

Science and Philosophy in Classical Greece
Edited with a Preface by ALAN C. BOWEN
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The Relation of Greek Spherics to Early Greek Astronomy

J. L. BERGGREN

My subject is the history of a tradition in the science of spherics that developed from the fourth to the first centuries BC. Spherics is a name which goes back to antiquity for that science whose subject is the mutual relations of arcs and angles formed by circles on a sphere. This science as it is found in texts of the fourth century has been variously described as 'sufficient for the astronomy of its time' [Heiberg quoted in Hultsch 1906], as 'fumbling attempts to obtain some quantitative and geometric insight' [Neugebauer 1975], and as the kind of literature that Plato thought typical of real astronomy [Mueller 1980]. In this paper I wish to examine these and other views of this tradition of spherics in the light of the ancient texts themselves and of the recent writing on this subject, in order to explore the relation of this tradition to the mathematics and astronomy of its own time.

All who have written on spherics have recognized its intimate relation with certain problems in ancient Greek astronomy. These problems, like all those which are at once beautiful and difficult, called forth a variety of solutions; and I should like to begin with a summary review of these problems and their ancient solutions in order to set ancient spherics in its proper context. I begin, then, with Ptolemy's *Almagest*.

After discussing some philosophical, physical, and mathematical preliminaries to the study of astronomy, Ptolemy devotes the end of book 1 and all of book 2 to solving a set of problems concerning spherical arcs and angles. The importance of the solutions of these problems for both astronomy and geography, is evident from the following sample:

1. Find the height of the Sun above or below the equator (its declination, δ) and its distance east or west of the equinoxes (right ascension, α) corresponding to any position of the Sun in the zodiac.
2. Given one of the three quantities—the maximum length (M) of daylight, the altitude (ϕ) of the north pole above the horizon, or the ratios of equinoctial and solstitial shadows to the length of the rod casting them—find the other two.
3. Find the variation in length of daylight during the year, given any one of the three quantities just listed.
4. Find the angles between the ecliptic and such great circles as the horizon or meridian.

Ptolemy's solutions to these problems are the earliest recorded that use the trigonometric methods developed between the time of Hipparchus of Bithynia (150 BC) and that of Menelaus (AD 100).

However, Vitruvius in *De arch.* ix 7 tells us that already by his time in the late first century BC the problem of the length of daylight had been solved by the geometrical method of the *analemma*, and Pappus in *Coll. iv* prop. 40 cites a (now) lost work on this method by Diodorus of Alexandria, an older contemporary of Vitruvius. Otto Neugebauer [1975, 301] suggests that Hipparchus, who lived more than a century before Vitruvius, may have used an *analemma* to determine the arc of a parallel of declination from a setting star of known declination to the point of the same declination on the meridian. Further, we have it on the authority of Synesius of Cyrene (*obit ca.* AD 415) that a century before Diodorus, Hipparchus described the method for solving problems about the sphere now known as stereographic projection. We also know from Vitruvius, *De arch.* ix 8.8–14 that by his time this method had given rise to an instrument, the anaphoric clock, which provides a solution to the problem of finding the length of daylight for a given position of the Sun in the ecliptic.

Not all efforts to solve the sorts of problem Ptolemy addresses relied on geometry, however. At about the same time as Hipparchus, the Alexandrian scholar, Hypsicles, in his *Anaphoricus* exploited a number of fairly weak assumptions about relations between arcs on the ecliptic and those on the equator in order to solve arithmetically the problem of the length of daylight by beginning with the maximum length of daylight as the sole datum. The methods utilize the linear increase and decrease of numbers; they are very different from the geometrically-based methods we have been describing, and their origins are to be found in Mesopotamia. However,

Hypsicles' computations worked with much smaller ecliptic arcs than we find in related, earlier works; and his clear recognition that the same procedures, further refined if necessary, could be employed in any of the seven *climata*, showed that arithmetic methods were entirely capable of producing what King [1987] has called universal solutions in spherical astronomy.

Thus, the mid-second century BC marks the beginning of a period when two important geometrical methods developed, the one based on trigonometry and the other on the *analemma*. It also marks the end of a period of development of arithmetical methods. There is, however, another geometrical tradition, one in which none of the writers we have mentioned participated, despite the fact that it began probably two centuries before the time of Hipparchus and continued as a source of new treatises for another century after him. This tradition comprised a body of theorems on solid geometry which pertain to the sphere and whose principal interest lay in their relevance to astronomy. It is known to us through the following works: *De sphaera quae movetur* (*The Rotating Sphere*) and *De ortibus et occasibus* (*Risings and Settings*),¹ both by Autolycus of Pitane, who flourished in the latter half of the fourth century BC; *Phaenomena* by Autolycus' contemporary, Euclid; and *Sphaerica, De diebus et noctibus* (*Nights and Days*), and *De habitationibus* (*Habitations*),² all by Theodosius of Bithynia, whose probable *floruit* of 100 BC would make him a younger contemporary of his countryman, Hipparchus. None of these writers makes explicit reference to any of the others; and all presuppose not only basic theorems of solid geometry such as one finds in Euclid's *Elements* xi, but theorems on arcs and angles of the sphere which are not found in the *Elements*. Apparently, then, none of these works stands at the beginning of this tradition.

Indeed, the often-remarked relation of the entire theory of spherics to some of the astronomical problems listed at the beginning of this paper shows that the mathematics in these treatises originated after the time when Greek astronomers began to try to derive explanations of observed phenomena as well as predictions from the model of a spherical Earth fixed at the center of a rotating cosmos. As to when this was, Goldstein and Bowen [1983] argue that ancient Greek astronomy may be profitably studied by dividing its history to 300 BC into two periods. The first of these two phases is an ancient tradition characterized by a concern with calendrical matters. In this tradition the dominant astronomical activity

¹ Our study concerns only the first of these works; for the relation of the two books of the latter work to each other, see Schmidt 1949.

