

8057 INCOMMENSURABILITY, MUSIC AND CONTINUUM: a cognitive approach.

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Introduction.

"How did the Greeks discover incommensurability?" This is one of the most puzzling 'scripts' of the history of mathematics. The 'movie' usually shows a Pythagorean (Hippasus or Archytas for example) drawing a square with its diagonals (or a pentagon with its golden sections on sides and diagonals) (von Fritz 1944), or drawing dots in a square (Knorr 1975), or even making some computations (successive subtractions, *ἐνqυφα...ρεσιj*) (Fowler 1987), and exclaiming: "But this is impossible! If I am not wrong, this means that geometrical, continuous magnitudes cannot be reduced to numbers, discrete entities! Pythagoras was wrong!". Then a meeting of the sect takes place and all Pythagoreans swear to keep the secret of the 'scandal'. A *Grundlagenkrisis* for the ancient Pythagoreanism follows, for whose solution a brand new axiomatization will be necessary, which will lead to Eudoxus and Euclid.

This movie deserves great attention because its scenes are the background of our modern perception of the mathematical continuum, and, at the same time, these scenes are forced by this perception. To question the movie means also to discuss the most basic and undisputed premises of our current philosophical view of mathematics.

It is the aim of this paper to try to shed some light on the connection between the historical interpretation of the discovery of incommensurability and our idea of *continuum*: this is what I mean by a 'cognitive' approach.

To this aim I am going to outline an alternative historical interpretation of the involved discovery, centered on *music* instead of *geometry*, which in turn suggests an alternative view of the *continuum* considered as something far from being 'natural' and 'empirical'. My aim is to address the music-theoretical origin of the discovery of incommensurability, together with the question why this origin was subsequently forgotten, while at the same time the theory of incommensurability was translated into a geometric form. I claim that there were two main reasons for both the oblivion and the translation: first, the negative character of this discovery vs. the positive character of the geometric version (side-diagonal of the square, or of the pentagon); second, the connected breakdown of the ancient musical-arithmetical idea of 'linear numerical magnitude'.

This alternative perspective looks at mathematics as deeply involved in the whole culture, its discoveries being always perceived by the protagonists through the glasses of the general cognitive aspects of a civilization, so that I can say that *nothing relevant in mathematics can be actually and distinctly discovered 'empirically' or 'by chance' or as a 'by-product'*. This does not mean to relativize mathematics because my question is not about the *objectively true* nature of the mathematical ideas, but about their becoming *intersubjectively certain*, my concern is on *phenomenology*, and not on *ontology*.

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Music and Incommensurability.

At the end of the 19th century Paul Tannery already addressed the problem of the role played by music in the discovery of incommensurability. More recently the question was raised by Arpad Szabo (1978, II), who suggested that the pre-Eudoxian theory of proportions initially arose in the Pythagorean theory of music.

Szabo supported this thesis with a deep analysis of the Greek technical terms of the theory (*diásthma*, *lógon*, *enlógon*, *lógon*, etc.) and their recognition in the supposedly Pythagorean experimental practice of a string stretched across a ruler, the so-called "canon", *kánon*, divided in 12 parts, which according to Diogenes Laertius (1962, VIII. I, 12), was invented by Pythagoras. The music-theoretical terminology, employed for example in the *Sectio Canonis* (von Jan 1895), was based on the canon: a 'note' was represented by the interval, *diásthma*, on the string whose pluck yielded that note, and usually the interval $(0, a)$ represented all the intervals $(d, d+a)$ and was represented by the number a .

The Pythagorean theory represented the consonance between the notes a and b by the *ratio*, *lógon*, $a:b$, that could also be ambiguously represented by the interval (a, b) . Hence the junction of two contiguous ratios/intervals (a, b) and (b, c) , equal to the interval (a, c) , corresponded to the product of the related ratios $(a:b)(b:c) = (a:c)$. The Pythagoreans found three main consonances between two notes: the 'octave' (*diplásion*, 12:6 or 2:1), the 'fourth' (*tétraktys*, 12:9 or 8:6 or 4:3) and the 'fifth' (*miólion*, 12:8 or 9:6 or 3:2). Each of these is a "superparticular ratio", that is a ratio reducible to the form $n+1:n$.

I have few remarks to add to Szabo's analysis. I cannot resist giving another example of that proportion-theoretical terminology, which is difficult to explain outside a musical background. Think of the term "continuous proportion" for the three-terms proportion $a:b = b:c$. Why *sunekýj*, "continuous"? This term seems to be reasonable only if we relate it to a canon where three points are considered. A four-terms proportion $a:b = d:c$ was the equality of two *ratios*, namely a relationship between the *ratios/intervals* (a, b) and (d, c) . With respect to this general case, in the 'continuous' proportion the right extreme of the first interval and the left extreme of the second are coincident and thus the intervals a, b and d, c share the same extreme point $b=d$, according to an idea of 'continuity' that appears in the Aristotelian definition: "a thing is continuous with another thing when those extremes by which they touch become one and the same thing" (Aristoteles 1831, *Physica*, 231, a24).

I must remind that all the involved numbers are integers. However, irrationality and incommensurability were inescapable in the Pythagorean theory of music. In fact, musical consonances were explicitly connected by Archytas with the means: "Now there are three means in music..." arithmetic $a-b=b-c$, geometric $a:b=b:c$, and harmonic or subcontrary $a-b:a = b-c:c$. (Diels and Kranz 1964, 47 B2)

The arithmetic mean in one octave 12:6 can be easily recognized in the fourth 12:9 or 8:6, in fact $12 - 9 = 9 - 6$ (such as C-F). The harmonic mean in one octave 12:6 can be easily recognized in the fifth 12:8 or 9:6, in fact $12-8 : 12 = 8-6 : 6$ (such as C-G). On the contrary the geometric mean does not correspond to any 'rational' consonance in the octave 12:6 because it would yield $12:(6 \times \sqrt{2}) = (6 \times \sqrt{2}):6$. Perhaps for this reason Aristotle (Aristoteles 1886, fragm. 43) recognized instead that only the arithmetic and harmonic means were related to musical harmony.

According to Szabo (1978, 174) "an octave cannot be divided into two equal subintervals by a number" (for Szabo 'equal intervals' (in consonance) means equal ratios and hence 'proportionality'). This is odd enough, because the geometric mean is the natural relationship between the multiple octaves: for example the interval of two octaves 12:3 can be divided in two intervals of one octave

by its geometrical mean, $12:3 = 12:6 \times 6:3$. Such a role for the geometrical mean seems to come to an end with the single octave, which according to Philolaus coincided with the perfect harmony (Diels and Kranz 1964, 44 b6) and could not be divided in two equal semi-octaves, so that its division in a fourth and a fifth ($12:6 = 12:9 \times 9:6$) was the better 'dichotomy' that could be accomplished on it.

Szabo (1978, 171) did not ascribe to the Pythagorean theory of music "more than a start". He added that the full development of the theory of proportions had to be found in the "geometrical arithmetic of the Pythagoreans", where a crucial role was played by the definition of "similar (ἴμοις) plane numbers" as "those which have their sides proportional" (Euclides 1956, *Elements*, def. VII,21).

Szabo (1978, 173-4) moreover conjectured that "the concepts of the musical theory of proportions were applied first of all in arithmetic... Furthermore the application of this theory to geometrical arithmetic contributed towards an understanding of the problem of geometric similarity, and this problem in turn soon led to the problem of linear incommensurability".

This translation of the problem from music to geometry, which in my opinion should be ascribed to the times of Archytas, Eudoxus and Theaetetus, as I shall try to explain below, was probably also the rationale of the term 'geometrical' mean, because it could produce no musical consonances, whereas it had precise and easy geometrical instances.

Music and arithmetic in ancient civilizations.

Could the discovery of consonant arithmetic ratios have been made in Egypt or Babylon? This question is intriguing, but, as far as I know, there is little archaeological evidence about this subject. There are many interesting ethnomusicology books about ancient Egypt and Babylonia as well as about Chinese and Indian or primitive civilizations, which show how deep was the religious, social and cosmological role of music (for example in the Upanishad). As for the mathematical aspects, Sachs (1943, II part, ch. 3) mentions two flutes dating back to the Middle Empire (c. 2000 B.C.) whose holes were pierced according to simple ratios. For one of them the ratios were the classical 12:10:9:8. In addition, I would like to remark that:

- Iamblichus (1975, 118) tells us that the musical sequence 6, 8, 9, 12 was taken by Pythagoras in Babylon. A further reference to the problem of 'cutting the tone' can be found in Plutarchus (1962, *De Iside et Osiride*, 367 e-f). Plutarchus ascribed it to the Pythagoreans, but there he was speaking about Osiris and the unlucky 17, which was the natural 'candidate' to cut the tone (between 8 and 9, i.e between 16 and 18).

- The discovery of mathematical ratios of basic consonances is quite easy on a lute, where the strings can be shortened by the finger, and the tuning is made 'by division' on the string, but it is virtually impossible on a lyre where there is no neck and the tuning is made 'by ear'. Well, the lute, the so called 'pandoura', was extremely unusual in Greek music but common in Egypt and Babylon where it was used since the III millennium B.C.

The view of an eastern origin of the mathematical ratios of the basic consonances fits quite well with Burkert's (1972) thesis about Pythagoras as a mathematician. We could consider him not as a great mathematician, but as a lay Greek intellectual, a figure quite common at that time for the weakness of Greek priesthood, and a traveler, as it was usual in the Greek Mediterranean sea in an 'open' society, where the break-up of the ancient 'enlarged family' (*Grossfamilie*) and the establishment of the Achaean people in the polis (Benveniste 1973, I) were spreading 'fragments' of Greek society in the colonization process leading to a brand new social, political and cultural environment. Here, the social reproduction of culture was no more in the hands of a class of scribes

linked to the temple and to the palace, but was somehow rid of those links and deeply embedded in the polis.

