the formal approaches of Euclid and Hilbert, Mueller [1981, ch. 1] seeks, rightly I believe, to distinguish the ancient and modern views.

⁵⁸ Among many discussions of the relation of Aristotle's theory of deductive science and Euclid's procedure in the *Elements*, one may note Mueller's account in ch. 5, above; Mendell 1986, ch. 6; Hintikka 1981; and Heath 1949, 50–57.

⁵⁹ For further remarks, see Burkert 1959, 191–192.

fluidity in the order of proof, which would obtain in the field before the appearance of a definitive textbook treatment.⁶⁰

The second notion of element is reminiscent of the position adopted by Aristotle in *Physics* i 1.⁶¹ In the effort to organize a field of experience into a deductive science, in his account, we take complex wholes more knowable to us through sense perception and break these down into their simple components more knowable by nature; the latter are the elements or first principles which form the basis of the science. Implied in this remark is a two-part process in which, first, the analysis of phenomena leads to the discovery of the primary conceptions, and then the field is reconstructed deductively on the basis of these conceptions. On this view, Euclid's *Elements* might be viewed as a synthesis of the latter sort, where the fundamental principles, already discovered, can be stated at the outset and their consequences worked out as propositions, in systematic order.⁶²

This sense of 'element' is linked with the modern views mentioned earlier, not only by virtue of its emphasis on deductive form, but also for an important nuance in its conception of the geometric field in relation to its principles. For they all imply an account of a formal system which assigns priority to its axioms and views its propositions merely as their deductive consequences. In the modern conception, in fact, the axioms define the system by specifying the entities it contains and their essential properties; whatever other properties can be deduced from these may be considered as already implicit within the axioms.⁶³ In effect, demonstration becomes a mechanical process, deriving consequences from the first principles in accordance with deductive rules. In the ancient mathematical terminology, this process is called synthesis, and it characterizes all the proofs one finds in Euclid's *Elements*.⁶⁴

⁶⁰ Barnes [1976] would take these remarks from Menaechmus to indicate the employment of circular reasonings in the proofs by mathematicians before Aristotle. This inference is, I think, too bold to be sustained on the evidence at hand.

⁶¹ On this passage, see Ross 1936, 456–458.

⁶² Other Aristotelian passages are cited in Heath 1956, i 116—*Top.* 158b35, 163b23; *Meta.* 998a25, 1014a35–b5: cf. also Heath 1949, 205–206. For additional discussion of the Aristotelian senses of 'element', see Mendell 1986, 492ff.

⁶³ One may think of Aristotle's characteristic distinction between the potential and the actual, where in the present instance one would refer not to the existence of derived terms, but to knowledge of their existence: cf. *Meta.* 1051a30–33, and the discussion in Mendell 1984.

⁶⁴ The principal ancient account of the method of analysis and synthesis is from Pappus [Hultsch 1876–1878, vii 634–636]. The modern literature on the method is extensive: see, for instance, Knorr 1986a, esp. ch. 8.2; A. Jones 1986, 66–70; Hintikka and Remes 1974; Mahoney 1968–1969; Gulley 1958; Robinson 1936.

This conception of the *Elements* as an inquiry about the first principles of geometry will undoubtedly sponsor interesting insights into the nature of formal studies. But it assigns too small a role to the technical content of the fields of Euclid's geometry, and thus misses a feature which, I believe, is critical for a historical understanding of his project.⁶⁵ In what follows, I will take up three aspects related to setting the genre of Euclid's work: its place as the paradigm of a particular type of technical treatise; the manner of Euclid's activity in editing his source materials; and the character of technical instruction toward which it was applied.

For these purposes, it seems more useful to pursue Menaechmus' first sense of the term 'element', namely, as any principle or theorem assumed within the proof of another. That is, *all* the propositions of Euclid's treatise are elements with respect to higher studies in geometry. As Proclus himself notes, this was precisely how Euclid was used by later geometers:

the most fundamental and simplest theorems and most kin to the primary hypotheses are here [*scil.* in the *Elements*] joined together, taking the appropriate order, and the proofs of the other (theorems) use them as thoroughly familiar and arise from them. Just as Archimedes... and Apollonius and all the others seem to use the things proved in this very treatise, as agreed on starting points. [Friedlein 1873, 71]

Moreover, the title *Elements* was not restricted to Euclid's treatise. Proclus [Friedlein 1873, 66–67] denotes by this term geometric works produced by Euclid's predecessors Hippocrates, Leon, Theudius, and Hermotimus. Even if the term derives from Proclus' possible source, Aristotle's disciple Eudemus, rather than from the titles of the works themselves, we can infer its generic use in the 4th century to denote works of Euclid's kind.⁶⁶ From another of Aristotle's disciples, Aristoxenus, we possess a treatise in

⁶⁵ A similar distortion of emphasis, I believe, accompanies the familiar view that the ancients' geometric constructions played the role of existence proofs: cf. Knorr 1983.

⁶⁶ Burkert [1959, 193] takes *perforce* the designation 'elements' as the actual title of the pre-Euclidean works. Mendell [1986, 493] observes, however, that what these authors titled their works is immaterial; the salient fact is that Proclus' source, Eudemus, could designate them as 'Elements', whence the works themselves must have conformed to a Peripatetic conception of the term. In Mendell's view, Eudemus would surely have held the second of Menaechmus' senses for 'elements', namely, the simple components into which a complex can be resolved.

two books, the *Harmonic Elements*.⁶⁷ Apollonius [Heiberg 1891–1893, i 4] describes the first four books of his own *Conics* as ‘falling into (the class of) elementary training’ [πέπτωκεν εἰς ἀγωγήν στοιχειώδη], in contrast to the contents of the last four books, of a more advanced character, which he terms ‘supplementary’ (περιουσιαστικώτερα); the commentator Eutocius [Heiberg 1891–1893, ii 176] cites the same work as the *Conic Elements*. Pappus [Hultsch 1876–1878, ii 672.12] denotes by the term *Elements* not only Euclid’s work, but also treatises by other geometers, e.g., the *Conic Elements* by Aristaeus.⁶⁸ Archimedes once cites his *Elements of Mechanics* for a result we know from his *On Plane Equilibria* (i 8) [cf. Heiberg 1910–1915, ii 350.21]; several of his references [Heiberg 1910–1915, i 270.24; ii 268.3, 436.3] to the *Conic Elements* are likely to denote the works by Euclid or Aristaeus, precursors of Apollonius.⁶⁹ Interestingly, Archimedes once uses the phrase ‘conic elements’ to denote three specific *propositions* (i.e., *Quad. parab. props.* 1–3) for which he provides only the enunciations, since their proofs can be assumed from a treatise on conics, itself also called the *Conic Elements* [Heiberg 1910–1913, ii 266.3, 268.3]. Among later writers, e.g. pseudo-Hero [Schöne and Heiberg 1903–1914, iv 14.1, 84.18] and Diophantus, the term στοιχειώσις becomes synonymous with ‘introduction’.⁷⁰