² I have used only the first of these three works in this study.

was the composition of *paraepgmata* in which the phases of important stars (that is, their first and last morning and evening appearances) were given in terms of some calendar along with meteorological phenomena that could be expected to accompany these astronomical phenomena. It was an activity that attracted men of considerable distinction, as Ptolemy's inclusion of Meton, Euctemon, Democritus, and Eudoxus, in his list of *paraepgmata* testifies. Indeed, Ptolemy mentions these predecessors in his own *paraepgma*, book 2 of his *Phaseis*, which is by far the acme of this tradition. It is almost a corollary of the intent of this kind of literature that there is no mention of eclipse-phenomena or planetary matters. In fact, it was concern with eclipse-phenomena and the dimensions of the cosmos that marked Greek astronomy after 300 BC.

Goldstein and Bowen also point to other activities, contemporaneous with the ancient tradition of *paraepgmata* and relevant to the history of astronomy. Among them were the numerological speculations of the Pythagoreans and their idea of explaining the heavens by a *ἀρμονία* (*harmonia*) of whole numbers. Indeed, as Aristotle reports, it was the primacy of numbers in their science and the way in which numbers seemed to reflect a moral order based on the properties and ratios of *ἀρμονία* which led the Pythagoreans to suppose 'the elements of numbers to be the elements of all things and the entire heaven to be a *harmonia* and number'.³ Also important, according to Goldstein and Bowen, were the cosmological speculations of the Presocratics, since they introduce elements that were later to become basic parts of scientific astronomy. Thus, some time before Eudoxus, the image of a sphere of stars rotating around a concentric, spherical Earth had been proposed; and such an image is found in the cosmological-moral tradition leading up to works like Plato's *Republic* and *Timaeus*. Indeed, in recent studies, Charles Kahn [1970, and in this volume] has argued that one of the principal achievements of Presocratic speculation is the construction of 'a cosmic model, including a spherical heaven, a spherical Earth, and a geometrical account of celestial motion'.

However, Goldstein and Bowen argue that this world-picture, held by some Presocratics, was not yet a mathematical model, in the sense of an explicit mathematical analogy between physical domains (both idealized mathematically) that served as a basis for computation, or at least comparison, of magnitudes. They believe that with the work of Eudoxus the second phase of early Greek astronomy began and that the distinguishing feature was what they call the two-sphere model, so named because

³ Aristotle, *Meta.* 985b23-986a3 [quoted from Goldstein and Bowen 1983, 333].

it placed a spherical Earth at the center of a spherical cosmos which rotates daily around an axis passing through the Earth's center. Although components of this model can be found in various Presocratic texts, what was new was the exploitation of the model with its reference circles, including the horizon, equator, and ecliptic as well as those below, to provide a *mathematical* explanation of phenomena associated with the risings and settings of stars and the length of daylight. It is this that differentiates Eudoxus' tradition from that exemplified by the Presocratic and the Platonic writings mentioned above, and it is this that gave rise to the science of spherics.

The mathematical utility of the model lay in its division of (the celestial) sphere into five regions concentric around the poles: the torrid region between the tropics, temperate regions on either side of these, and finally the frigid zones around the poles. Although these names reflect climatological characteristics, the origin of the zones is astronomical. The tropics are bounded by the parallel circles defining the northern and southern limits of the Sun's annual motion, and the boundaries separating the temperate from the frigid regions are defined by the circle of always-visible stars and the circle of always-invisible stars, circles dependent on the latitude of the observer. All these circles and the regions between them were then transferred to corresponding circles on the Earth bearing the same names.

Eudoxus and his successors elaborated this model to include not only a sphere for the fixed stars but spheres for the Sun, Moon, and planets; but this system of homocentric spheres lasted only until astronomers realized it could account neither for the retrograde motion of Mars nor for the variation in apparent sizes of such luminaries as the Moon and Venus. After that time the general theory had no more impact on mathematical astronomy; however, the two-sphere model itself became part of the professional astronomer's stock in trade and the science of spherics took the place it occupied in the education of the astronomer throughout antiquity. Since, however, not all recent writers have seen the Platonic writings as being so distinct from the mathematical tradition initiated by Eudoxus, we must here take some account of Plato's idea of what spherics should be, as indicated in a well-known passage, *Republic* vii 528e-530c.⁴ Here Socrates rebukes Glaucon for his notion that simply looking at stars is somehow the same as gaining knowledge through higher, rational speculation. Glaucon accepts the rebuke and, on asking how Socrates thinks the study of astronomy should be reformed, he learns that man must use his reason to conceive 'the true realities—the real relative velocities, in the world of pure

⁴ The following account and all translations are taken from Cornford 1951.

number and all perfect geometrical figures, of the movements which carry round the bodies involved in them'. The real astronomer, we are told, will admire the observed sky as a geometer might admire 'diagrams exquisitely drawn by some consummate artist like Daedalus'. But,

when it comes to the proportions of day to night, of day and night to month, of month to year, and of the periods of other stars to Sun and Moon and to one another, he will think it absurd to believe that these visible material things go on for ever without change or the slightest deviation, and to spend all his pains on trying to find exact truth in them.

After Glaucon assents, Socrates concludes that the genuine study of astronomy proceeds as does that of geometry, by problems, and proposes to leave the starry heavens alone.

What Plato meant by this is sufficiently illustrated in the following section of the *Republic* where there is a discussion of ἀρμονία, and Socrates and Glaucon lament the folly of those who in seeking to understand it 'waste their time in measuring audible concords and sounds one against another'. Such people 'do not rise to the level of formulating problems and inquiring which numbers are inherently consonant and which are not, and for what reasons'. Presumably, such principles when applied to the study of astronomy as recommended earlier, would produce a calendar like Philolaus' where numerological considerations forced a 59-year cycle in which each year had $364\frac{1}{2}$ days. This absurd number was chosen only so that the number of months in the resulting cycle would be a Pythagorean number for the Sun, 729, where $729 = 27^2 = 9^3$ and 27 is the number for the Moon, and 9 the number for the Earth [Neugebauer 1975, 619].⁵

In a recent paper, Ian Mueller [1980] considers the passages in the *Republic* cited above and argues that Plato's assimilation of astronomy to geometry and harmonics to arithmetic is 'not unreasonable' given 'certain Greek scientific texts which, I believe, make clearer the kind of astronomy and harmonics Plato has in mind in the *Republic*'. The texts to which he refers are Theodosius' *Sphaerica*, Autolycus' *De sphaera quae movetur*, and Euclid's *Phaenomena*.