Pythagoras was credibly in touch with aspects of the mathematical Egyptian culture that he simply imported in his new environment, turning them in a lay learning - "a scheme of liberal education" as Proclus (1970, 65) called it – and embedding them in one of the usual Greek religious frames, the sects of the mystical cults. Orphism is often considered a sort of paradigm for the Pythagoreanism. Reminding of the role of music in these cults, we can realize that *the connection between arithmetic and music had a deep social, cultural and political role to play*. Even the mythical fame Pythagoras won in antiquity hints more at such a religious, political, social and cultural role than at his own mathematical achievements.

Apparently, the lack of a finite numerical expression for the ratio of side and diagonal of the square was already known to the Babylonians. There is a clay tablet (Plimpton 322) containing Pythagorean triples, i.e. integer values for the sides of a right-angled triangle, listed for increasing difference between the acute angles. The first was relative to an almost-isosceles triangle, but the isosceles triangle was not included. Another tablet (YBC 7289) shows a square with its diagonals, and a number, which is a very good approximation of the square root of 2 (both tablets analyzed in Neugebauer 1957, ch.2, for a sharper analysis and a different interpretation of Plimpton 322 see Robson 2002). We can be quite sure that the Babylonians knew that in the isosceles right-angled triangle the ratio between the different sides did not have a precise value, but probably they considered this 'absence' not different from the analogous phenomenon for some inverses (for example, of 7). The difference is that in the latter case the approximating number is periodic, whereas it is not so in the former. In the latter case the situation changes when changing the numerical base, whereas this does not happen in the former.

Geometrical and musical background for the discovery.

It is probably wrong to look for the 'first' rigorous proof of the incommensurability, because such a proof would require the introduction of the "reductio ad absurdum" in an earlier visual and constructive mathematics, so that *the theorem had to be proved together with the establishment of its method of proof!* In particular the theorem could not be included in those pre-Euclidean books of "Elements" which did not include proofs by "reductio ad absurdum" (Proclus 1970, in *Euclidem Comm.* 73). Instead, if we look just for the starting of the enquiry about incommensurability the hypothesis of a 'musical' background for the discovery acquires great relevance. Actually, a 'negative' proposition asserting the incommensurability in musical and arithmetic terms can be found in prop. 3 of the *Sectio Canonis*, ascribed to Euclid or to some other mathematician of the end of the IV century B.C. (Barbera 1991). (See Appendix 2).

The same proposition can be found in Boethius' *De Institutione Musica* iii, 11 (Boethius 1867, Diels and Kranz 1964, 47 A19), who ascribed it to Archytas: "No mean proportional number can ever be found between two numbers in a superparticular ratio". 'Superparticular ratio' is the $\frac{n+1}{n}$ or $\frac{m+1}{m}$, that is the ratio $n+1:n$ or $(m+1):m$, for n, m integers (Appendix 1). It is evident that no integer can be found between $n+1$ and n . Could there be a mean proportional number between $2(n+1)$ and $2n$, or in general between $m(n+1)$ and mn ? The answer is negative, and Boethius' and *Sectio Canonis*' texts seem to consider it to have been proved.

These texts do not give perfectly rigorous proofs. As for Archytas' text, to get a rigorous proof some theorems included in the arithmetic books VII, VIII and IX of Euclid's *Elements* are required, such as VIII,3: "If as many numbers as we please in continued proportion be the least of those which have the same ratio with them, the extremes of them are prime to one another"; VIII, 8: "If between

two numbers there fall numbers in continued proportion with them, then, however many numbers fall between them in continued proportion, so many will also fall in continued proportion between the numbers which have the same ratio with the original numbers" and VIII, 20: "If one mean proportional number falls between two numbers, the numbers will be similar plane numbers" (remark that the reference to 'similar plane' numbers could be put in purely arithmetical terms) (Euclides 1956).

In the simplest case, for $n=1$ in the superparticular ratio, the above results prove the non-existence of an integer x such that $2m:x=x:m$, and then such that $2:(x/m)=(x/m):1$, i.e. the earliest form of an incommensurability example.

(A detailed, possible reconstruction of the proof can be found in Knorr (1975, VII.1). A similar, but simpler reconstruction written in Euclidean style on the track of Archytas' text, is given in Appendix 4. This reconstruction is added in order to show how easily one could get the proof with the techniques of book VIII of the *Elements*. Although these lacking theorems are not a problem for the *Sectio Canonis*' text, there is insufficient evidence to ascribe those results to Archytas.)

This 'negative' theorem in Archytas' form was then credibly translated in the geometric side-diagonal-of-the-square form and progressively substituted by a rigorous proof by "reductio ad absurdum", according to the Euclidean style such as the spurious X.117 of the *Elements* (Appendix 3) and the sketch of proof in Aristotle (Aristoteles 1831, *Anal. Priora* I.23, 41a 26-7).

I do not believe that a 'rigorous' music-theoretical proof ever actually appeared in Greek mathematics. Probably the lacking steps in Archytas' original proof were beyond the level of his logistic, and were credibly substituted by numerical examples, as is usual also in the *Sectio Canonis*: something like the numbers in the figure attached to the Appendix 4.

Apparently, such an approach was abandoned before it was rigorously developed. However, this did not occur because of technical reasons, but because of other cognitive difficulties, which I am going to outline in the following sections.

Szabo's conclusion is that

from a historical point of view the Greeks originally thought of the problem of irrationality as belonging to the theory of proportions. I hope that part II has shown how the theory of proportions, whose initial development took place in the Pythagorean theory of music, may in fact have led by way of arithmetic to the problem of geometrical similarity and thence to problem of linear commensurability...We have not yet shown how they reached the stage of being able to prove the existence of linear incommensurability in a rigorous manner" (Szabo 1978, 181)

The musical-logistic approach to incommensurability

In part III of Szabo (1978) the crucial role of Eleatism for embedding mathematics within a deductive framework is very well argued by Szabo, but it is not my goal to discuss it now. Instead, I would like to stress the 'musical-arithmetical' origin of the problem of incommensurability. More precisely, I believe that for Archytas *the core of mathematics was the 'logistic'*, which was not simply a practical art of computation, but the science of the relationships between numbers, the *theory of the logoi*, including substantially a pre-Eudoxian theory of proportions on numbers. The possible reconstructions of the proof quoted above show that it was solid enough to deal with the theoretical aspects of incommensurability. To this aim I would like to remark that music and logistic were considered Archytas' main fields of interest, and the fragment "logistic seems superior to the other sciences ... even to geometry" (Diels and Kranz 1964, 47 B4) in my view supports this

hypothesis. As Tannery stated, "It is characteristic of this ("Pythagorean") tradition that it apprehends the numbers themselves directly in the visible world, but their ratios in the audible world" (Tannery 1902, 70).

In order to underline the role of Archytas' logistic it is enough to recall Klein's analysis of the relationship between arithmetic and logistic, which overcomes the common idea of an opposition between a purely practical logistic and a purely theoretical arithmetic. Aristotle's remark (Aristoteles 1831, *De Anima*, II, 2, 413a) that squaring is better defined as the finding of a mean proportional than as the construction of a square equal to a given oblong figure, because the former states the *cause*, underlines the major role that this 'theory of logoi' in arithmetic form played until in the first half of the 4th century.

Following Knorr (1975), in order to underline the continuity between Archytas and the proposition X.117 of the *Elements*, I would call attention to a notational aspect which is common to Boethius' fragment ascribed to Archytas (Appendix 1), to the Euclidean prop.3 of the *Sectio Canonis* (Appendix 2) as well as to the post-Euclidean proof of the incommensurability of side and diagonal of the square (Elements X.117, Appendix 3, and so far the first rigorous proof I know).

The two incommensurable intervals/numbers are not homogeneously denoted, the larger being denoted by two letters, the smaller by one. The rationale in Boethius is clear: the greater number (or numbered part of the Canon, according to Szabo) is the sum of the smaller and one of its parts, represented by the pair of letters. In both the *Sectio Canonis* and the *Elements* XII.117 the pair of letters are instead the extremes of the greater interval/number and the single letter is the smaller interval/number. The greater interval/number is divided during the proof in two intervals by one point (and this use is usual in Euclid and can be found for example in the proposition IX.35). The difference between the second and third text is that the former deals with numbers represented as segments, the latter deals with intervals and thus the starting point is explicitly geometric. There is also another difference: the first and second text deal with the general superparticular ratio, the third only with side and diagonal of the square, that is with the specific superparticular ratio 1:2.

The permanence of the syntax with a change in meaning is intriguing, and seems to mark a historical evolution which changed only the 'embedding' of the proof from the musical-arithmetical to the geometrical form, but preserved its basic arithmetic structure. I shall return below to this changing the 'semantic' interpretation of a fixed 'syntax'.

I think that the discovery of incommensurability was *perceivable in Greek Mathematics* because it was deeply *involved in Greek culture*. First and foremost one has to underline that in the ancient Pythagorean philosophy, music was something central, much more than geometrical constructions, which could only appear in special problems as the duplication of the cube or in the construction of solids, e.g. Platonic solids for astronomical modeling.

We can read the echo of Damon's learning in Plato: "never musical modes can be changed without changing the most important laws of the polis" (Plato 1964, *Respublica*, 424c). Music was the ground of the Platonic theory of education, and appeared both in elementary (with gymnastic) and superior (as a part of the *Quadrivium*) education. According to Philolaus (Diels and Kranz 1964, 44 B 6) the structure of the kosmos' harmony reflected musical consonances.

Cutting the musical intervals by the geometric mean of superparticular ratios substantially meant finding the way to connect the seven modes of Greek music (Dorian, Phrygian, Lydian, etc), which were considered by Plato (and credibly by the Pythagoreans too) to be basic for the harmonic behavior of the citizen and of the polis as well.