These instances make clear that Euclid’s *Elements* was the paradigm for a literary genre which embraced technical treatises extending beyond the specific field of Euclidean geometry. Proclus indicates this when he refers to the ‘numerous compositions’ falling into the category of elementary treatise [στοιχειώσις] in the areas of arithmetic and astronomy. In speaking of the general activity of producing such treatises, he remarks on the diversity of their editorial styles [Friedlein 1873, 73]:

It is a difficult task in any science to select and arrange properly the elements out of which all other matters are produced and into which

⁶⁷ See Barker’s discussion in ch. 9, below. Note that Aristoxenus’ *Harm. Elem.* is discursive in style, thus in marked contrast to the rigorously deductive format of Euclid.

⁶⁸ Cf. also Hultsch 1876–1878, ii 552.4 (possibly referring to Theodosius or Euclid: cf. 553n), 608.2 (referring to Euclid’s *Phaen.*), 660.19 (taken by Hultsch to refer to Apollonius’ *De plan. loc.*). It is possible, however, that some of these passages are interpolations. For other uses of the term, see Hultsch’s index s.v. στοιχείον.

⁶⁹ Note that two citations of Euclid’s *Elements* at Heiberg 1910–1915, i 20.15 and ii 444.28 are certainly interpolations.

⁷⁰ A scholiast [cf. Tannery 1893–1895, ii 72.16–20] to Iamblichus refers to Diophantus’ *Arithmetica* by this term. In Diophantus’ preface, the initial materials are said to be elementary (ἐχόντα στοιχειώδως) [Tannery 1893–1895, i 16.3]. Of course, Proclus frequently refers to Euclid’s work as στοιχειώσις.

they can be resolved. Of those who have attempted it [Morrow 1970, 60: 'scil. for geometry'] some have brought together more theorems, some less; some have used rather short demonstrations, others have extended their treatment to great lengths; some have avoided the reduction to impossibility, others proportion; some have devised defenses in advance against attacks upon the starting-points; and in general many ways of constructing elementary expositions have been individually invented.⁷¹

This passage has sometimes been taken to refer to treatments of elementary geometry;⁷² but one would surely find remarkable the implied proliferation of ancient editions alternative to Euclid.⁷³ Further, Proclus is unlikely to have had access to pre-Euclidean versions. Proclus' source in the present discussion may be Geminus; but only Eudemus could provide Proclus information on the pre-Euclidean tradition, and Eudemus of course could not compare such versions with Euclid's treatment [cf. Heath 1921, i 114].⁷⁴ But as Proclus makes clear his interest in the expositions of 'any science', we must suppose that he is considering the whole range of technical treatises, not just those in geometry. Among extant treatises, reduction-avoidance happens to be characteristic of Menelaus in the *Spherics*, while the inclination toward lengthy proofs typifies Theodosius' *Spherics* in contrast with parallels in Menelaus and Euclid (*Phaenomena*) [see Heath 1921, ii 248–249, 263, 265].⁷⁵ The preface to one of the editions of Euclid's *Optics* forms

⁷¹ Cf. Morrow 1970, 60, which I here follow. The characterization here of elements as that 'out of which all other matters are produced and into which they can be resolved' is reminiscent of Aristotle's passages on elements (in particular, the material kind of elements): cf. *Meta.* 983b8–11; *Phys.* 194b24, 195a16–19.

⁷² Cf. Morrow 1970, 60n which explains Proclus' phrase 'of those who have attempted it' by the remark: 'scil. for geometry'. Artmann [1985] begins with this very passage as the basis for developing his thesis that some of the proportion-avoiding proofs in Euclid's books 1–4 derive from a pre-Euclidean treatise. The difficulty, however, is that Proclus is not speaking here about pre-Euclidean precedents of the *Elements*, but rather of the whole ancient tradition of elementary mathematical treatises.

⁷³ Of course, there were many *commentaries* on Euclid, such as those by Hero, Geminus, and Proclus: cf. Heath 1956, i 33–45.

⁷⁴ We return below to the issue of the pre-Euclidean precedents for the proportion-avoiding proofs in the *Elements* [see n83, below].

⁷⁵ Similarly, Autolycus' *De sphaera quae movetur* is more abstract and verbose than the parallels in Euclid's *Phaen.*: cf. the specimens cited by Heath [1921, i 351–352]. For further discussion of these treatises, see Berggren's essay [ch. 10, below], Berggren and Thomas 1992, Knorr 1989b.

a sort of defense of its postulates, in that notions of the nature and transmission of visual rays are explained in a counterfactual manner: hypotheses different from those implicit in Euclid are refuted through consideration of associated physical phenomena.⁷⁶ Thus, Proclus' remarks on the variety of styles in this tradition of technical writing can be related to some extant works, even though most of the works he refers to must be ones that are now lost.

The *Elements* is thus the model, if not the first exemplar, of a particular type of scientific treatise, in which the content of the science is presented in a formal, systematic, and deductive manner. The exposition develops as a series of propositions in which the demonstration of each depends on those preceding and in its turn can serve toward the demonstrations of those following. The whole series is initiated by the statement of certain primary terms and propositions (the elements in the absolute sense), in the form of definitions and postulates, for instance, which in the context of the work may be taken as inderivable.⁷⁷

In studying a treatise of the synthetic type, one will follow its deductive order. But the project of producing the treatise entailed the *discovery* of the appropriate deductive sequence and this heuristic phase will invariably proceed in the reverse order, that of analysis. In its usual sense in ancient geometry, 'analysis' refers to a method for finding solutions of geometric problems.⁷⁸ A targeted result (e.g., a construction satisfying certain specifications) will first be assumed as known; the quest for a derivation or proof will then bring forward certain other results, these in their turn still others, until one obtains a sequence linking the desired end result to ones

⁷⁶ The preface to one recension of the *Optics* is thought by Heiberg [1895, vii 144–154] to be based on introductory lectures by Theon. I discuss aspects of the preface in Knorr 1985b, sect. 9, and argue against Heiberg's assignment of the recensions in Knorr 1992.