In regard to astronomy, two problems arise from Mueller's arguments. One of these concerns the interpretation of Plato's intent in the passages quoted and the other concerns chronology. I shall discuss them in turn. First, as concerns intent, I proposed earlier that what Plato meant when

⁵ For other examples of numerology in Greek astronomy, see Neugebauer 1975, 630–631, 659–660, and 693.

he talked about 'real' velocities and 'perfect figures' is sufficiently illustrated by his remarks on ἀρμονία, where we are urged to study not the consonances we hear but numbers which are inherently harmonious; that is, Plato's intent is uniform in the two sections I have mentioned. If this is correct, then, it is hard to see in the geometrical and kinematical idealizations of the three treatises mentioned above, the spatial analogue of the sort of numerological speculations Plato suggests for harmonics. That is, if Plato really had texts like those of Theodosius in mind, he could not have intended the same sort of thing in his comments on astronomy as he did in his remarks on ἀρμονία. Second, there are serious chronological problems in assuming that texts like those of Autolycus were available at the time Plato wrote his *Republic*. Under the current view of the composition of Plato's dialogues, the *Republic* was written before the year 370 BC. But comet-observations reported by Aristotle [see below] suggest that the two-sphere model was probably introduced after 372 BC and before 340 BC. Thus, there is every reason to believe that the introduction of the mathematical model, which must have antedated the three texts Mueller addresses, did not occur until some time after Plato had written the *Republic*; consequently, these three texts could not have been the genre of literature that Plato had in mind when he wrote the *Republic*. Thus, I prefer to take Plato at his (or at least Glaucon's) word, when Glaucon asks Socrates, 'How do you mean the study of astronomy to be *reformed*, so as to serve our purposes?' [emphasis added]. It is a reform of astronomy that Plato advocates; he is not describing a current genre of literature. In fact, Plato underlines how far his ideal is from current practice, at the close of that section when Glaucon says, 'That will make the astronomer's labour many times greater than it is now.'

If, then, the mathematical two-sphere model was introduced too late in the fourth century to have any effect on the composition of the *Republic*, when was the model invented? In this regard Goldstein and Bowen refer to Aristotle's report that

in the archonship of Nicomachus (*scil.* 341 BC) a comet appeared for a few days about the equinoctial circle (this one had not risen in the west), and simultaneously with it there happened the storm in Corinth. That there are few comets and that they appear rarely and outside the tropic circles more than within them is due to the motion of the sun and stars.

That Aristotle [*Meteor.* 343b1-7] refers neither of the other comets he mentions (427/26 and 373/72) to any reference-circle increases our faith that this one detail was not added in later times and that it thus supplies us

with a *terminus ante quem* for the introduction of the mathematical two-sphere model, namely, 341 BC. A *terminus post quem* is, to stay on the conservative side, 372 BC, the date of the latest unspecified comet-sighting; and so we have an interval of approximately thirty years within which the mathematical two-sphere model was developed by Eudoxus. (Goldstein and Bowen provide evidence that Eudoxus not only used this model in his astronomy but actually invented it.)

There is good evidence, moreover, that the science of spherics developed fairly rapidly. Indeed, two of the three treatises that will concern us were written by two contemporaries who were active, it seems, at the end of the fourth century. I mean, of course, Euclid and Autolycus, whose treatises on the subject of spherics are the earliest we have.⁶

It is often said of Autolycus' *De sphaera quae movetur* that it represents the earliest extant Greek mathematical treatise; but one must agree with Neugebauer that there is so little hard evidence to separate Autolycus and Euclid chronologically that all we can say with any confidence is that their treatises were all written at roughly the same time, probably in the second half of the fourth century BC. Even Germaine Aujac [1984], who places Autolycus and his writings some thirty years before Euclid, agrees that in any case Autolycus was not a source for Euclid; and there seems to be unanimity on the central point that both Autolycus and Euclid rely on an earlier work for the basic theorems of the subject. This earlier work must have appeared between the years of, say, 360 and 320 BC; of its content and range we may form some notion indirectly from the writings of Euclid and Autolycus.

We turn, then, to the nature and apparent purpose of these treatises. As early as John Philoponus, who in the sixth century of our era provides the first reference to the title of Autolycus' work, *De sphaera quae movetur*,⁷ commentators have seen the difference in character between this treatise and Euclid's *Phaenomena*, for Philoponus points out that Euclid's work is the more 'physical' of the two in that it considers not only motion, a distinguishing concern of ancient physics, but οὐσία (substance) as well, that is, the Earth and the stars. (It also mentions by name all the principal astronomical circles on the sphere.) Furthermore, Euclid's treatise

⁶ Autolycus' *floruit* is securely dated to the last quarter of the fourth century BC: see Mogenet 1950, 5–7 for details. Regarding Euclid's date, we have nothing more secure than the claim that he lived in the period between Aristotle and Apollonius. A later date would only strengthen the arguments I will give subsequently about the purpose of Euclid's work.