In addition, an overwhelming Pythagorean interest about the general reduction of the continuum to discrete atoms is difficult to maintain. The "numerical atomism" was never a theory about some

kind of 'least magnitude'. From Philolaus up to Diophantus square, triangular and polygonal numbers were studied without any trace of 'incommensurability' troubles, and Aristotle reminds us that Euritos drew pictures of animals and plants as others did for triangles and squares (Aristoteles 1831, *Metaphysica* 1092b). All of them were full of incommensurable magnitudes, diagonal of squares and heights of equilateral triangles, but nobody worried about that. Summing up, I claim that *5th century geometry was 'insensitive' to the problem of incommensurability, and this was linked to the absence of a technical idea of continuum.*

The idea of a 'least interval' was instead common in the ancient music theories. In addition, music was the core of the Pythagorean philosophy, and also the basis of the parallel between Pythagorism and Orphism, witnessed by Herodotus, Plato, Plutarchus, Iamblichus, and so forth.

Plutarchus (1976, *de an. procr. in Tim.*, c17, 1020E) reminds us that the crucial problem was the division of the tone (9/8, i.e. the interval between the fourth and the fifth) in two 'equal', i.e. 'proportional', parts, and that the Pythagoreans discovered it to be impossible because 9/8 was superparticular. An equivalent question was whether the octave could be divided either in 6 tones (following Aristoxenus, who rejected the relevance of the mathematical impossibility of cutting the tone) or 5 tones and 2 not joinable semitones (following the Pythagoreans).

Besides, we could remember the words of Plato (1964, *Respublica* 531 a-c) against the musicians who try to find the "least interval" by ear "placing the ear before the mind", and, at the opposite of the Pythagoreans, Aristoxenus' defense of the musical practice to reveal the real consonances, and his search for something like our "equal temperament" to allow the connection in one framework of the different modes, because "there is no least interval" (Aristoxenes 1954, II,46).

In Aristoxenus music had almost completely lost its Pythagorean awe, and was placed instead in the realm of perception, ἀἴσθησις. Aristoxenus was proud of his Pythagorean education, but his idea of science and music was thoroughly Aristotelian. By that time music had lost any ethic flavor, and was just a psychological tool for politics, for the pleasure it gave (Aristoteles 1831, *Politica*, VIII). And Aristoxenus was Aristotelean also in the rejection of the ancient arithmetic perception of the linear magnitude, something I am now going to discuss.

Negative arguments and *Epinomis*' passage

The main reason for rejecting the 'musical' hypothesis about the discovery of incommensurability is probably given by the fact that in Plato and Aristotle there seems to be no trace thereof, whereas it is always connected to geometrical or arithmetical questions. In addition, in Plato even the theory of proportions seems to belong to geometry.

The 'demusicalization' of the theory of proportions by Plato is shocking. In Plato (1964, *Timeus* 35-36) the group of music-theoretical proportions, 3/2, 4/3, 9/8, 27/24, 256/243 and the role of harmony which can be found in Philolaus' music theory (Diels and Kranz 1964, 44 A26) is translated in a cosmologic theory without any trace of musical reference. If we consider the likely Pythagorean and Philolaic origin of the *Timeus* this 'removal' seems really astonishing!

However, I think I can prove that in the Platonic Academy there was a trace of this earlier approach, with a tight connection between music, numerical means and similarity, and without any reference to geometric figures, such as square or pentagon. To this aim I offer in Appendix 5 a new translation of a passage of Plato (1964, *Epinomis*, 990d-991b), which is usually read in 'not technical' translations, so that ὁμο...οῦ is 'likening' or even sometimes 'commensurable', ὁμο...α is 'analogy', and σῶμμετρον is 'proportioned'. In my translation I employ instead the technical terms, respectively: 'similar', 'proportion', 'commensurable'. *Epinomis* is usually ascribed to Philippus of Opus, a member of the Academy, approximately contemporary of Aristotle and Eudoxus.

Knorr (1975, 93 and note 109) gave another translation of the passage on 'similarity' whose syntactic construction I share (see Appendix 5). However he did not employ the Euclidean terminology, justifying his choice by an analogous usage in Thales and Nichomachus. In addition, he underevaluated the value of this passage because "most of the mathematics Plato brings up in his dialogues does not correspond with the style or the subject matter characteristic of Archytas' studies. Moreover, Plato's contacts with Archytas were not such to have enabled a significant influence on Plato's earlier conceptions" (Knorr 1978, 89).

This is true. By and large, Plato was well acquainted with some technical results of geometric incommensurability, but his general approach was more Philolaus-like than Archytas-like. In particular, Plato did not seem to appreciate the tendency to employ mathematics as representation and model of physical reality, as done by both Archytas (for example when he interprets musical tone as speed of vibration) and Eudoxus (for example when he employs cycles and epicycles in astronomy). Plato seemed to reject this approach in *Respublica* (Plato 1964, *Resp.* 531c), where he criticizes those who in astronomy and music analyze the numerical ratios without looking to the 'problems' and 'causes'.

However, in my opinion Philippus' acquaintance with Archytas could have been greater than Plato's, and Philippus was very far, both chronologically and culturally, from Thales and Nichomachus. The translation of the *Epinomis*' passage gets a very clear meaning if I assign to Σύμμετροι and Δμο...οι the Euclidean meaning (except in one case for Δμο...οι where for the context I incline toward a more general meaning).

The passage is based on the reduction of the problem of commensurability to the problem of similarity. At the beginning it includes the statement that linear different numbers cannot be similar and that similarity can be instead recognized among planar numbers. The passage continues by stating that in stereometry we can make similar two numbers by two mean proportionals (according to Archytas' algorithm (Diels and Kranz 1964, 47A 14)). Then, after a (likely) reference to the general role of dichotomy in the framework of dialectics, the passage continues with a reference to the Pythagorean means on the canon, and comes to an end with the recognition that the musical harmony must be restricted only to commensurable intervals.

In front of the relatively rich fragments about musical questions related to the incommensurability credited to the Pythagoreans, we cannot find in the fragments ascribed to them any trace of incommensurability deriving from geometrical constructions. On the contrary, in Plato and Aristotle incommensurability is exclusively referred to geometrical constructions. Why these silences? And why this sudden and radical change?

Why the Pythagoreans' silence? The "secret of the sect"? So well kept by Archytas on the geometrical side but betrayed on the musical side, rigorous in Taras while at the same time it was a favorite theme in Plato's Academy and in Theodorus' lessons in Athens? Boethius' text that I am going to analyze shows that the early musical theory of incommensurability was somehow known, even though it had been overcome by a sudden rupture. I can explain this 'rupture' as revealing a sharp passage from the musical to the geometrical framework in Archytas and Eudoxus, ultimately established in the environment of the Academy with the vanishing of the earlier approach. Such translation was easy because geometric similarity was well known, and the connection between the duplication of the square and the mean proportional between 1 and 2 was known as well, since Hippocrates of Chios could reduce the duplication of the cube to the search of two mean proportionals and Archytas could accomplish it geometrically.

Why this sharp change? I think the first reason was that the musical proof was *only negative*, whereas the geometrical approach allowed the *effective construction* of incommensurable

magnitudes.

This argument is employed by Knorr (1975, 216) to claim "that the harmonic theory of irrationality was a derivative from the geometric theory rather than the converse". Frankly, in my view this seems a non-sequitur. Knorr's argument would rather explain the quick prevailing of the geometric approach on the musical-arithmetical one, all the more because a purely negative result (speaking about "something which is not") had to fall under the blows of the *negative judgment paradox*. Such paradox forbade speaking *about what is not*, and played a crucial role from Parmenides to the Sophists and Plato, widespread between the end of the V and the beginning of the IV century in Athens, as witnessed in Plato's *Sophista* (Plato 1964).

In addition, Knorr (1975, 223) correctly argued that there could be some logical flaw in Archytas' proof. More likely, some steps could have been proved by simple 'visual evidence' or by numerical examples, for want of a more rigorous style. Instead, I would stress that such a flaw was not a limit inherent in the musical-arithmetical approach, because a rigorous proof could be built with the techniques of the VIII book of the *Elements*, and without reference to strictly geometric similarity, as shown in Knorr (1975) and in the Appendix 4 below. And, more important, these flaws could be relevant for a referee who had to accept Archytas' paper for publication, but not for understanding the role of the connection between Greek music and the beginning of Greek mathematics.

If my translation is not wrong, *Epinomis'* passage is the Platonic fragment most influenced by Archytas and his theory of proportions. Consequently, one could conclude that in Plato's and Archytas' times the strict geometric version of commensurability (as given in Plato's *Meno* (Plato 1964)) was playing an increasing role, probably justified by the geometric model of the concepts of similar (plane and solid) numbers, whereas the earlier musical approach was still present in the arithmetic theory of proportions, but was quickly vanishing together with the influence of Pythagorean philosophy. Thus, during Plato's life the translation of the musical version of incommensurability into the geometrical one was a mathematical 'work in progress' which took place in the mathematical milieu of the Academy. For those reasons the names of Archytas, Eudoxus, Theaetetus seem inescapable as the authors of the geometric translation of the incommensurability theory.

Boethius' "history" of incommensurability.

Boethius' *De Institutione Musica* is a technical text with very precise and detailed references to the Pythagoreans (Hippasus, Philolaus and Archytas) largely coherent with each other, and with other fragments of these authors. In book III it also includes long and detailed computations which show the development of the ancient Pythagorean music theory since the first 'naive' numerological efforts, continuing with the empirically negative 'tone cutting' results on the different ratios involved in the theory, before giving Archytas' main proof.