⁷⁷ In the absolute sense, such inderivables would be postulates or definitions. But in advanced treatises, such as Archimedes' *Quad. parab.* and *Meth.*, one typically permits as initial assumptions the results established in more elementary treatises (such as the *Con. elem.* or *De plan. aequil.* i).

⁷⁸ Pappus [Hultsch 1876–1878, ii 634–636] distinguishes two types of analysis: the problematic (*scil.* analysis of problems) and the theoretic (that of theorems), and clearly he assigns priority to the theoretic type. In this I believe he has severely undervalued the significance of the analysis of problems: cf. Knorr 1986a, ch. 8.2. Hintikka and Remes [1974] also find the analysis of problems (constructions) to be the more fruitful domain for examination.

that are already known or admissible *per se*.⁷⁹ Although this method is usually employed in the investigation of individual constructions or propositions, a heuristic procedure of the same kind can be applied toward the organization of systems of constructions and theorems.⁸⁰ A nuance we have noted in Menaechmus' first account of 'elements' is that the deductive order is relative: within certain limits, the researcher can make the *choice* as to which results will be prior and which derived from them. The options will emerge as one pushes further into the analysis of the principal theorems and constructions of the field.

In the specific instance of the *Elements*, however, Euclid is not so much its composer as its editor. As Proclus informs us (following, it appears, the authority of Eudemos), other geometric compilations in the form of 'Elements' were produced in the century before Euclid, so that his treatise is a consolidation of several generations of geometric study. Euclid's sources must have provided expository models, not only for the proofs of individual propositions, but in some cases for the substance of major sections and even of whole books. It seems clear, for instance, that the structure of his proportion theory in book 5 existed in much this form among disciples of Eudoxus, and that similarly extensive prototypes existed for his number theory in book 7, his theory of irrationals in book 10, his exhaustion theory in book 12, and his constructions of the regular solids in book 13.⁸¹

⁷⁹ In analysis the examination pursues the deductive consequences of the assumed target; the formal synthesis reverses the logical order. In his account of the method, Pappus is ambiguous, at one time asserting that the reasoning in the analysis is deductive, but then describing it as a search for appropriate antecedents. Older accounts [e.g., Robinson 1936] attempt to reconcile the two views by emphasizing the convertibility of geometric propositions. Hintikka and Remes [1974] look toward the special nature of reasoning *via* geometric diagrams. Knorr [1986a, ch. 8] prefers a textual explanation, in which Pappus has merged a mathematical and an Aristotelian account conceived along contrasting lines.

⁸⁰ Burkert [1959, 195] suggests that the bi-directional character of research in geometry—the investigation of deductive consequences on the one hand and the search for prior principles on the other—gave rise in the 5th century to an application of the term στοιχείον (until then used only to denote a 'column' or 'file', as of soldiers) to designate the ordered sequence of geometric propositions. Knorr [1985] adopts an analytic strategy to explicate the development of the theory of irrationals in Euclid, *Elem.* x.

⁸¹ van der Waerden [1954, 115, 123–124; cf. 1979, 352–353] has striven to assign the prototypes of books 2, 4, and 7 to the Pythagoreans, that of book 8 to the Pythagorean Archytas [1954, 112, 149, 153–155], that of book 10 to Theaetetus [1954, 172], and those of books 5 and 12 to Eudoxus [1954, 184–189]. Neuen-schwander [1972–1973] sustains some of these claims on the Pythagorean provenance of substantial parts of books 1–4. But I think the effort to view whole Euclidean books as, in effect, mere transcripts of treatises written a century or

Presumably, Euclid did not merely transcribe his sources verbatim, however. Each of the books has a few key problems and theorems [e.g., *Elem.* i prop. 47, the 'Pythagorean theorem'], and Euclid must have been able to choose from among several variants for his own constructions and proofs. We can only speculate as to the manner of his examination of these materials, but one can readily suppose that it took the form of an analysis. That is, inspecting the preferred variants, Euclid would determine which prior results were necessary for establishing them; taking these up in their turn, he would determine what they required; and so on. Eventually, the backward sequence must terminate in results which can be accepted as primary, without further proof or justification. In this way, the elements or first principles, suitable as the basis of the corresponding synthetic exposition, would emerge as the last terms in the analytic inspection of the major theorems of each book. One may note, for instance, how the sequence of problems of construction in Euclid's book 1 traces back to the three postulates of construction which preface the book; or further, how the parallel postulate first enters within the proof of *Elem.* i prop. 29, and in fact is stated in precisely the terms that this proof requires [cf. Knorr 1983].⁸²

One of the remarkable features of Euclid's formal style is his deferral of the methods of proportion until book 5. This commits Euclid to presenting congruence proofs for all the propositions on plane figures in books 1-4. At times this results in intricate congruence proofs where the use of the similarity of figures would be straightforward [cf. *Elem.* i prop. 47, iii props. 35-37, iv prop. 10]. Indeed, one would presume that simpler variants of the latter type were employed in Euclid's sources.⁸³ From the technical

more earlier drastically oversimplifies the likely process of transmission and editing. One will freely admit that the *content* of Euclid's propositions was, for the most part, entirely familiar among mathematicians by the middle of the 4th century and, for much of the more elementary material, far earlier than this. But the *organization* into treatises closely resembling Euclid's books was surely still in progress until very near Euclid's time and Euclid himself must be assigned a major role in establishing whatever stylistic unity one can discern in the *Elements* over all. Such an account is surely what is suggested by Proclus' survey [Friedlein 1873, 65-67] of the pre-Euclidean writers on elements.

⁸² Note that the account in sect. 2 above of the development of proportion theory follows a similar pattern of analyzing proofs (i.e. those in the 'Eudoxan' style) back to prior principles which, in turn, become the basis for the synthesis of an alternative form of the theory (namely, that in Euclid, *Elem.* v).