⁷ For the text and a summary, see Mogenet 1950, 160.

is preceded by a long introduction whose purport is that the (mathematical) two-sphere model applies to our world; and it begins with definitions of 'horizon', 'meridian', and 'tropic circles', while such purely geometrical objects as tangent circles on the sphere and angles between great circles on the sphere are not defined. I would conclude that in defining circles of astronomical and physical importance and in passing lightly over purely geometrical objects, Euclid is telling his readers what are the really important ideas in his treatise relative to his intent in writing it. Indeed, he does not mislead the reader: *Phaen. prop. 1* (*The Earth is in the middle of the cosmos and occupies the place of the center in relation to the cosmos*) puts us in no abstract, geometrical setting but in our own cosmos. The proof uses a dioptr pointed at the beginning of Cancer rising and is obviously intended to put an important physical image in the reader's mind. The structure of the proof, however, reflects in all details the structure of a Greek geometrical proof; and I conclude from this that Euclid intends us to take the proof as seriously as any of his proofs in the *Elements*. That we have trouble doing so reflects our tastes and not Euclid's, nor those of his time. Indeed, Galen tells us, 'Euclid showed in Theorem I of the <Phaenomena> in a few words that the earth is in the midst of the cosmos, as a point or center, and the students trust the proof as if it were two and two is four' [quoted from Neugebauer 1975, 748]. In any case, we know from proposition 1 onward that we are dealing with a demonstrative science whose subject-matter is astronomical phenomena.

These phenomena are further defined by the following propositions. The first part of proposition 2 states that a great circle through the pole is, in one rotation of the sphere, twice at right angles to the horizon; and the second part concerns the angles made by the ecliptic during its daily rotation, with the meridian and horizon—the latter angles being of importance for the phases of a star. Euclid's remark that 'this has been shown' has often been taken to refer to an earlier treatise, but it seems unlikely that out of the large number of results on spherics which his treatise presupposes Euclid should have chosen only this particular one to cite. Following this are four propositions [*Phaen. 3–6*] on star-risings, of which we may take proposition 5 as typical: *Of stars on the circumference of a great circle that cuts the always visible circle the ones more to the north rise earlier and set later*. The method of proof in both *Phaen. props. 4 and 5* is to project the stars from one parallel circle to another by means of great circles that are rotations of the horizon, thereby reducing the argument to the case when the two stars lie on the same parallel circle. (The parallel circles are not, however, identified as such in Euclid's treatise, though they are in Autolycus' proposition 8.) As this technique is applied, there is abundant

appeal to visual evidence from the diagrams,⁸ and the lemmas needed to establish the necessary assumptions are never stated.

The same technique as that employed in *Phaen.* props. 4 and 5 is fundamental to Autolycus' *De ortibus et occasibus*. In the *Phaenomena*, rotations of the horizon are used to find points on the day-circle of one star that rise and set simultaneously with a star on another day-circle. In Autolycus' work, on the other hand, rotations of the horizon are used to find points on the ecliptic that rise and set simultaneously with a given star. When such points are located, then the points 15° behind and ahead of them mark the points that determine the four phases of the star.

The next two propositions, *Phaen.* props. 7 and 8, deal with arcs of the horizon where the whole zodiac or individual signs rise, and would have been of interest in the theory of lunar eclipses, that is, in what was called the *prosneusis* of eclipses.⁹ The inclusion of such a theorem could indicate that when the treatise was written some of the elements of the theory of eclipses were in place. However, it also provides an introduction to the idea of ortive amplitudes, that is, of the angle on the horizon north or south of the East-West line indicating the point where the Sun rises, and may equally well have been included for that reason. As is the case with the whole treatise, the results here are entirely qualitative. Aujac [1984, 100] has pointed out that a good example of the more geometrical language employed by Autolycus occurs in *Phaen.* prop. 7, where Euclid relies on the theorem, *That the zodiac rises and sets at all places of the horizon between the tropics*—the tropic circles are assumed to be at least as large as the always-visible and always-invisible circles. Compare this with Autolycus' statement of the same theorem in *De sph.* prop. 11:

If in a sphere a great circle, which is inclined to the axis, demarcates [ὀρίζων] the visible (half) of the sphere and the invisible, while some other inclined great circle is tangent to larger circles than the horizon is tangent to, then it makes its risings and settings along the whole arc of the horizon between the parallel circles that it (the other great circle) is tangent to.

Perhaps Autolycus includes this proposition in order to introduce the idea of the Sun's ortive amplitude.

⁸ For a discussion of the principles on which the diagrams in extant mss of Greek texts on spherics appear to be drawn, see Neugebauer 1975, 751–755.

⁹ '*Prosneusis*' refers to the angle formed by the ecliptic and the great circle passing through the centers of the Moon and the Sun (in the case of a solar eclipse) or the Moon and the center of the Earth's shadow (in the case of a lunar eclipse): see Neugebauer 1975, 141–144.

With *Phaen.* props. 9–13 we come to what appears to be the central purpose of the treatise, a consideration of the rising-times of the signs of the zodiac. The notion appears in proposition 9 but is nowhere defined, so a word of explanation may be useful here. The rising-time of some arc of the zodiac is simply the time it takes that arc to rise over the horizon; and one may measure it either in equinoctial hours or in equatorial degrees (since the equator rises over the horizon at a uniform rate), where the relation between equinoctial hours and degrees is $1^h = 15^\circ$.¹⁰ Since in the period from sunrise to sunset of a given day, 180° of the ecliptic must rise, one may use rising-times to compute the length of daylight at a given locality on a given day by finding the rising-times of arbitrary semicircles on the ecliptic.

The first two propositions on the subject, *Phaen.* props. 9 and 10, provide qualitative results concerning the rising-times of semicircles,¹¹ while the next four, *Phaen.* props. 11–13 and the lemma, concern rising- and setting-times of equal arcs placed variously. The notion of a rising-time was obviously considered well understood, in contrast to the basic spherical apparatus which was so carefully defined at the beginning of the treatise. One imagines an audience already familiar with the idea of rising-times from linear arithmetic schemes, but which still needed to be schooled in the geometrical elements of the science of spherics.

Phaen. prop. 12 is a good example of the theorems demonstrated:

Equal arcs of the semicircle that follows Cancer set in unequal times. Those nearest the tropics set in the greatest times while the following set in lesser times. Those nearest the equator set in least times while those equidistant from the equator rise and set in equal times.