Nowadays Moritz Cantor's negative opinion about Boethius seems excessive. According to Burkert (1972, 397) Boethius seems trustworthy and a crucial personality in the history of Middle Ages who coined the word 'quadrivium', or 'quadrivium'. We can state that Boethius was a good editor of Greek mathematics, well acquainted with the original sources. His mathematical codices, coherent and uniformly present throughout Europe, date from the 9th century, that is very close to Boethius' time, a contiguity in time that is absolutely exceptional for ancient authors, whereas the earliest *Sectio Canonis'* manuscripts can be dated to the 12th-13th century, fifteen centuries after having been written. Thus, Boethius could perhaps be considered as the most reliable source about Greek mathematical theory of music (Barbera 1991, 104).

In Boethius' work we can read the evolution of the musical road to the discovery of incommensurability. In *De Inst. Musica* (Boethius 1867, II,18-19) Boethius ascribes to Hippasus, Eubulides and Nicomachus the analysis of the connection between superparticular ratios, multiples and unitary fractions. Book II continues by displaying the troubles Pythagoreans faced in embedding the 'logarithmic' nature of the musical consonances in a 'linear' canon arithmetic, with the final conclusion that "diapason consonance is smaller than six tones" (II, 30).

In Book III Aristoxenus' theory is criticized from a Pythagorean point of view. The crucial question is 'cutting the tone', i.e. finding a middle proportional between 8 and 9. It is easy to see that 17 is not a middle proportional between 16 and 18. In III,5 and III,8 Boethius describes Philolaus' attempt to cut the tone. Philolaus set out to solve the problem starting from purely 'numeric' considerations by displaying an idea of 'ratio' as a generic relation between two numbers and by analyzing the music intervals as ratios between two numbers or as differences between the same numbers or as single numbers. There we can find the statement of the problem in terms of dichotomy of the tone, of the comma and of the diesis (Diels and Kranz 1964, 44 A26, B6).

Some authors (Burkert 1972, Huffman 1993) consider the naivety of this approach as a sign of Boethius' unreliability. Instead, I would like to point out how well it fits with Szabo's analysis of the ancient meaning and employment of words like *diásthma*, *lógoi* and *ánalog...*a (1978, 114-170), and how closely Philolaus' words agree with Plato's numerology in *Timeus* 35-36 (Plato 1964). Other fragments from Porphirius (Diels and Kranz 1964, 47 A17) confirm the existence of an earlier 'naive' purely numerological approach to the idea of ratio, echoed even in Plato's *Timeus*, 47 (Plato 1964), where we can recognize the deep connections Pythagoreans and Plato instituted between numbers, proportions, music, astronomy and human knowledge under the basic headline of "harmony". This 'naivety' had to be quite common in earlier times if Ptolemy reminds us the 'ridiculous' idea that consonance could be measured by the sum of the two terms of the ratio (Ptolemaeus 2000, I, 14.2).

There seems to be evidence that in Philolaus the theory of proportions was in a first phase of development, when *lógoi* meant just a vaguely characterized relation between two numbers somehow connected with its music-theoretic actualization as *diásthma* characterized as both a pair of tones and a pair of extremes on the canon (or just one extreme if we consider the other one the rightmost or the leftmost by default). Finally in Book III,11 we read Archytas' proof about the division of superparticular ratios, where we find the discovery that these dichotomies in music consonances are impossible (Diels and Kranz 1964, 47 A19).

According to Burkert (1972, 399) "the material introduced by Boethius must have belonged to Pythagorean musicology before Archytas". The fact that Boethius neither mentions incommensurability nor connects these results to arguments other than the defense of Pythagoreans' music theory makes fake reconstructions or forgeries unlikely. The whole book is only about the Pythagorean theory, mostly against Aristoxenus, and is oriented toward the main results involving the 'cutting the superparticularis ratio'. I think it reports quite exactly the ancient Pythagorean mathematical theory of music since its beginning, and its conclusion is the negative face of the incommensurability, with a proof of superparticular ratios' indivisibility whose original rigor, however, can not be ascertained. The techniques of book VIII of Euclid's *Elements* are sufficient to give a perfectly rigorous proof, but we do not know how close to them was Archytas' logistic.

Many authors (Szabo, Knorr, Burkert) have discussed Boethius' text, and all of them have recognized a musical phase in the discovery of incommensurability. Even the details of their interpretations are quite similar. According to Burkert (1972, 463), "Music theory advances as far as the problems of the irrational, but stops there and declares them nonexistent. The irrational belongs to the domain, not of arithmetic, but of geometry". The real issue is the 'historical relevance' to be

assigned to such a phase. Knorr is the most skeptic, and I am going to discuss his objections below.

Actually, we can read in Boethius the 'musical' approach with all its stages, steps and reasons, but the final 'geometric' step appears nowhere and, moreover, the mathematical music theory is "one of the few fixed points in the reconstruction of Pythagoreanism before Plato" (Burkert 1972, 371). On the other hand, we know nothing about the steps and the reasons of the geometric tradition. We have just later simple statements, for example in Aristotle, of the incommensurability of side and diagonal of the square with a short reference to an arithmetical proof. In addition, there is strong evidence that also the *Sectio Canonis* underwent during the Middle Ages modification from a substantially arithmetic to a more geometric style with geometric diagrams (Barbera 1991, 102).

The only argument I know to support the antiquity (at least, to the second half of the 5th century) of the 'geometrical' approach is provided by Theaetetus' words in *Theaetetus* 147c- 148b (Plato 1964): "Theodorus was proving for us via diagrams something about powers, in particular about the 3-foot and the 5-foot, demonstrating that these are not commensurable in length with the 1-foot, and selecting each power individually in this way up to the 17-foot, but in this one for some reason he encountered difficulty". No doubt this proposition seems to be embedded into a geometrical approach.

There has been a great debate about the 'geometric' reason of the 'difficulty' encountered by Theodorus with the 17-foot (Knorr 1975). The narrated episode had to happen in 399 B.C. but the dialogue was written probably around 367 B.C. Hence, when he wrote this dialogue, Plato had credibly a good acquaintance with the well established 'geometric' theory of incommensurability, most of all after Theaetetus, and probably he remembered something about its 'state of art' in his youth, taught in the lessons of Theodorus, and 'some difficulty' with 17. But I can guess that he could have been wrong in the details. The trouble with 17 could be connected not to some geometrical construction, but to the role this number played in the Pythagorean arithmetic, where it was called the 'obstacle', because it "broke the proportion of 9/8 in not equal intervals" (Plutarchus 1962, *De Iside et Osiride*, 367 f). This hypothesis was considered even by an anonymous commentator (Diels, Schubart, Heiberg 1905), but ignored by Knorr (1975, note 79) for the 'inappropriateness for the context': the geometrical context was unsuitable for a musical interpretation, but this was clear also for the commentator. In order to put forward that musical interpretation credibly the commentator had good reasons to think that a contamination between music and geometry was possible.

There is also a second oddity in the episode above. Why did Theaetetus mention 3-foot and 5-foot among the 'powers not commensurable in lengths with the 1-foot', but ignored 2-foot, the most important one? (Burnyeat 1976). If my hypothesis is not wrong the answer is easy: 3 and 5 are the natural candidates as arithmetic means for the dichotomy of the octave (1:2) and of the fifth (2:3). In other words 3, 5 and 17 are the most natural values for cutting the most important musical ratios, whereas 2 plays no role in this musical problem.

Last but not least, my interpretation could even make clear a third problem in Plato's text: the ambiguous employment of the word *dýnamis* and its derived terms. According to the geometrical interpretation, within few lines they can mean 'square', 'side of the square' and 'possibility' (Knorr 1975, 62-68, Szabo 1978, 36-45), i.e. heterogeneous magnitudes and concepts. I have to remark that in a musical interpretation all the magnitudes are homogeneous, and *dýnamis* could have consistently meant 'possible musical value', i.e. tone that can or cannot cut an interval.

Thus, we can imagine Pythagorean mathematics at the end of the 5th century tackling the difficulties of these cutting problems, and experiencing the failure of the 'empirical' and 'naïve' solutions. Archytas was looking for a general proof of the non-existence of such solutions, and an echo of this tradition appeared in Theodorus' lessons. In the subsequent decades the musical

approach yielded to the geometrical one, that allowed the construction of the mean proportional (*Elements*, II. 14). Theaetetus was the most brilliant author of this shifting by producing his most important achievements (book X of the *Elements*). At that time arithmetic was quickly losing the leading-edge of Greek mathematics, and Plato recognized in geometry the paradigm of his philosophy and described incommensurability in the geometric language we can read in his *Meno*.

Many years later, when he was sixty years old, Plato wrote his dialogue named after Theaetetus, and recalled Socrates' first encounter with this recently died mathematician, which occurred more than thirty years before. There is nothing strange in guessing that Plato's memory could have mixed some numbers and terms heard in that encounter with Theaetetus' later great geometric achievements. The above episode seems scarcely credible even for other reasons. For example it entails that even the idea of 'figured numbers' had to be ascribed to Theodorus' students! "This story is a fiction" wrote Burnyeat (1976). I would rather say it is an overlapping of memories.

The next question I would like to discuss is the following: why is the traditional but vague and not supported hypothesis of a prevailing geometrical origin of incommensurability so largely and (usually) unquestionably accepted, as if the music-theoretical approach - even though so well documented and leading to the discovery - were just a historical accident, without any role to play in a otherwise completely 'geometric' problem?

The Continuum

According to the above reconstruction, I have to advocate two historical theses of great foundational relevance.

The first one is that incommensurability was not the casual result of simply technical, philosophical or mathematical problems, "an accidental by-product of the Pythagoreans' study of square figures" (Knorr 1975, 298), but it rose instead from the basic cultural and political questions of the (not only Pythagorean) ancient Greek culture, where music and the relationship between musical modes and state/society played a role that is difficult to understand nowadays, and were essential ingredients of both society and culture. In my opinion, *if incommensurability had been just a by-product of some technical enquiry, it could not even be 'perceived' as a relevant problem.*

The second thesis is that at the time of Archytas and Plato a sharp rupture occurred that fostered a shift from musical to geometrical incommensurability, so that we can find nothing about the geometric approach in the extant Pythagorean fragments as well as nothing about the musical approach in Plato or Aristotle.