⁸³ A very good *résumé* of these proofs and their alternative versions is given in Artmann 1985. As noted above, however, the evidence from Proclus [Friedlein 1873, 73] does not deal specifically—and perhaps not at all—with treatises from the early geometric tradition. Thus, we have no grounds for assigning the project of devising the proportion-avoiding proofs to any pre-Euclidean editor of *Elements*.

viewpoint, this is an entirely artificial project, since the results established in either event are the same. Why does Euclid divide the materials of plane geometry in this manner? Is it for philosophically interesting reasons, e.g., adherence to a principle of economy in setting up a deductive system? Or for the mathematically interesting reason of determining the precise domain accessible under specified postulates (e.g., the axioms of congruence)?⁸⁴ Yet the aesthetics of deciding between a simple proof which uses proportions and a complicated proof which does not would appear to be unclear. Moreover, Euclid is elsewhere not so concerned over formal niceties, as in his unexplicated assumptions on continuity in books 5 and 12,⁸⁵ or his appeal to geometric motions in book 13.⁸⁶ At best, Euclid's success in maintaining such formal restrictions would appear to be uneven.

On the other hand, the attempt to root Euclid's deferral of proportion theory in historical reasons does not face the immediate issue. It might be the case that the avoidance of proportions was recommended after geometers realized how the existence of incommensurable magnitudes rendered invalid the use of theorems established through an integer-based definition of ratio. But this state of affairs could have held only briefly, during the earlier part of the 4th century.⁸⁷ By Euclid's time Eudoxus had long since

Artmann is right, I believe, to see in these Euclidean proofs a subtle project: to establish as much geometry on as few assumptions as possible. But we can as well assign this effort to Euclid himself as to any of his precursors, and his motives for undertaking it could be other than purely mathematical (e.g., pedagogical), as is maintained below.

⁸⁴ One could pose similar questions about the motivation for restricting the means of construction to compass and straightedge: see n88, below.

⁸⁵ We have noted already Euclid's assumption of the existence of finite multiples of magnitudes greater than other given magnitudes. At several places he also assumes the existence of the fourth proportional of given magnitudes. Neither assumption is covered by explicit postulates.

⁸⁶ The definition of 'sphere' in book 11 and its applications in book 13 conceive this figure as a solid of revolution. Euclid could have defined it statically—as the locus of points in space equidistant from a given point—by analogy with his definition of circle in book 1. This is, in fact, the way the sphere is defined by Theodosius. Attempts to explain away his inconsistency, e.g., as occasioned by the specific exigencies of the solid constructions in book 13 [cf. Heath 1956, iii 269], seem motivated by the desire to save an interpretive principle (*scil.* the desire to avoid assumptions of motion) which did not actually bear on the ancients' view. It is clear, for instance, that Archimedes, Apollonius, and Pappus have no qualms in retaining and exploiting the generational conception of the sphere and other solids of revolution.

⁸⁷ It is doubtful, however, that any such dislocation of mathematics ever took place in antiquity. See the discussion of the alleged foundations crisis in the pre-Euclidean period in Knorr 1975, ch. 9.

resolved these difficulties. It would seem merely clumsy on Euclid's part to persist in the avoidance of proportions in his pre-Eudoxan sources (if indeed he did work directly from older sources, rather than contemporary editions), when the mathematical difficulties had been resolved.

An account along the lines of the pedagogical intent of the *Elements* seems possible: for Euclid may have judged that the simple notions of the congruence of figures constituted a manageable body of material for introductory purposes. The use of similar figures could extend the field and facilitate many of the proofs, and these could have been admitted on the basis of a naive conception of proportion. But Euclid's plan for the *Elements* includes the presentation of proportion theory in a fully rigorous manner. Set early in the sequence of books, instruction in the logical subtleties of the general proportion theory would distract from the geometric material, so that one is well advised to postpone this theory until after a basic block of geometry had been covered. Having made this choice, Euclid would be compelled to find alternative congruence-based proofs, however intricate those constructions might turn out.⁸⁸

The strategy of avoiding proportions in the earlier books of the *Elements* is thus occasioned by the combination of the rigorous proportion theory with the whole field of plane geometry. Neither body of material, taken separately, would compel such alternative demonstrations. Presumably, none of Euclid's predecessors had attempted to compile such a large portion of the geometric field within the limits of a single treatise. Hence, our view carries the implication that the proportion-avoiding proofs were due to Euclid himself.⁸⁹

In setting the genre of the *Elements* as a systematic geometric treatise, we thus perceive two different formats as it were, the research monograph and the introductory textbook. That a study of the elements of a field is the objective of a work of the latter sort is already attested in Aristotle, *Top.*

⁸⁸ One can account for the restriction to planar constructions (that is, circle and straight line) on similar pedagogical grounds: the use of other methods, such as *neuses* (sliding rulers), conics, mechanical curves, and the like, would require additional postulates and an appreciable body of lemmas, before they could be admitted into a formal work of the type of the *Elements*. The narrow base of the Euclidean postulates opens up a domain of constructions rich enough to make such additions unnecessary. Of course, the ancients studied the alternative constructions extensively [see Knorr 1986a, for a survey]. One might view Apollonius' *Conics* as an effort to axiomatize the field of conics on the Euclidean model. But it is unclear whether any of the other approaches received a comparably rigorous elaboration. Despite this, they were freely admitted into researches on advanced problems.

⁸⁹ Contrast Artmann 1985: cf. n83, above.

viii 3. As noted above, Apollonius divides his own *Conics* into two parts, the first half being elementary—that is, devoted to the systematic presentation of familiar results, presumably for the purposes of instruction—and the second half being an advanced supplement comprising new material. Most of the writings in the Archimedean corpus were produced as research works, as one can gather from the prefaces;⁹⁰ the first book *On Plane Equilibria*, however, fits better into the instructional category.⁹¹ In the case of Euclid's work, it was in fact adopted as the standard textbook in its field.⁹² This is evident in the manner of its citation throughout the later geometric tradition and the appearance of a substantial body of commentary on it, as by Hero, Apollonius, Geminus, and others.⁹³ Proclus [Friedlein 1873, 74], for instance, often alludes to its uses in teaching, as when he lists its advantages over other textbooks.