The evidently qualitative nature of this result and those accompanying it should not, however, mislead us into thinking that they are of little use; for, in fact, these theorems justify all the inequalities on rising-times that Neugebauer [1975, 712] shows to be sufficient for deriving arithmetic sequences for rising-times under either Babylonian System A or B. Indeed, the inequalities Neugebauer uses are just those Hypsicles (who was a contemporary of Hipparchus) presents for the linear scheme in his *Anaphoricus*,

¹⁰ In making this definition I do not mean to imply that Euclid or his contemporaries had defined equinoctial hours. The expressions used in the Greek text may be translated as 'in the time that arc X rises arc Y has also risen' or 'arc X rises in more time than arc Y', where X and Y are arcs of the ecliptic. Nothing is said of the units in which time is measured.

¹¹ If one knows the rising-time for any semicircle, one can find the day-length for any day of the year.

and the scholiast to that treatise saw clearly that the theorems at the end of the *Phaenomena* were just what were needed to justify the assumptions Hypsicles stated [see De Falco and Krause 1966, 41–45]. I maintain, then, that the *Phaenomena* represents an effort to provide a more systematic basis for those intuitively plausible symmetry-considerations needed to justify linear schemes for rising-times. This would at least explain both Euclid's casual reference to the subject as well as his qualitative treatment of it.

Hypsicles lived some one hundred fifty years after Euclid; but his *Anaphoricus* was certainly not the first treatise of its kind, as is shown by his mention of 'those who occupy themselves with rising-times'. According to Neugebauer,¹² the earliest methods for calculating the length of daylight were schemes which used linear interpolation to compute, month by month, the length of daylight. These schemes began with the length of the shortest day and increased this by a constant amount each month, an amount calculated so as to arrive at the correct value for the length of the longest day. These schemes are found in ancient Egypt in a Rameside papyrus (12th century BC), in the Hibeh Papyrus and the so-called Eudoxus Papyrus (both containing views from the 3rd century BC); and they survive well into the Middle Ages. However, the more sophisticated arithmetic approach to the problem, which sums up the rising-times of ecliptic-arcs (these being taken to be in an arithmetic progression), was used not only by the Babylonians of the Seleucid period but also by Epigenes (perhaps ca. 250 BC) in Alexandria and pseudo-Berosus (1st century BC: cf. Kuhrt 1987, 36–44). Neugebauer suggests that a lunar tablet dating from about 400 BC, which gives length of daylight as a function of the Sun's position on the ecliptic, may represent a transitional stage between linear schemes for length of daylight and an approach which uses linear schemes for rising-times and then sums these to obtain length of daylight.

It seems, then, that the method of calculating the length of daylight by rising-times was introduced around the end of the fourth and beginning of the third century. I suggest, therefore, that Euclid's *Phaenomena* reflects the kind of mathematics that was needed in contemporary astronomy. In this I am following Heiberg's evaluation of the *Phaenomena* as a treatise sufficient for the astronomy of its time [cited in Hultsch 1906, col. 1048]. While I agree with Neugebauer [1963, 530b] that 'Euclid and Aristarchus... demonstrate the inadequacy of traditional mathematics to cope with spherical astronomy and trigonometry at the end of the fourth century', I would suggest that Euclid at least was not trying to cope with spherical astronomy or trigonometry but rather was aiming to give the

¹² For the following remarks, see the extensive discussion in Neugebauer 1975, 706–733.

student the theoretical background to understand methods that had been developed in astronomy.

In any case, with the inequalities of rising-times given in *Phaen.* props. 11–13, Euclid is in a position to finish the treatise with a comparison in *Phaen.* props. 14–18 of the time it takes equal arcs of the ecliptic to leave the hemispheres above and below the horizon. (I see no reason to doubt that this material is genuine, despite the fact that the last two propositions are found only in recension B.) The idea of leaving (viz. changing: ἐξαλλαγή) a hemisphere is defined at the end of the introduction to the *Phaenomena* as the passage of an arc of the ecliptic from its first point being on the eastern horizon to its last point being on the western horizon. Such a concept would arise in the calculation of rising-times as follows.¹³ When the Sun rises, it is at a certain point *P* of the ecliptic which is on the horizon. Since the Sun always moves slowly westwards along the ecliptic, it happens that, when the rotation of the heavens brings *P* to the western horizon, the Sun is not yet on the horizon because it has traversed a certain arc *PQ* on the ecliptic (on the order of half a degree) in a direction opposite to that of the daily rotation of the heavens. Thus, the arc *PQ* must leave the hemisphere for the Sun to set, so the true length of daylight is the time it takes *PQ* to leave the visible hemisphere.¹⁴ Since this is just the rising-time of the arc from *P* to *P* + 180° added to the rising-time of the 1/2°-arc diametrically opposite *PQ*, we may calculate a good value for the true length of daylight by interpolating linearly between the values for rising-times which Hypsicles gives at intervals of 1°.

Thus, again, one may see in Euclid's *Phaenomena* a geometrical account of topics that may have been known from the arithmetic methods of his day. Although there is no evidence that points to such fine calculations in Euclid's time, it is still true of ancient mathematics—and more so than of later times—that 'absence of evidence is not evidence of absence'; and the hypothesis of a relation with the arithmetic methods of the time at least gives some point to an exercise which otherwise seems rather mad—worrying about the rising-times of arcs on the order of 1/2° when one is, in fact, unable to determine even approximately the rising-times of whole signs.

Quite different from Euclid's *Phaenomena* in character and subject is Autolycus' *De sphaera quae movetur*. I have already quoted the remark

¹³ This is pointed out in Schmidt 1943, a study from which I have derived great benefit. It is a pity that it has never been published.