This second thesis deserves a deeper analysis. Why did Greek mathematics and philosophy 'replace' the musical approach with the geometric one? To this question I am going to give two answers. The first answer, as I have already remarked, is that the break was fostered by the "negative judgment paradox": *Given that an affirmative statement corresponds to a fact in the world, something that is, a negative statement corresponds to something that is not. But a statement about what is not is about nothing, and hence is impossible.*

Another possible answer involves the Greek *perception* of both the "linear numerical magnitude" and the idea of continuum.

To formulate this second answer, I have to address a foundational issue which involves the problem of *the nature of the opposition between arithmetic and geometry, and between discrete and continuous* in ancient Greek mathematics.

The opposition discrete/continuous is the background of a 'topology' of mathematics, which has been considered 'natural' or 'obvious' ever since its statement in the *Quadrivium*, which is the core of book VII of the *Respublica* (Plato 1964). It also occurs in Proclus (1970, 35-36): arithmetic and

music concern the 'multitude' (pŏson), geometry and astronomy concern the 'magnitude' (phl...kon). In the first pair arithmetic deals with multitude in itself, music with relations between multitudes. In the second pair geometry deals with magnitudes at rest, astronomy with moving magnitudes. Analogous relations are described by Archytas (Diels and Kranz 1964, B1) where we can also recognize in the pairs geometry/astronomy and arithmetic/music the template 'theoretical/applied' with the first pale appearance of the idea of 'model'.

No doubt incommensurability was perfectly fitted in the discrete/continuous opposition in the *Quadrivium*. The problem is: was such opposition 'caused' by the discovery of incommensurability or was it already a basic feature of Greek mathematics?

To recognize the sharp distinction between "multitude" and "magnitude", and its deep roots in Greek culture and language, it is enough to consider the linguistic independence of the relative terms and take Aristotle seriously. He states explicitly that the number is not listed among the geometric magnitudes (Aristoteles 1831, *Metaph.* 992a 16) and he does not find a term characterizing the common genus of numbers and magnitudes (Aristoteles 1831, *An. Post.* 74 a17).

The hypothesis that such common genus could be the 'quantity' (pŏson) is not absurd, at least in Aristotle, but its mathematical role is undermined by the inclusion of the "speech" among the discrete quantities, for which there is no proportion theory at all. In addition, it is worth remarking that such an hypothesis is not employed in other parts of Aristotle's writings and, moreover, that the term pŏson was traditionally ascribed only to the discrete.

This opposition can be also recognized in Euclid's *Elements*: the 'Eudoxian' book V is frequently referred in the geometrical book VI, but never in the arithmetical books VII, VIII, IX.

The same sharp opposition can be found in neo-Pythagorean authors, too. Nicomachus wrote of the "the natural contrariety (ϵntipepŏnqhsin) of these two genera" (Nicomachus 1866, I, VII), and Iamblichus of their "opposite (ϵntipŏscon) nature" (Iamblichus 1975, 23). This opposition was never connected by them to incommensurability.

As for the opposition between discrete and continuous, against the thesis of a mathematical *Grundlagenkrisis* due to the discovery of incommensurability I must underline that this discovery *was experienced more as a great confirmation than as a rupture*.

From a comparative historical point of view, it is worth remembering Iamblichus' claim that the musical proportions were introduced by Pythagoras but originated in Babylon. According to Needham (1954, IV, 177), even the Chinese musical theory concerning pitches had the same origin, even if in China they were organized not as a "scale", but as a "court" and a "spiral of fifths".

In Greece and China the "sister sciences", music and astronomy, tackled a similar problem: to fit a set of regularities, basic for the whole social and intellectual life, from calendars to measuring systems and musical practice, within numerical schemes. *Incommensurability was the key for the Greek and European answer, which grounded European mathematics on the Quadrivium and on the opposition between discrete and continuous*. These two 'sister sciences' are always deeply connected in the Pythagorean-Platonic tradition. Even the choice between heptachord and octochord was linked to a 7- or 8-spheres (adding the stars sphere) Kosmos, while the Earth was 'almost silent' (Boethius 1867, *De Inst. Musica* I, XXVII); in both sciences 'motion' caused the potentially infinite differences and 'harmony' caused their finite establishment.

The question is: why did Greek mathematics and philosophy 'remove' the musical approach for the geometric one? Before continuing my analysis, I have to set now another, related question: why did the history of mathematics 'remove' the 'musical way'? In my opinion, the answer is given by an ancient prejudice concerning both *the presumably 'empirical', 'natural', 'phenomenological' character of the idea of "continuum" and the opposition discrete/continuous*. This is a common

prejudice shared not only by historians and philologists, but even by almost all of the mathematicians. The 'natural' embedding of incommensurability in geometry is undeniably linked to the presumably 'natural' idea of continuum!

The connection between incommensurability, infinite and geometrical continuum is explicitly stated by Proclus: "If there were no infinity, all magnitudes would be commensurable and there would be nothing inexpressible or irrational, features that are thought to distinguish geometry from arithmetic" (Proclus 1970, 6), and in a scholium to Euclid: "whereas for the numbers the common measure is the monad, the Pythagoreans were not able to find a common measure for the magnitudes... the magnitude is infinitely divisible" (Euclides 1969, X, sch. 1, V). This connection is clear in Aristotle, in Euclid, and in Philippus of Opus as well, if my translation of *Epinomis*' passage is correct. This was the positive face of incommensurability. If the idea of continuum was something 'natural', it had to be natural to consider the geometrical continuum the right embedding for the discovery of incommensurability, in the analogously natural organization of mathematics in the discrete/continuous opposition of the *Quadrivium*.

In Aristotle we find the 'modern' characterization of 'continuity': 'divisibility', i.e. the potentially infinite divisibility of a finite magnitude, and the 'holding together of its parts', i.e. the existence of a unique common limit between any pair of its contiguous parts. In these two concepts we can recognize the ancestors of the modern concepts of, respectively "denseness" and "Dedekind's continuity". In addition, Aristotle points out the opposite behavior of the quantities with respect to division and addition: discrete quantities are potentially infinite by addition and limited by division to the monad, continuous quantities are limited by addition to the finite universe and potentially infinite by division, the point not being a magnitude.

In my opinion, this characterization was *not the last refinement of a 'natural' idea, but a complex and late way out from the network of paradoxes the idea of quantity bore till Plato*.

I have called a 'prejudice' our perception of 'continuity' as 'natural' because from a 'cognitive' point of view it is hard to believe it.

- According to cognitive psychology, the developed idea of continuity occurs very late in children's cognitive development and its appearance seems to be connected to a general process of cognitive development more than to empirical experiences. Its first intuitive appearance is perhaps only a 'negative' of the idea of "object": there is continuity where different objects cannot be recognized. The idea of infinite divisibility does not seem to appear before the formal phase. In addition, intuitive discrete and continuous quantity have to be distinguished to yield the idea of 'number', whereas in earlier phases of the cognitive development a row of objects changes its 'numerousness' if the distances between the objects are increased (Piaget and Szeminska 1941).
- Chinese mathematics did not experience the opposition discrete/continuous. As for the problem of incommensurability, Chinese mathematicians were "neither attracted nor perplexed by irrationals, if indeed they appreciated their separate existence" (Needham 1954, III, 90). In particular in the late school of Mohist logic, which is the nearest to our standards among the Chinese schools, the same pair of terms rendered pairs sharply different for us: unit/total, member/class, part/whole, undermining any discrete/continuous distinction. In addition the Mohist idea of point did not face paradoxes analogous to Zeno's and Sophists' antinomies (Graham 1978). For two thousand years the opposition between discrete and continuous created troubles to the development of western science, until the construction of the real numbers; when such a construction was accomplished, intuitively in the 17-th and formally in the 19-th century, it triggered an epochal change that the Chinese science never experienced.

-After Dedekind, Cantor, Hilbert, Zermelo, Skolem, Gödel, Cohen we know that Euclidean geometry admits denumerable models, that we cannot give the modern continuum a first order categorical axiomatization, that the geometrical continuum cannot be proved coincident with the numerical one, that it cannot be empirically verified, and that the place of the numerical continuum in the transfinite hierarchy is one of the most debated questions, which is linked to the most disputed axiom of set theory, the axiom of choice, whose general version implies paradoxes connected to the idea of 'measure', due to Vitali, Hausdorff, Banach-Tarski (Wagon 1985, Moore 1982).

If the continuum has never got empirical evidence or logical naturality, if it stems from complex processes of cognitive development and if it does not appear outside western civilization, we can suppose that also its evolution in Greek mathematics had to be quite complex.

To outline this evolution, I must first and foremost remind that the Pythagorean monad could not be divided. However, Aristotelian empiricism could not remain insensitive to the idea of 'one' as a measure unit, and Aristotle's philosophy had to remove the antinomy inherent the idea of 'One', both *indivisible* - the 'Being' in the Eleatic framework, and the monad, both unit and point, in Pythagorean Mathematics - and *divisible*, the measure unit of the 'magnitude'. The solution required that the idea of continuity based on the divisibility-of-the-magnitude had to be connected to the idea of continuity based on the singleness-of-the-separating-extremes through the idea of sign/point. A sign/point can always and everywhere, potentially but not actually, distinguish/divide the continuum, whereas a unity actually is an already distinguished and well defined object to be considered as a whole.