Indeed, for Proclus, the *Elements* defined the scope of an introductory course in geometry. Euclid's formal manner would be especially welcome within the curriculum of the Neoplatonic Academy, ultimately geared toward training in Platonic philosophy. But as a technical introduction, the *Elements* is surely remarkable—for its sophistication on the one hand, and its opacity on the other. For a presentation of the basics in geometry, one would find Hero's *Metrica* a more likely text.⁹⁴ In *Metr.* i, for instance, Hero sets out the different kinds of plane figures in order (triangles, regular polygons, circles, parabolas and ellipses, conical and spherical surfaces) with arithmetical rules for computing their areas, and likewise for the volumes of solid figures in book 2. (Book 3 is devoted to problems in the division of plane and solid figures.) In a very few cases derivations are provided (as for the circle-segment rule in *Metr.* i). But for the most part, the rules are only stated, together with details of the working of explicit problems; for formal justifications the student is referred to the appropriate writings by

⁹⁰ This does not preclude that some of his writings eventually found their way into school use. Archimedes' *De sph. et cyl.*, *Dimen. circ.*, *De plan. aequil.*, and *Meth.* are frequently cited by later commentators like Hero, Pappus, Theon, and Eutocius, who thus assume their availability to students of higher mathematics.

⁹¹ Berggren [1976–1977] suggests that the extant *De plan. aequil.* i is an adaptation for school study. The manner of the origin of the extant text of *Dimen. circ.*, as a reworking of materials from Theon, would indicate the same for this work: cf. Knorr 1986b.

⁹² A similar status would seem to apply for the *Data*, *Optics*, *Catoptrics* (whether or not this is an authentic Euclidean work), and the *Phaenomena*.

⁹³ On Euclid commentaries, see Heath 1956, i ch. 3–4.

⁹⁴ For the text, see Schöne and Heiberg 1903–1914, iii and Bruins 1964. For a survey, see Heath 1921, ii ch. 18.

Euclid, Archimedes, and others. The same concrete approach is adopted for an introduction to arithmetic problem solving in Diophantus' *Arithmetica*. Here one encounters sequences of problems, set in terms of explicit numerical parameters, with complete working out; formal justifications of any underlying arithmetic or algebraic relations are usually omitted, save for occasional references to companion treatises.⁹⁵ As another example of an introductory text, Ptolemy's *Almagest* covers the more advanced field of mathematical astronomy, where a thorough grounding in plane and spherical geometry is assumed [see Toomer 1984, 6]. While Ptolemy's exposition shares some features of Euclid's in that proofs of important geometric relations are often provided, it includes guidance in the more practical aspects of the field, like instrumentation, observations, numerical methods, tables, and so on.⁹⁶

In contrast with these examples, Euclid provides no insight into the application of his theorems, nor does he take up any of the related practical aspects, like the nature and manipulation of instruments for the construction of problems. For all his theorems on prime numbers, perfect numbers, square and cube numbers, irrational lines, and so on, not one concrete example is provided of any.⁹⁷ Because he adopts the synthetic mode exclusively, the reasons behind the steps in his proofs and constructions—why, for instance, an auxiliary term is introduced or a particular proportion is used—are left unexplained. At times one is awed, even mystified, at the *dénouement* of an especially complicated proof.⁹⁸ Without the heuristic

⁹⁵ In some instances, Diophantus cites his own *Porisms* for the derivations of assumed results: cf. Heath 1910, ch. 5.

⁹⁶ On Ptolemy's procedures, see Pedersen 1974. Ptolemy takes pains to derive from observations the numerical parameters of his planetary models. But he offers perfunctory explanations at best to justify the specific basic geometric configurations themselves (e.g., eccenters, epicycles, equants, and so on). Presumably, the basic geometric options were fixed in the older technical literature, particularly, the work of Hipparchus, so that only the refinement of parameters needed detailed commentary.

⁹⁷ Numerical examples turn up in the scholia to the *Elements*. By contrast with Euclid, the arithmetic expositions of the neo-Pythagoreans, following Nicomachus (2nd cent. AD), are based almost entirely on specific examples. Here, general results must be inferred on the basis of incomplete inductions.

⁹⁸ Note, as particularly striking instances, the proof of *Elem.* v prop. 8, the indirect limiting arguments in xii props. 2, 5, 10–12, 18, and the solid constructions in xiii props. 13–17. The Euclidean proof of the 'Pythagorean theorem' on right triangles [*Elem.* i prop. 47] is frequently held up as a model of contrivance. In the case of problems, a reconstructed analysis mitigates the element of surprise, for it reveals, as the synthetic proofs do not, the reason for the critical auxiliary steps: cf. Knorr 1986a, 9.

insights that one would obtain from the analyses, Euclid's moves often seem arbitrary. Some parts of the *Elements*—most notably, the elaborate classification of irrational lines in book 10—evade description as introductory at all [see Knorr 1985]. In all, the student is drawn into a passive appreciation of Euclid's often imposing reasoning, rather than stimulated to develop active expertise in solving problems.

The presence of such advanced features in the *Elements* indicates that Euclid can already suppose the student's understanding of the basics of practical arithmetic and geometry, that is, properties and rules of the type set out by Hero. Although Hero cites Euclid and other authors in the formal tradition, that need not imply his students' prior exposure to these works; these could as well (indeed, more fittingly) be viewed as references forward to works of a more advanced nature for future study. Precedents for this more concrete, application-oriented mathematics are firmly established in the older Egyptian and Mesopotamian traditions.⁹⁹ The Greeks themselves, from Herodotus and Eudemus to Proclus look to practical contexts for the origins of mathematics, and modern scholarship tends to support their view of a transference of the older techniques to the Greeks sometime in the pre-Euclidean period.¹⁰⁰ The achievement of the Greeks in the 5th and 4th centuries, culminating with Euclid, would then lie in their provision of the rigorous deductive foundation of this geometric lore, not the creation of an abstract geometry *ex nihilo*.

The difficulty in assessing Euclid's aims in the *Elements* thus appears to result from what we would term a confusion of genres. For Euclid has fashioned his introductory textbook along lines which we more readily associate with research treatises. Instead of offering models of analysis, whereby the student would learn the arts of geometric inquiry, he provides the formal exposition of results in the synthetic manner. To be sure, the instructor would be free to supplement the text with suitable motivating

⁹⁹ For surveys, see van der Waerden 1954, ch. 1–3; Neugebauer 1957, ch. 2, 4.