¹⁴ I am surprised to find nowhere in the literature a paradox of the Achilles-and-the-tortoise type, since one is suggested by the fact that, by the time *Q* gets to the horizon, the Sun has again moved away from it.

by Philoponus on the more geometrical nature of this treatise, so we may proceed immediately to a brief account of the subjects it treats. To begin, there is no long preface as with Euclid; instead Autolycus starts by defining the uniform motion of a point [but see Aujac 1979, 42nn1, 4] and then turns immediately to a series of twelve propositions which treat the following topics:

- 1-3 Generation of parallel circles, which are perpendicular to the axis, and similar arcs by points on the surface of a uniformly rotating sphere.
- 4-6 Cases when no points, all points or some points rise and set. Introduction of the *sphaera obliqua*, that is, the sphere for an observer not at the equator.
- 7 In a *sphaera obliqua* points rise and set on the same parallel and all parallels are equally inclined to the horizon.
- 8 Great circles tangent to the same (parallel) circles as the horizon are rotations of the horizon.
- 9 The co-risings and co-settings of stars in a *sphaera obliqua*.
- 10 In a *sphaera obliqua* a rotating circle that passes through the poles is only twice perpendicular to the horizon.
- 11 Where, on the horizon, does a circle tangent to circles larger than the horizon rise and set? [See my earlier comments on *Phaen.* prop. 7.]
- 12 If a fixed circle always bisects a moving circle, neither circle being perpendicular to the axis or passing through the poles, then each of these is a great circle.¹⁵

It is apparent from this list of theorems that this work, unlike the *Phaenomena*, is not dedicated to a goal more specific than discussing various phenomena arising in the *sphaera obliqua*. Certainly, some of the same topics are broached, for example, the perpendicularity of circles through the poles to the horizon and the arcs of the horizon which the ecliptic circle passes by during its daily rotation. There is also a discussion of the risings and settings of stars, a discussion which begins with rare cases and then turns to the *sphaera obliqua*. But, on the whole, one senses that this

¹⁵In his preface Euclid cites a weaker version of this theorem, in which the moving circle is given as a great circle, in order to show that the horizon is a great circle. His wording is slightly different and he neglects the necessary restrictions which Autolycus states here. It is certainly the sort of result one would appeal to in trying to apply the abstract model of spherics to observed phenomena.

treatise, with its smattering of topics—all fundamental but none pursued very far—is the sort of text that one would have a student read prior to reading the *Phaenomena*. The half-physical, half-abstract nature of this treatise is clearly exemplified by the fact that although the horizon and the always-visible and always-invisible circles are mentioned, there is only an allusion to the ecliptic and tropics in propositions 11 and 12, where these circles are described rather than named.

From the mathematical point of view, there are several features of both the *Phaenomena* and *De sphaera quae movetur* worth mentioning. First of all, on a formal level, the Euclidean treatise defines astronomically important circles whereas Autolycus defines uniform motion. Moreover, the similarity of the structure of the proofs of the propositions in both treatises to that familiar to us from Euclid's *Elements*—*πρότασις*, *ἐκθεσις*, *διορισμός*, *κατασκευή*, *ἀπόδειξις*, and *συμπέρασμα*—shows that both writers were writing within a mathematical tradition, whatever the degree to which physical notions entered. Finally, the formal incompleteness of these treatises, both of which quite casually cite results they need as apparently well known, is evidence that these treatises are but individual patterns in a larger mosaic: both writers are evidently working in a background of familiar results and methods, not only in the special area of spherics but generally in solid geometry as found in book 11 of the *Elements*. Both writers may have aimed to train readers in special topics, but neither wrote for beginners in geometry.

Turning now from the formal aspects of the treatises to the mathematical methods they employ, let us consider Hultsch's suggestion [1886] that the theorems from Theodosius' *Sphaerica* which Euclid and Autolycus refer to as known and available for their use, together with the geometrical results used to establish those theorems, may be taken as the kit of mathematical tools available to writers on spherical astronomy at the time of Euclid and Autolycus. According to Olaf Schmidt [1943, 11–12], however, the story is not so simple. Certainly, if a result proved by Theodosius is cited word for word by Euclid in the *Phaenomena*, one may assume that the result in that form was part of the mathematics available to Euclid; but this does not at the same time justify arguing that, because we know how Theodosius proved the result, we may extract from the theorems used in this proof other theorems Euclid must have known.

Schmidt's example of how such an assumption can mislead concerns proofs using tangent circles on the sphere. Neither Euclid nor Autolycus defines these circles, but it is apparent from such proofs as *De sph.* prop. 6, which shows that the parallel circles touching the horizon are either always visible or always invisible, that Autolycus considers two circles

as tangent if they have only one point in common; and there is no reason to suppose that Euclid's conception was any different. On the other hand, at the beginning of *Sphaerica* ii, Theodosius defines two circles to be tangent at a common point if the line through that common point and in the plane of each circle is tangent to each circle. Theodosius then uses this to establish a group of propositions, *Sphaer.* ii props. 3-5, which together could be called the Fundamental Theorem of Tangency, and whose import is that two circles touching at a point are tangent if and only if that point and their poles lie on a single great circle. Admittedly, the last of these three, *Sphaer.* ii prop. 5, is used without proof by Euclid [*Phaen.* prop. 2] and by Autolycus [*De sph.* prop. 10]; however, in light of the fact that neither of these treatises contains any hint of the Theodosian definition of tangency, it would be unwise to assume that the source Euclid and Autolycus used for the theorem proved it from the same definition of tangency that Theodosius used.

Another important group of theorems that is basic to the theory of tangency is the group *Sphaer.* i props. 13-15, the import of which is that if G is a great circle and S a small circle on a sphere, then the following statements are equivalent: (1) G bisects S , (2) G is perpendicular to S , and (3) G contains the poles of S . The proofs of these propositions are built on several previous propositions of book 1, which utilize *Sphaer.* i props. 1, 7, 8, and 9, and these propositions are, in turn, used in the construction-problem, i prop. 21 (*To construct the pole of a given circle on the sphere*) and in i prop. 17, which is itself used in the proof of i prop. 21. *Sphaer.* i prop. 15 is also used in Autolycus, *De sph.* props. 5, 6, 7, and 10, while *Sphaer.* i props. 13 and 15 are both used in Euclid, *Phaen.* prop. 2.