There is nothing 'natural' or 'obvious' in this connection, which required the whole apparatus of Aristotelian philosophy. More precisely, it required the only potential existence of the points, which meant the rejection of the simultaneous complete divisibility, and the impossibility of finding a point contiguous to a given point. In addition, we can even recognize in these two issues the modern themes of the rejection of the general axiom of choice and of the well-ordering of the straight line as well. *The essential ingredients of our idea of Continuum appeared just as brand new establishments in Aristotle's philosophy, and they have never changed since then.*

The double characterization of the 'one' as both logic predicate and measure unity (Aristoteles 1831, *Metaph.* XIII.8) entailed two different approaches in Greek mathematics. The first one considered that "An unit is that by virtue of which each of the things that exist is called one" (Euclides 1956, VII, def.1) and cannot be divided, so that One is not a number and, rigorously speaking, can not be a 'part' of a number because part/parts are relationships between two (integer) numbers (Euclides 1956, VII, def. 3,4). The second one required an idea of number as measure, and hence as a sort of magnitude. This ambiguity in the idea of 'one' is immediately linked to the problematical idea of "linear numerical magnitude" (that is numbers represented on a geometric line).

Our idea of (*continuous*) *numerical magnitude* is based on the idea of *real number*, a sequence of digits split by a decimal point:

$$\dots a_3 a_2 a_1 . a_{-1} a_{-2} a_{-3} \dots$$

It includes two infinite sharply different sequences: the left one is potentially infinite and in the realm of discreteness, the right one is actually infinite and in the realm of continuity.

Thus, *continuous* and *discrete* appear somehow intertwined: discrete magnitudes are special continuous magnitudes and a continuous magnitude can be represented by an infinite set of discrete magnitudes.

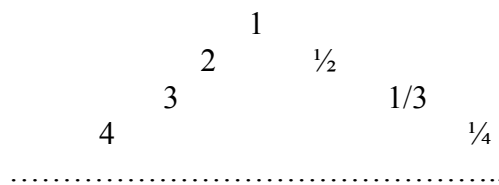
Since Descartes the *continuous numerical magnitude* is the basic *symbolic form* of modern science. Its career has been so astonishing to get the status of a Kantian *apriori* in the *axioms of intuition* by a double reduction: the reduction of every perceptual quantity in space and time to the *geometric magnitude*, as in the first Kantian axiom, “all intuitions are extensive quantities”, and the reduction of this magnitude to the *numerical magnitude*, as in the second Kantian axiom, “in all phenomena the perceived real has got an intensive quantity, a degree” (Kant 1781, I, II part, 2nd book, sect. 3).

Probably the idea of "linear numerical magnitude" underwent a slow evolution in Greek mathematics. As far as we know, in earlier Pythagoreanism the "linear numerical magnitude" was simply a line of indivisible monads and the difference between monad and point was only in "having position". This peculiar ‘perception’ of the points/numbers was clear in the abacus, and was linked to the already mentioned role of the One, which played in Greek philosophy a double role as both a quantity and a logical predicate. In the logical role any number was ‘one’ number and ‘one’ was indivisible and could not be a number. As unit of measurement ‘one’ was divisible but it had to change the ‘genus’, because it and its parts were heterogeneous, and in the original genus it was indivisible (Aristoteles 1831, *Metaph.* XIV.1). Even in modern non-decimal systems we have ‘foot’, ‘inch’, etc. i.e. heterogeneous units connected by a multiplicative factor.

Euclid's definition of Monad (VII.1) was biased toward a purely logical and arithmetical view. Iamblichus (1975, 11) criticized it explicitly, adding the words "even though it is a collection" to the Euclidean definition VII,1: "An unit is that by virtue of which each of the things that exist is called one".

Nevertheless, Euclid's arithmetic (Book VII of the *Elements*) appears as a step toward the modern idea of "linear numerical magnitude" in that it deals with the connection between the arithmetical ideas of *parts/multiples* and the more geometrical (and undefined) idea of *measure*. Many theorems of book VII are devoted to establish such a (for us) trivial connection.

A different representation, I should say a different ‘perception’, of the numerical magnitude is revealed by the tradition of neo-Pythagorean Greek arithmetic. It can be displayed by Iamblichus’ *lambdoid figure*, a lambda with 1 as root and two infinite branches, the left one given by the sequence of the integer numbers, the right one given by the corresponding unitary fractions (Iamblichus 1975, 14),



The two branches are in a radical opposition, and there is evidence that even their respective numerical bases were different: 10 for the integers, 12 for the parts. For example, Heath (1921, 47) reports the existence of a Greek abacus whose integer part had four tokens in each zone so that the numerical base had to be 5, and whose fractional part had instead five tokens in each zone so that the numerical base was 6. Moritz Cantor (1907, I, 530) describes an analogous Roman abacus.

Noteworthy, in the lambdoid figure any “part” is always a part of the One, which is not a number but the ‘seed’ of both numbers and parts.

Iamblichus' lambdoid seems to suggest another step forward in the direction of the modern view of "linear numerical magnitude". An analogous 'perception' appears in Nichomachus’ (1866)

Introductionis Arithmeticae I, VII, when he defines 'even' numbers by requiring that their splitting could be "greatest in size and smallest in quantity". From the context we realize that the 'linear number' was therefore seen as a sort of 'magnitude', whose division in parts produced a 'multitude' of '(integer) magnitudes', so that the even number could be divided in the smallest quantity ('two') of greatest (integer) equal magnitudes (the 'half').

There was a slow evolution in the Greek idea of 'linear numerical magnitude', even though it is not easy to outline its details. Euclid's basic view can be contrasted with Archytas', according to 1) the differences I have pointed out among the proofs of incommensurability in Boethius, *Seccio Canonis* and the proposition X.117 of the *Elements*, and 2) the Euclidean connection between *parts* and *measures*.

In my opinion, it is impossible to understand the history of ancient mathematics without considering how far is our *intertwining* of discrete and continuous, arithmetic and geometry, number and point, pictured by the idea of "real number", from the radical *distinction*, even *opposition*, of these concepts which can be recognized without exceptions in classic Greek mathematics. It is impossible to understand the oddities of Greek mathematics without taking into account the deep contradiction contained in the idea of One.

As for the history of the evolution of the idea of *continuum*, it would be enough to emphasize that:

- According to Homer and Hesiod, *sunec»j*, "continuous", means "without interruption", both temporal ('continuous raining') and material ('fastened buckles')
- In Pythagoreanism the distinction between 'monad' and 'point' lay just in "having position". Probably, this reflects only the employment of points as pebbles on abaci or in polygonal shapes. This ancient distinction among quantities had to be quite deep in Greek arithmetic. It still appears in Aristotle's *Categoriae*, 6 (Aristoteles 1831) together with his 'modern' discrete/continuous dichotomy, as well as at the beginning of Pseudo-Iamblichus' *Theologoumena Arithmeticae* ([Iamblichus] 1975). We can suppose that the *Quadrivium* in its earlier Pythagorean version did not know any discrete/continuous opposition.
- According to Parmenides, continuity is 'homogeneity' of the Being, and is completely inside the Being/not-Being and One/Many quandaries. Zeno showed that the idea of infinite divisibility and the concepts of point and instant were at the center of a network of paradoxes related to Being and not-Being. Among the Atomists, continuity disappears or is reduced to 'contact'. Democritus' antinomy on the sections of the cone (if they are different the cone is 'stepwise', if they are equal the cone is a cylinder) reminds of Zeno's paradoxes.
- In Anaxagoras the 'absence' of a minimum is necessary to allow the change by mutual mixing of everything in everything (Diels and Kranz 1964, 59 B11) in homogeneous bodies. It is only a sort of 'physical' continuity. Still, for Aristotle physical infinite divisibility is easier to be accepted than mathematical infinite divisibility (Aristoteles 1831, *De Caelo* 306 a28-30): just the opposite of our view! Probably, the reason is in that "matter" was connected to "measurement" and the physical divisibility was naturally potential, whereas the "unit" was a 'logical' idea and the mathematical divisibility could have been conceptually simultaneous and actual (and hence to be rejected).
- Even in Plato the continuum preserves the ancient common meaning. It does not show any technical employment, and in addition it is applied to discrete quantities as well (the 'continuous' sequence of integer numbers or of names in a sentence). In Plato we can also catch the extent of the paradoxes concerning the 'One' in the dialogue *Parmenides*.
- As the Mohists did, even Plato always considered a 'point' existing as the 'beginning of the line', a simple 'sign', *sÁma*. The first traces of the process leading to the Aristotelian idea of

continuum, can be found in his idea of 'instant' as expressed in *Parmenides* 156 d-e (Plato 1964): the instant is both motion and rest, being and not being, out of time and extraordinary in nature. Eventually, in Aristotle's *Physica* (Aristoteles 1831, 218-219) points and instants are not parts of time and space, and cannot be close to each other. Instants are the borders of time but not its parts, and can be enumerated but cannot enumerate. In addition, the words for 'instant' in Plato and Aristotle are substantivized adverbs: TMxa...fnhj, 'suddenly', nân, 'now', and this suggests a relative novelty.

-Even Plato and the Platonists sometimes did not distinguish between point and monad. For example, they put the monad, instead of the point, at the beginning of the hierarchy line-surface-solid (Aristoteles 1886, fragm.28 and Aristoteles 1831, *Metaph.* 1085 a8).

In mathematics continuity developed in Eudoxus' theory of proportions and in the method of exhaustion named after him, where we can recognize what we call today Eudoxus' lemma, the Archimedean magnitudes and the property of denseness. This 'weak' form of continuity can be tracked down from Anaxagoras' fragment 1 to Aristotle (Aristoteles 1831, *Phys.* 266b 2) without any trace of other aspects of the Aristotelian idea of continuity. More precisely, the uniqueness of the separating point, the impossibility of the contiguity between two points and the potential/actual distinction, which are the final and essential ingredient in Aristotle, play no role in mathematics.

The Aristotelian idea of "continuity" was not the technical refinement of the naïve, pre-technical concept that we can find in Plato. The fact that for Plato even a sequence of consecutive integers is continuous, whereas it is discrete in the technical terminology of Aristotle, displays this cleavage. As Hermann Weyl (1917, 70-71) stated, "das anschauliche und das mathematische Kontinuum decken sich nicht; zwischen ihnen eine tiefe Kluft befestigt".