¹⁰⁰ The Egyptian precedent is cited by Proclus [Friedlein 1873, 64–65: cf. Morrow 1970, 51–52]. The Mesopotamian precedent, although largely unnoted by the ancients, is standard in the current historical literature: cf. van der Waerden 1954, 124; Neugebauer 1957, ch. 6. I have proposed that the channel for transmission of Mesopotamian techniques to Greece in the pre-Euclidean period was Egypt during and after the Persian occupation (6th–5th cent. BC) [Knorr 1982b, 157]. Scepticism about the Mesopotamian element in early Greek geometry, however, has been expressed by Berggren [1984, 339]. To similar effect, D. H. Fowler [1987, 8, 285] maintains that the Mesopotamian characteristics evident in Heronian metrics entered the Greek tradition in the Hellenistic period, hence, well after the Euclidean manner had been established through the prior researches of the Classical period.

and explanatory insights, as he saw fit. But the *Elements* itself provides no guidance along these lines, and one perceives from the extensive technical commentaries by later writers like Pappus, Theon, Proclus and Eutocius, that the formal aspects of geometry dominated the course of university-level mathematics.

In effect, Euclid transformed the study of geometry and the other technical disciplines into a scholastic enterprise—the appreciation and criticism of standard texts.¹⁰¹ While creative research of a high level was pursued in the immediate circles of the greatest figures, like Archimedes, Apollonius, Hipparchus, and Ptolemy, the tension between the aims of research and criticism emerge early. Already Archimedes can chide Dositheus and his Alexandrian colleagues:

Of the theorems addressed to Conon, about which you continually write me to send the proofs, some I send to you in this book. . . . Do not be amazed if I have taken a long time before issuing their proofs. For this has occurred through my desire first to give them to those proficient in geometry and committed to their investigation. . . . But after Conon's death, though many years have passed, we sense that none of these problems has been moved by anyone.¹⁰² [Heiberg 1910–1915, ii 2.2–3, 5–10, 18–21]

One senses that the Alexandrian group had come to take greater pains over the assessment of proofs than the discovery of new results. By contrast, Archimedes' concern is to stimulate inquiry, as his praises of Conon here indicate.

Our view of the genre of Euclid's *Elements* expands as we move from the prefatory first principles—definitions, postulates, and axioms—and into the main body of problems and theorems. Initially, it may seem plausible that Euclid's aim is to articulate the absolute principles of geometric science, and to elaborate from these the content of the field. But Euclid withholds all commentary on the developing character of his system; the relation of any given proposition to the first principles is never an explicit issue for remark, but only the justification for each of the steps of its proof. Moreover, the presence and position of any construction or theorem are determined by what can be proved at that point; there is no indication

¹⁰¹ One may observe that Alexandrian scholarship in the early Hellenistic period similarly transformed the study of other fields of learning, in particular, literature: cf. the surveys in Reynolds and Wilson 1968, Russell 1981.

¹⁰² A similarly defensive tone, balancing the heuristic and apodictic aims of research, is evident in the *Method* [Heiberg 1910–1915, ii 428–430].

that what is absent is excluded on absolute grounds—e.g., that certain entities whose construction cannot be given do *not exist* within the system defined by the initial postulates [cf. Knorr 1983]. Euclid must be more pragmatic: he is not possessed of those algebraic techniques which can establish definitively which constructions fall within the Euclidean domain; he can only know which constructions have been worked out, and which (so far) have not.¹⁰³ Of those in the latter category, some may have been constructed via alternative means (e.g., cube duplication or angle trisection by means of mechanical curves or conics); but the *possibility* of effecting them via the Euclidean postulate remains for him unresolved. In view of this, Euclid's *Elements* could not be, even for Euclid, an exposition of the whole geometric field.

Thus, as we make our way through his treatise, we perceive how the structure is the vehicle for presenting a body of fundamentals. The structure, however, is not itself the subject of Euclid's interest. Euclid's meticulous attention to formal detail may well connect the *Elements* with sophisticated mathematical and philosophical inquiries into foundations. But this does not make it a treatise on foundations. The effort to interpret it as such a treatise qualifies as a philosophical critique of the *Elements*, but not as an *exegesis* of its own objectives.

One should consider remarkable—and perhaps unfortunate—Euclid's decision to adopt this formal geometric style in the context of a work intended as an introduction to higher studies. Insight into heuristic techniques is omitted, like the scaffolding scuttled upon completion of the edifice.¹⁰⁴ The *Elements* sets out the finished product for our contemplation; it is not a builder's manual. Yet Euclid must surely intend his work to serve as an introduction to the study of geometry—specifically, the formal, demonstrative type of geometry. Demonstrations in effect set out the causes which justify geometric procedures. What is striking is that Euclid considers this manner of presentation, namely, the formal exposition of finished results, to be the basis for instruction. That points to a fundamental difference between his views of research and pedagogy from our own. Euclid expects that

¹⁰³ For a discussion of the ancients' views on the classification and solution of problems, see Knorr 1986a, ch. 8.

¹⁰⁴ Cf. the related observation in Hirsch 1967, 78: 'Genre ideas... have a necessary heuristic function in interpretation, and it is well known that heuristic instruments are to be thrown away as soon as they have served their purpose.' Hirsch is here not expressing his own position, however; he goes on to argue that the notion of genre, specifically 'intrinsic genre', is indispensable for the articulation of meaning.

the learner will acquire expertise in geometry through the contemplation of its finished form, rather than through exercise in its production.

5. Conclusion

The scholarship on ancient mathematics invariably adopts an intentional vocabulary in framing interpretations. In discussing passages from ancient writings like Euclid's *Elements*, scholars present their views as if they were Euclid's own. Theorists of literary criticism have long recognized the difficulties that attend a naive conception of authorial meaning, however, and have generated a spectrum of positions on the admissibility of this concept. I have attempted to apply some ideas from E. D. Hirsch, Jr., whose defense of authorial meaning offers, I believe, a position more fruitful for the practicing historian than would the many varieties of literary scepticism. My aim has been, not to win a consensus for certain interpretations of my own on debated points about Euclid, but rather to show how Hirsch's insights can reveal the methodological assumptions implicit in the different views and thus inform the process of judging among them. His four criteria of legitimacy, correspondence, genre appropriateness, and coherence are especially helpful in determining whether a given view is likely to represent the ancient writer's meaning, or might instead be a criticism of it, that is, a projection of the text into the environment of the critic's concepts and concerns.