However, it is instructive to compare the different ways in which *Sphaer.* i props. 13 and 15 are used by Autolycus and Theodosius. The latter uses them in the proof of the first part of what we have called the Fundamental Theorem of Tangency, namely, in *Sphaer.* ii prop. 3, which states that two intersecting circles are tangent if the point of intersection and their poles lie on a single great circle. His argument is that, by *Sphaer.* i prop. 15, the two intersecting circles are perpendicular to the great circle joining their poles, so that the line in which their planes intersect will be perpendicular to the great circle. Also, by *Sphaer.* i prop. 13, the great circle bisects each of the intersecting circles so that each circle intersects the great circle in a diameter. Accordingly, the line of intersection of the planes of the two circles is perpendicular to each diameter and is, therefore, tangent to each circle. By definition, then, the circles are tangent.

In contrast, Autolycus, when he must prove in *De sph.* prop. 6 a particular case of *Sphaer.* ii prop. 13, namely, that the horizon and greatest

always-visible circle are tangent, begins just as Theodosius does but ends in a very different way. Autolycus starts by remarking that, since the meridian contains the pole of the horizon, it follows [cf. *Sphaer.* i prop. 15] that the meridian is perpendicular to the horizon and bisects it. Now, however, he takes a different tack. He observes that a section of a circle (the meridian) is upright on the diameter of a circle (the horizon) and is divided into unequal parts at the north pole. Next, as we remarked earlier, Autolycus applies the result later proven by Theodosius in *Sphaer.* iii prop. 1 to show that the greatest always-visible circle is tangent to the horizon. It is significant, I think, that although *Sphaer.* iii prop. 1 uses the notion of tangent circles, Theodosius' proof uses only the Pythagorean theorem and some elementary results in solid geometry concerning one plane's being perpendicular to another—in other words, Theodosius' proof makes no appeal to his definition of tangency. This, then, is one result in Theodosius' *Sphaerica* which is relevant to tangency and whose proof could go back to a pre-Euclidean source. In short, it would appear that *Sphaer.* iii prop. 1 forms part of the ancient theory of tangents, but that at some time after Autolycus someone saw how to apply *Sphaer.* i props. 13 and 15 to establish a new theory of tangents.

Theodosius himself uses *Sphaer.* ii prop. 5 in the proof of ii prop. 13 which introduces the idea of disjoint semicircles of great circles on a sphere that are tangent to the same parallel circle. This notion is used by Euclid, *Phaen.* props. 4–7, 12, and 14 to prove that certain arcs are equal. It is also applied in Autolycus' *De sph.* prop. 8 which states that 'great circles tangent to the same circles as the horizon will, when the sphere is rotated, coincide with the horizon'. Here Autolycus uses precisely the same terminology of disjoint semicircles as found in Theodosius.

On the other hand, *Sphaer.* ii prop. 13 introduces a sequence of propositions, *Sphaer.* ii props. 13–16, which provides constructions [ii props. 14 and 15] and theory [ii prop. 13 and its partial converse, ii prop. 16] relevant to comparing arcs on parallel circles. The centerpiece of this group, which finds important application in the *Phaenomena*, is *Sphaer.* ii prop. 15 on the construction of a great circle tangent to a given parallel circle and passing through a point between that parallel and another which is parallel and equal to it. As the table of the logical structure of *Sphaer.* ii shows [see Table 1], this proposition is well embedded into the structure of book 2 and makes use of the Theodosian theory of tangent circles. Euclid, however, appeals to this theorem on three different occasions [*Phaen.* props. 4, 5, and 12], and of crucial importance to these appeals is the result proven in *Sphaer.* ii prop. 13 concerning arcs of two parallel circles cut off by non-intersecting semicircles. Autolycus, too, states the enunciation of *Sphaer.*

ii prop. 13 word for word in his treatise. Here again it seems plain that the results of *Sphaer.* ii props. 13–16 represent¹⁶ part of a pre-Euclidean chapter on the subject, which some writer later than Euclid put on a different basis.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
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Table 1. The logical structure of Theodosius, *Sphaerica* ii

A • at row m in column n means that the proof of proposition m uses proposition n . For example, proposition 4 in book 2 relies on propositions 2, 3, and 4. This table does not show that ii prop. 1 uses i prop. 10; that ii prop. 2 uses i prop. 10; and that i prop. 15 is cited in the proofs of ii props. 3, 9, 10, and 21.

Further evidence of progress in the treatment of spherics between the time of Euclid and that of Theodosius consists in the much more sophisticated treatment of the angles of inclination between the ecliptic and horizon as it is discussed in *Sphaer.* ii prop. 22.¹⁷ We are still dealing here, it is

¹⁶ See Aujac 1984, 104–105 for details and a convincing discussion.

¹⁷ According to Theodosius, one plane is more inclined to a base plane than is another plane if it makes a smaller angle with that base than the other does.

true, with angles between planes and not with angles on the surface of the sphere; but, even given this, Theodosius' treatment of the way the angle in question varies monotonically from a maximum to a minimum, together with its discussion of symmetries, goes far beyond the beginning steps in the solution of the problem that we see in Euclid's *Phaen.* prop. 2.

Theodosius' method of proving the result is of some mathematical interest in that it measures the variation in one magnitude, the angle between ecliptic and horizon, by that of another, namely, the height of the pole of the ecliptic relative to the horizon. The standard phrase for 'height of the celestial pole' is ἔξαρμα τοῦ πόλου, which refers to the arc of the great circle through the zenith and the pole contained between the pole and the horizon. The phrase does not occur in the *Sphaerica*, however: in comparing two positions of the pole of the ecliptic, Theodosius describes the one pole as μετρωρότερος than the other, and measures the elevation of the pole by the perpendicular from it to the horizon. The result is that in going from greater elevation of the pole to greater inclination of the ecliptic, the arc mentioned earlier is introduced secondarily and at the expense of some complication in the proof of *Sphaer.* ii props. 21–22, as a quantity that varies monotonically with the pole height and that allows one to pass from it to inclinations of planes.