The point is that *the Aristotelian definition does not cover the phenomena belonging to the pre-Aristotelian concept, but is tailored to remove the ancient paradoxes*, and to this extent it depends on concepts such as 'potential/actual' or 'category' in the framework of Aristotelian philosophy.

When music theory paved the road toward the discovery of incommensurability, the idea of geometric continuity was too clumsy to develop and even to understand such discovery. Perhaps, it was exactly the possibility of the geometric drawing of a not-existent music interval to foster the development of the Aristotelian idea of continuity. Such not-expressible existence was a breach in the rigid Parmenidean isomorphism between *being*, *thought* and *language*: "...for the same thing is there both to be thought of and to be...it is not to be said nor thought that it is not" (Diels and Kranz 1964, 28 B8). Parmenides and Zeno are not so awful in Aristotle as they appear in Plato. This breach rid the continuum of the being/not-being paradoxes, where they dwelt before.

A trace of this rupture is given by the fact that incommensurability in Aristotle is the paradigmatic example of a "being as true", $\hat{\text{O}}\text{n } \alpha\epsilon\text{j } \epsilon\text{l} \text{h}\alpha\text{q}\rangle\text{j}$ (Aristoteles 1831, *Metaph.* VI,4), a kind of "being" which is warranted in a purely theoretical way. Analogously, the Aristotelian potential infinite is something that cannot become actual, because it can occur just in 'thinking', but it *exists*. *Autonomy of thinking is the crucial step for overcoming the Parmenidean deadlock we can read in the negative judgement paradox.*

The refusal of speaking of "what is not", a crucial topos in the Sophists-Platonic times, was the reason why the musical incommensurability fell into oblivion. Instead, the geometric embedding of incommensurability was the example of a *not being* (the cut of the tone) which could become a *being* (actually constructed segments) as a 'theoretical' truth.

Incommensurability caused a *Grundlagenkrise not in the Pythagorean mathematics, but in the main stream of Greek philosophy*, the philosophy of *being* grown from Parmenides to the Academia through the Sophists' paradoxes and their analysis in the Platonic dialogues.

In my opinion, the historical preference for the 'geometrical' approach to the discovery of incommensurability reveals how deep is the difficulty in understanding Greek mathematics. For example, Knorr (1975, 242) writes: "Why then should the music theorists declare the impossibility of the equi-division of the tone and engage in detailed alternative designations of various categories of subtones expressible as integral ratios? They ought rather to have constructed the geometrical proportional as the exact semitone, and, if need be, approximate it to whatever accuracy was called for".

There Knorr employs the Cartesian idea of integers as peculiar real numbers, and real numbers as limits of rational numbers. According to his argument, Archytas had to extend geometrically the canon with a 8 interval from the extreme of a 9 interval. He had to find the middle point of this extended interval, and to trace a semicircle with center in this point and diameter the extended interval. Then he had to inscribe a triangle in this semicircle by finding the third vertex *via* the intersection of the semicircle with an orthogonal segment from the extreme of the 9 interval, and finally to translate the height of such triangle on the canon. Nothing more obvious in a Cartesian framework, and something of this kind became reasonable in 14th century, when music was shifting from *number* to *sound*, but not in *quadruvium*'s age, when music was embedded in arithmetic and had nothing to do with geometry (Hentschel 1998).

In Greek mathematics such 'geometrization of music' shows its first steps that began in Aristoxenus and can be recognized in Euclid and in Ptolemy's 'helicon' (Ptolemaeus 2000, II,2). This seems to be unlikely in Archytas, when numbers and magnitudes were sharply different entities and music belonged to the arithmetic side.

These remarks raise the question of the difference between the ancient Pythagorean 'musical' *perception* as displayed in the Pythagorean idea of 'linear number' in Boethius or in Nicomachus, and the modern 'geometrical' *perception* of the linear numerical magnitudes.

I have already pointed out that in modern perception the "number" is a "point" on the geometrical continuum. However, there is no trace of this perception among the Greek mathematicians. Euclid stated in both the *Elements* and the *Sectio Canonis* that a magnitude can be denoted by a single letter (if the magnitude is to be considered as a whole), or by two letters, its extremes (if the magnitude has been or is going to be divided), and thus the point can be only an extreme. Actually even in Archytas (Diels and Kranz 1964, 47 A14) we find the same style, even though this could be a later restyling (by Eutocius?).

According to Barker (1989, II: 7-8), in all Pythagorean harmonics "notes are treated as entities one of whose attributes, that of pitch, varies quantitatively and can be expressed in numbers. Intervals between notes are to be expressed as ratios of numbers. Notes, then, are items possessing magnitudes of some sort. They are not points on a line".

This 'discrete' perception of music is not only Pythagorean. Aristotle singles out the quarter of tone as the 'unit' in music, to be compared with the "letter" in a speech (Aristoteles 1831, *Metaphys.* 1016 b22, 1053 a12). Even Aristoxenus (Aristoxenes 1954, 27.20) employs the same comparison and claims that "that there is no interval which we divide ad infinitum in melody is one that commands assent: there is some greatest number of parts into which melody divides each of the intervals" (Aristoxenes 1954, 53. 20-22).

It is worth remarking that in Aristoxenus this least interval is just for perception and production of sounds, whereas it is clear that infinite different sounds are possible. This was a major breakthrough that followed the Aristotelian concept of continuity. However, this trade-off between discrete perception and continuous reality never appears to be analogous to our ideas of perceptive resolution power, error and approximation of a continuous magnitude that could be reduced as much

as we please by technical instruments: the unit of measure, also in Aristotle, is always indivisible (Aristoteles 1831, *Metaph.* 1052 b 32).

In fact, for modern science real properties are continuous and expressible by real numbers, whereas the language of science is discrete. The bridge is given by our ideas of scientific experiment and scientific measure in terms of discrete approximation of real numbers, as well as by our idea of laws of nature written in algebraic language.

There is nothing like this in Greek science, where continuity does not play any role. Continuity appears only in Aristotle as a philosophical concept framed in the distinction of matter/form and power/act to overcome the paradoxes about infinite raised by Sophists. The objects of science are 'indivisible by form' (ἄδιαρετόν τι ἐκδύ) and immediately perceived as such, their continuity is always perceived only as their being a 'whole', irreducible to, and not actually divisible in, parts, and hence also measure units are indivisible. The discrete description of reality is grounded on such units, and science requires 'minimums' (like the letters of the alphabet and the quarter of tone in music), not connected to physical constraints of the organs of sense. *Continuum is not only inexpressible, but also external to the knowledge of reality.*

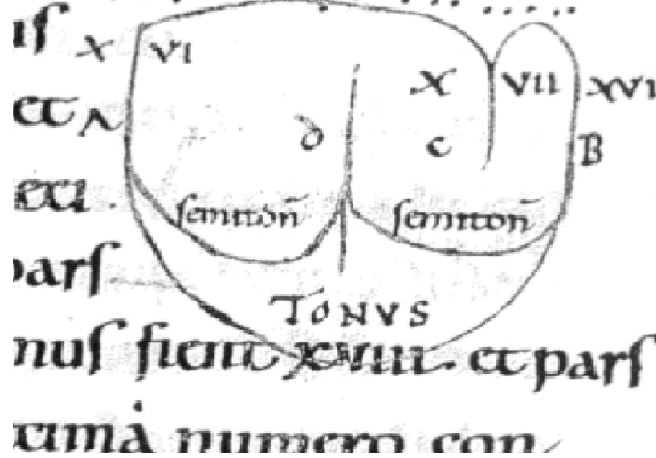
According to Aristoxenus, the infinite sounds are never 'actual', and the discrete perception of sounds can be compared with the finite number of letters in the alphabet. Hence, it concerns immediately the basic structure of science. From this point of view Aristotle's gnosiology is not substantially different from Plato's one as expounded in the *Philebus*.

In order to build his musical theory in terms of magnitudes, Aristoxenus rejected any numerical or metrical features. Euclid's geometry is not only non-metrical, but it also gives continuity no explicit role. At the beginning of the *Sectio Canonis*, Euclid or its anonymous author embeds music and "everything is composed of parts" in a rigid arithmetic approach.

Boethius writes at the end of the ancient world. By that time the earlier Pythagorean perception of music, Philolaus' confusion between ratios and numbers, the sharp opposition between metrical and theoretical aspects of geometry were substantially over. Already Claudius Ptolemaeus described the experiences on the canon correctly and coherently in terms of ratios between intervals defined by points. However, the earlier tradition was somehow still present in Boethius, perhaps because he had no empirical musical practice and was just a commentator, not a true mathematician.

Be that as it may, Boethius' diagrams are not easy to reconstruct in their original form. Sometimes an 'integer number' appears as a little interval, occasionally coincident with its second extreme. In other cases, in drawing the arcs on the canon by which Boethius denotes the musical intervals, the extremes of the arc are the left or the right extreme or an inner point of the interval.

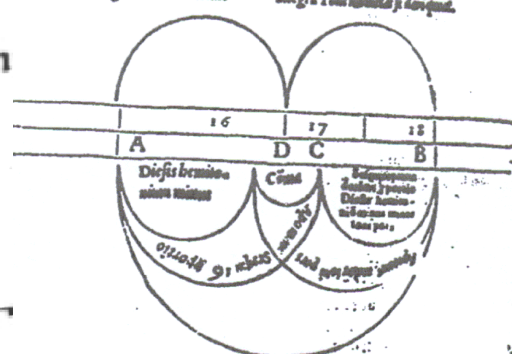
...ag. o numero nul
na sup sesq xvii. ma.



medietas inter sesquiseptimdecimam, ac sesquiseptimdecimam ult.
ac in integro numero nullo modo poterit inueniri.