The standard views on Euclid's proportion theory (e.g., those cited from de Morgan and Heath) tend to read it in the context of modern notions of real number. In the definition of 'having a ratio' [*Elem.* v def. 4], it is maintained, Euclid's intent is one (or perhaps all three) of the following: to restrict ratios to pairs of homogeneous magnitudes, or to exclude non-finite magnitudes, or to incorporate incommensurable as well as commensurable magnitudes. All these views fit the technical demands applicable to any general theory of proportion. But each fares poorly as a rendering of Euclid's text: *legitimacy*—implicit meanings must be assigned to his terms, where there is no clear cause why he should not have made these meanings explicit, if such was indeed his intent; *correspondence*—aspects of the text are left out of account or appear superfluous (why, for instance, is the 'exceeding of multiples' set out as a reciprocal relation?); *coherence*—the definition is viewed in isolation from its textual position among the other definitions and propositions of Euclid's theory.

Like Heath, Mueller takes the intent of the definition to be the inclusion of incommensurables; but he transcribes it in a form different from that adopted by previous commentators. In his version, the definition is set

firmly in the context of the comparison of equimultiples, characteristic of Euclid's form of the theory. As I have maintained, this effectively displaces all three of the former proposed views of the definition's intent and establishes an alternative one, namely to serve as a specific precondition for the definition of 'having the same ratio' which immediately follows. From this, one can develop a view of the origin of Euclid's fourth definition as a by-product of the recasting of a precursor version of the proportion theory of book 5. The resultant view, I believe, works well as a historical account of Euclid's meaning and editorial method. The standard views, while hereby displaced as accounts of Euclid's meaning, retain value as critical instruments. For we can now use them to judge Euclid's procedure in contrast to the standard modern theories.

With respect to their arithmetic theories, did Euclid and the other ancients conceive fractions as one does in modern mathematics—that is, did they intend by their terminology of fractions the same things we do? Fowler advocates the provocative thesis that they did not, that the unit-fractional mode adopted in the Egyptian and early Greek calculations acted as a barrier against formulating the more general notion of the fractional number. For instead of presenting the result of dividing an integer m by another integer n merely as the corresponding fraction (i.e., the equivalent of our m/n), the ancients habitually engage in further computations, casting the quotient as a sum of unit-fractions. Ostensibly, a textual analysis like that given in the preceding example would confirm the separation of the ancient and modern concepts. But here I perceive a difference, in that we are not dealing with a specific text set in a clearly defined textual domain, but instead, with a wide family of texts spanning the whole of Egyptian, Mesopotamian, and Greek antiquity. Thus, the criterion of coherence is ambiguous and its application circular: if we minimize our constraints as to what would be an acceptable fraction concept and insist only on an operational equivalence with modern fractions, then texts from later antiquity (e.g., Diophantus and Hero and the writers in their practice-oriented tradition of mathematics) would certainly qualify, and one would naturally date the concept back, in the absence of any clear signs of innovation in the late authors. But if we adopt the stricter sense advocated by David Fowler (where, if I understand his position correctly, one ought to accept fractions as forms of ἀριθμοί, thus conflating the categories of λόγος and ἀριθμός), then either the ancients never advanced to such a conception; or, if they did, this occurs only in late Hellenistic texts influenced by the assimilation of Mesopotamian sexagesimal methods and thus should be sharply marked off from the early arithmetic tradition of the Greeks.

The criterion of *genre appropriateness* raises a further difficulty. Our principal evidence for ancient practical arithmetic comes from the mathematical papyri. The two millennia from the Egyptian Rhind Papyrus to the Graeco-Roman papyri embrace a remarkably uniform tradition of school arithmetic, within which fractions are commonly handled in the unit-mode and applications of the general manner, while arguably present, are not expounded systematically as a separate technique. But should we expect otherwise in a genre of school writing consisting of solved examples, rather than exposition, proof, and commentary? But if we turn to the 'high tradition' of Archimedes and comparable writers of formal geometry, computations of this sort are assumed as part of the students' elementary training. Moreover, when notations for terms like $1838 \frac{9}{11}$ appear in the formal tradition, are they merely the artefacts of Byzantine scribal conventions, or do they provide insight into the arithmetical expertise of centuries earlier?

Euclid's arithmetic theory in book 7 of the *Elements* falls within the latter genre of 'high geometry'. What its underlying concept of fraction is may be difficult to determine, since no such term appears there, beyond the undefined notion of measuring ($\mu\epsilon\tau\rho\epsilon\acute{\iota}\nu$).¹⁰⁵ But it does deal centrally with ratios of integers, and we of course recognize how to establish an equivalence between ratios and fractions.¹⁰⁶ Further, the practical writers cite Euclid for the theory underlying their arithmetic procedures. One would naturally infer, for instance, that Euclid's problems on the finding of least common multiples [*Elem.* vii props. 36, 39] were intended to provide formal justification of techniques familiar within the practical arithmetic field. To be sure, our evidence does not directly sustain this view, nor would we expect it to. But I find it more plausible than the converse view that Euclid (or, more precisely, the theoretical tradition he consolidated) elaborated his arithmetic theory purely as an abstract exercise, while the later practical

¹⁰⁵ Already in *Elem.* vii def. 3, the relation of measuring ($\kappa\alpha\tau\alpha\mu\epsilon\tau\rho\epsilon\acute{\iota}\nu$) is assumed for defining 'part' ($\mu\acute{\epsilon}\rho\omicron\varsigma$); cf. def. 5. The notion of one number's being measured ($\mu\epsilon\tau\rho\acute{\omicron}\mu\epsilon\nu\omicron\varsigma$) by another is exploited in the definitions of evenly even, evenly odd, oddly odd, prime, relatively prime, composite, and relatively composite numbers (defs. 8–15), even though a neater strategy could have been followed, applying the notion of part given in def. 3. The problems set out in vii props. 2–3 show how to find the greatest common measure of given integers; but neither 'measure' nor 'greatest common measure' has yet been defined. Similarly, one is assumed to understand the notions of measuring and measure as background for the definitions of commensurables and incommensurables in book 10.

¹⁰⁶ By this I mean that any operation on fractions can be re-expressed as a relation among integers.

writers serendipitously discovered its utility for their computations with fractions.