Any study of the mathematical methods of ancient spherics would be incomplete, however, without some consideration of the methods used in the theorems requiring some construction.¹⁸ A *propos* of such theorems in Theodosius' *Sphaerica*, Schmidt [1943, 13–14] sees in them evidence that those who worked with spherics in an astronomical context used constructions on a solid sphere. This is no doubt correct, but there is more to be said on this point.

Schmidt's principal arguments concern the group *Sphaer.* i props. 16–21 which provides the basis for the construction of a great circle through two given points and for finding the poles of a given circle. However, the proof of i prop. 18 assumes that one can draw lines on a circle within the sphere, and this theorem serves in i prop. 19 which requires one to find the diameter of a given sphere. Moreover, the very first construction-problem, namely, i prop. 2 (*To find the center of a sphere*), is obviously not solved by construction on the surface of the sphere; indeed, one wonders what use it would be for working with a solid sphere. It seems to me that i prop. 2 is an instance of a theorem that occurs in the *Sphaerica*, and, more to the

¹⁸ The question of the status of construction-problems relative to that of theorems in Greek geometry has been discussed in Bowen 1983; the question of the motivation for constructions is discussed in Knorr 1983. These are recent considerations of issues raised in Zeuthen 1896.

point, was developed in the first place, because it was thought that such a proposition belonged in an elements of spherical geometry.

Theodosius' view of construction-problems, however, goes deeper than a simple desire to do problems on the sphere analogous to those one wants to do in the plane; and this is shown by the solution of *Sphaer.* i prop. 20, which requires the construction of a great circle through two given points on a sphere. He begins the proof as follows:

Let *A* and *B* be the two given points on the surface of the sphere. It is required to draw the great circle through *A* and *B*. When *A* and *B* lie diametrically opposite it is clear that arbitrarily many great circles can be drawn through *A* and *B*, so let it be supposed from now on that *A* and *B* are not diametrically opposite.

Now if Theodosius' object had been to show astronomers how to do useful constructions on a solid sphere, he would hardly have omitted a case which is at least as useful as any other. Again, if the object were simply to write a treatise containing what a treatise on geometry ought to contain, then one would expect to see all cases treated. In fact, what the proof of this problem suggests is that one function of some of the constructions was to guarantee the existence of certain objects.¹⁹ In this case the existence of a great circle through two diametrically opposite points would have been clear to any reader who had understood *Sphaer.* i prop. 6 and the meaning of 'diametrically opposite'.

Finally, Theodosius' motives for including construction-problems are well illustrated by the proof of problem *Sphaer.* i prop. 21, which requires finding the pole of any given circle on the sphere. The existence of poles is guaranteed by *Sphaer.* i prop. 8 together with the existence of the perpendicular to any given plane at any point on it (assumed, in any case, in problem i prop. 2); yet Theodosius does not give such a proof. His procedure for the construction is certainly such that one could carry it out on the surface of the sphere. Yet, if it were intended simply as a 'how to' recipe for the novice astronomer working with the solid sphere, it is

¹⁹Zeuthen [1896] suggested that this was the motivation for construction-problems in Greek geometry. Knorr [1983] argues, on the other hand, that such a purpose accounts for only a very few of the extant constructions and is largely a backward projection of mathematically trained 19th-century historians of then-current concerns onto an ancient canvas. Knorr's arguments are generally convincing, but my point here is that there is internal evidence in the proof of *Sphaer.* i prop. 20 that one of the motives for including the theorem was to prove the existence of great circles satisfying certain conditions. That other motives were also operative is evident from the construction-methods Theodosius uses.

curiously incomplete, since it assumes, without explanation anywhere in the treatise, the bisection of arcs of circles. Of course, the existence of the midpoint of an arc is clear from the considerations of continuity, as is also the case with the existence of a fourth proportional to three given arcs, something Theodosius assumes in the proof of iii prop. 10. Given that Theodosius says nothing more about the midpoint, it may well be that continuity was what he was tacitly appealing to; but constructive proofs are also at hand with the material at his disposal, so it seems we are in no position to argue that Theodosius held any one attitude uniformly towards all his constructions.

Such then are the mathematical methods of spherics in the fourth century BC, partly as they are found in the texts from that century and partly as we have reconstructed them from a text of the first century BC. The goal of these methods was to explicate astronomical phenomena and their origin lay in the mathematical two-sphere model of Eudoxus. That the explanations were entirely qualitative should not surprise us given the state of astronomy in fourth-century Greece, since astronomers only had available to them at that time a collection of data based ultimately on rough observation and qualitative estimates.

On the other hand, interacting with this material in a complex way were arithmetic methods of varying degrees of sophistication. We have seen how, for example, in the case of the length of daylight, all such schemes were based on the idea of interpolating a sequence of numbers between the annual minimum (m) for the locality and the annual maximum (M). The different levels of sophistication lay not in the values for M but in:

1. the rules according to which the sequence of numbers was chosen,
2. the density of the derived sequence of numbers within the interval $[m, M]$, and
3. the integration of one sequence with another in a scheme such as that of the *climata* which would, so to speak, cast a numerical net over the whole οἰκουμένη (inhabited world).

This is but one example of how the precision of the ancient exact sciences lay not in the exactitude of their observations, which was poor by modern standards [see Aaboe and Price 1964], but in the fact that the scientists developed or used mathematical methods to turn, if I may overstate matters slightly, observational dross into scientific gold. Thus, it is entirely consistent with the qualitative, approximate character of ancient observational data that the first attempts to geometrize the world-picture should

themselves display a similarly qualitative character. In fact, the subject of spherics produced exact results only when it was combined, by means of trigonometrical tables, with numerical procedures which had long preceded it. But that belongs to the history of trigonometry and is another story.