One of the contributions of criticism is to articulate the subtle implications inherent in our assumptions. Fowler performs this service in reminding us of nuances in our concept of fraction: we learn early and thereafter take for granted that ' m divided by n ' is the fraction m/n , a term which can be manipulated with other such terms according to the familiar rules of arithmetic. But can one impute the same conception in ancient arithmetic texts? Fowler's negative position gives rise to the questions of when, by whom, and under what circumstances the general conception was introduced. Not only do our ancient sources provide no assistance toward answers, however, they seem unaware of any such questions. Like ourselves, they appear to take for granted the nuances implicit in their procedures for fractions.¹⁰⁷ It seems to me preferable, then, to take this silence seriously. The aspect of the fraction concept here at issue, the notion of the general fraction, is not a discovery in a simple sense, it would appear, but rather a concomitant of the basic notion of parts. To be sure, the persistence of the unit-fraction methods tends to obscure this general notion in many of our texts, but I would not take this to be conceptually significant. One might perceive a parallel in the persistence of 'English' standards in the United States today, despite the availability of the more efficient metric system. However much American learners might complain about the difficulties of the metric system, it is clear that no conceptual issue is involved, but merely the perpetuation of an outmoded technique. Doubtless, the ancients had comparable reasons of economic expedience for retaining the unit methods, long after the advantages of alternative procedures should have been evident. But this would not as such signify conceptual limitations.

The issue of *genre appropriateness* also illuminates one's understanding of Euclid's aims in the *Elements* as a whole. Euclid presents geometry according to a carefully worked out deductive structure—but his treatise is

¹⁰⁷ By contrast, it is clear when the commentators must assume that notions or techniques are not familiar to their readers. For instance, in his account of sexagesimal operations, Theon expounds the procedures at length, including detailed accounts of particular examples and full statements of individual cases (e.g., minutes times minutes, minutes times seconds, seconds times seconds, and so on): cf. *In Ptol. ad i* 10 [Rome 1936, 452–457]. This, of course, does not indicate the novelty of these techniques at Theon's time; they became available to Greek mathematical astronomers with the reception of Mesopotamian methods around the time of Hipparchus (2nd cent. BC) or earlier. But if techniques that are novel merely to the special group of learners entail such elaborations by Theon, all the more would actually new techniques receive such treatment near the time of their first introduction.

not *about* deductive structure. It is not in the genre of, say, Aristotle's *Posterior Analytics*, even though the juxtaposition of these two works may bring forth interesting details about the ancients' views on formal systems. As I have proposed above, the motivations underlying Euclid's arithmetic theory can be grounded in practice, and one can argue similarly for other parts of the *Elements*.¹⁰⁸ Indeed, much of its material on the measurement of plane and solid figures reappears in an arithmetical form suited for practical application in Hero's account of metrical geometry. Since this aspect of Hero's geometry may be seen to perpetuate the practical procedures of the more ancient Egyptian and Mesopotamian traditions,¹⁰⁹ it would follow that Euclid also developed his geometric theory on a comparable practical base. Ancient views on the term 'elements' link studies of this type to the introductory teaching of technical disciplines. It is, thus, plausible to associate Euclid's own intent in compiling the *Elements* with its actual use within the subsequent technical tradition, namely, as a basic textbook in geometry. But the *Elements* is not in the same category as the Heronian or Diophantine textbooks: Euclid appears to assume a practical grounding in the discipline, for which he aims to provide the appropriate formal demonstrations. In effect, the *Elements* is a treatise on the causes relevant to the geometric field; it offers the learner models of how to secure the results of geometry as deductive consequences ultimately rooted in certain notions (namely, the postulates and axioms) of figure and quantity. The learner is expected to gain expertise in geometric theory through the study of finished models of formal exposition.

Conceivably, Euclid himself was responsible for the decision to adopt the formal style in an *introductory* textbook. But the deductive form of geometric theory was a conception he owed to his predecessors. This emerged through the interaction of philosophical and mathematical specialists over the course of the 4th century. Although it is tempting to try to make

¹⁰⁸ The physical phenomena to which geometric propositions are related are often manifest in the cases of Euclid's *Optics* (and certainly the *Phaenomena* which utilizes the terminology of observational astronomy) and of Theodosius' *Sphaerica*. That parts of Euclid, *Elem.* i arose in the context of practical mensuration and instrumentation is noted by Proclus [cf. Friedlein 1873, 283, 352, for his remarks on Oenopides and Thales], while he claims that Pythagoras made mathematics an abstract study, whereas among the Egyptians and Phoenicians earlier it had been developed in the interests of commerce and surveying [Friedlein 1873, 64–65]. It is clear that the geometric constructions presented by Euclid and other writers draw directly from experience in construction with instruments, and many texts provide explicit information on practical execution: see Knorr 1983, 1986a.

¹⁰⁹ On the Greeks' debt to the older Mesopotamian and Egyptian traditions, see nn49–50, above.

explicit connections between Euclid's system and the epistemological pronouncements of earlier philosophers, particularly Aristotle and Plato, the situation was hardly this straightforward. Euclid had access to a variety of exemplars from the preceding generation of technical writers, and he was surely more likely to take his expository model from them than embark on a conscious effort to create a formalism satisfying the prescriptions of one or another philosopher. To the extent that Euclid is consistent with philosophical precursors, this can be assigned to their shared acquaintance with that technical corpus.

These three examples from the study of Euclid turn about a common methodological recommendation—that the historian of mathematics should give priority to the critical examination of the texts before undertaking a wider exploration of their philosophical and mathematical ramifications. This may sound too obvious to warrant special comment. But the combination of fragmentary evidence with a subject area readily associable with modern fields of mathematics and philosophy has made the study of ancient mathematics an arena for ambitious interpretation, where reconstruction overwhelms textual criticism. The result has been a striking use of intentionalist terminology in accounts so heavily dependent on the critics' special predispositions (mathematical or philosophical), that the ancient authors could hardly have actually intended what is claimed for them.¹¹⁰ If the undesirability of that situation is now clearer and the potential of the alternative textual method evident, I shall have accomplished my purpose here.

¹¹⁰ Ancient scholarship was hardly immune to the same charge. Russell [1981, 97] styles Hellenistic criticism as intentionalist, even among those Stoic and Neoplatonist writers committed to allegorical readings. In the ancient critics' view, then, Homer (to cite a specific example) actually intended the allegorical meanings they deduce from his texts.