Integra Toni medietas.

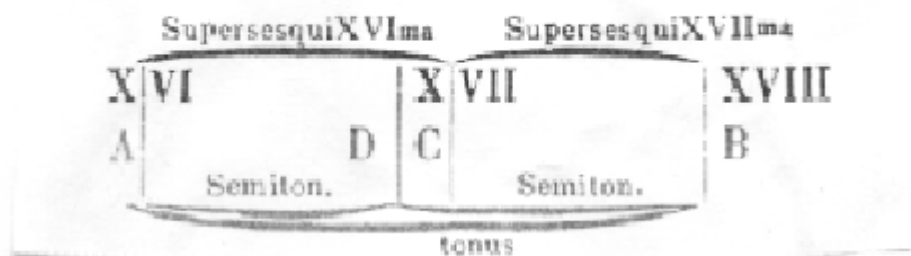
Integra Toni medietas si desquid.



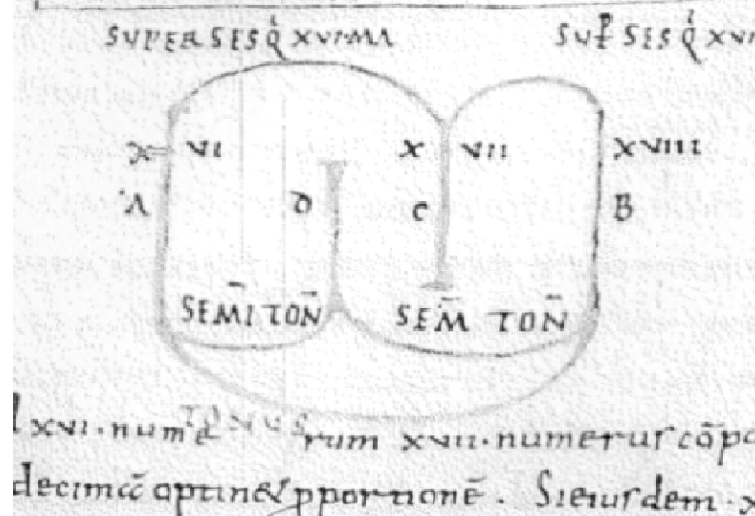
Tonus sesquioctauus proportio totum hoc spectant

am uero ad 16 numerum 17 numerum et 18 numerum. Sicut sesquiseptimdecimam.

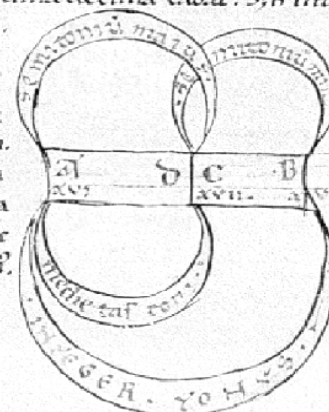
INST. MUS. III, 1.



poterit inueniri.



et septimā decimā cadat. Si h. integro
ueniri.
arar?
ia opti
de xvii.
regra
unita
ia hac
ngani?
xvi.



In the figures 1-5 I show the diagrams employed to describe the ‘cut of the tone’ in Friedlein’s edition (Boethius 1867, 271), in Migne’s edition (Boethius 1844, 1225) and in three codexes that include the diagram, *Monacensis* 18480, 14523, 367 (Bayerische Staatsbibliothek). Some of them (fig. 1, 3, 4) seem to denote correctly the numbers as points, but others are more ambiguous: Migne’s (fig. 2) reflects a Pythagorean perception and M 367 (fig.5) appears to mix both views.

The same ambiguity appears in Boethius’ description of the musical experiences on the monochord. For example, to play a ‘fourth’ (that is a ratio 3:4), the canon had to be divided in seven parts, “divido spatium ... septem partibus”, i.e. in seven intervals, and then played on the fourth, “et ad partem quartam septimarum appono punctum”. Obviously, the ‘punctum’ had to be the point between the fourth and the fifth interval, but the above description is at least ambiguous (Boethius 1867, IV, 18).

I think that *in ancient music, whose background was substantially arithmetical, the naïve perception of the linear numbers as a sequence of monads contrasted with the 'geometrical' nature of the canon and, to some extent, with the musical experience, which entailed a 'pointwise' press of the string.*

I think that this contrast became more and more troublesome when incommensurability broke down the ancient arithmetic perception of the numerical magnitude, and triggered what we could call a *Grundlagenkrisis*, where the term ‘Grundlagen’ has nothing to do with our axiomatic ‘foundations’ but *involved the basic Pythagorean ‘perception’ of the linear number.*

This could have been the second reason for reshaping incommensurability in a geometrical view, and the second form of *Krisis*, to be dealt with by the development of the Euclidean style. This style allowed an easier representation of (incommensurable) geometrical magnitudes, based on the ‘shift’ from a musical representation of the ‘linear numbers’ to a free multiplication and division of geometric magnitudes by numbers. This new representation can be recognized in Eudoxus’ theory of proportions, in his exhaustion method, as well as in Aristotle’s characterization of the One as a measure unit (Aristoteles 1831, *Metaph.* 1087 b33).

Conclusion

The best reference to the ‘cognitive’ approach is perhaps given by Netz (1999). There Netz defines *cognitive history* as something that appears “at the intersection of history of science and the cognitive sciences. Like history of science, it studies a cultural artefact. Like the cognitive sciences, it approaches knowledge not through its specific propositional contents, but through its forms and practices” (Netz 1999, 7). I agree with Netz’ idea that this approach allows the analysis of ‘too global’ cognitive processes, for example “reasoning”, which can not be understood in cognitive science, but can be analyzed historically.

According to Netz, historians may see this approach as “over-theoretical and too eager to generalize”. Is my ‘cognitive’ analysis actually stranger to the historian’s methods? I don’t think so. In my opinion, without a cognitive hypothesis it is impossible to interpret an ancient fragment of mathematics because it has no meaning outside a general architecture of knowledge in which the involved mathematical concepts are embedded. Who claims it useless, simply assumes some ‘standard’, but often thoroughly anti-historical, cognitive hypotheses.

For example, the passage (Diels and Kranz 1964, 47 B4) where Archytas claims the superiority

of logistic on all the other sciences, geometry included, is rejected by some authors (W. Burkert, for instance) on the basis of a cognitive prejudice: how could a great mathematician consider a practical art of computation superior to the great Greek geometry? However, if we admit the possibility that the very idea of divisible continuous magnitude is far from being 'natural', and that until Eudoxus it was nothing more than a soup of paradoxes, then geometry had to be little more than its Egyptian and Babylonian legacy, i.e. simple similitude properties, superposition techniques to compute areas, figurate numbers and simple properties of geometric algebra, connections between geometric figures and Gods, plus the first rough results about squaring the circle and doubling the cube.

On the other hand, under the impulse of music theory the 'theory of the logoi' could have been the main stream of mathematical research, producing the results of book VIII of the *Elements*, including the first negative proof of incommensurability and more advanced geometric applications like Archytas' algorithm to find two mean proportionals (Diels and Kranz 1964, 47 A14).

Thus, different cognitive hypotheses give completely different meanings (and authorship) to the same fragment. At the same time I think that history of science, and history of mathematics in particular, must become an essential ingredient of cognitive science, because it can give inspiring hints for the analysis of higher cognitive processes.

Acknowledgements. I would thank Julio Gonzales Cabillon and Umberto Bottazzini for their comments and suggestions.

Appendices.

1. Boethius' "De Institutione Musica" iii,11.

Archytas' proof that a superparticular ratio cannot be divided into equal parts.

A superparticular ratio [proportio] cannot be divided into equal parts by the interpolation of a mean proportional number... Let A B be a superparticular ratio; I take the least terms C, DE in the same ratio. Since C DE are the least in the same ratio and are superparticulars, the number DE exceeds the number C by a part of itself. Let this be D. I say that D will not be a number, but a unit. For if the number D is also part of DE, the number D measures DE; whence it measures the number E, so that it also measures C. Thus the number D measures both of the numbers C and DE, which is impossible. For those numbers which are the least in the same ratio as any other numbers are relatively prime, and they have the unit as sole difference. Thus D is the unit. Thus DE exceeds C by a unit. Whence between them no mean number lies which cuts their ratio equally. Thus, neither between those which have the same ratio can there be found a mean number, which cuts the same ratio equally.

2. prop. 3, "Sectio Canonis"

No mean number, neither one nor several, may be interpolated in proportion [ἐν λόγῳ] in an superparticular interval [δι᾽ ἄσθμα].

Let BΓ be an superparticular interval, and let the least terms in the same ratio be ΔΖ, Θ. Then they are measured only by the unit as common measure. Let HZ equal to Θ be subtracted, (and since ΔΖ of Θ is superparticular) the excess ΔΗ is a common measure of ΔΖ and Θ; it is thus the unit. Whence, there may not be interpolated between ΔΖ, Θ any mean number; for the mean would have to be less than ΔΖ but greater than Θ, (so that the unit would be divided, which is impossible. Thus,

none may be interpolated between ΔZ , Θ). Now however many numbers may be interpolated in proportion between the least terms, the same number may be interpolated in proportion between those having the same ratio. But no number may be interpolated between ΔZ , Θ ; whence none may be interpolated between B, Γ .

3. "Elements", X, 117

Let it be proposed to us to prove that in square figures the diameter is incommensurable in length with the side.

Let ABCD be a square, of which AC is the diameter. I say that AC is incommensurable in length with AB. <the figure includes a square ABCD with the diagonal AC, and two segments, the first with extremes EZ and with the letter T in the middle, the second with the letter H in the middle>.

For if possible, let it be commensurable. I say that it will follow that the same number is odd and even. Now it is manifest that the square on AC is the double of that on AB. Since AC is commensurable with AB, then AC will have the ratio to AB of one number to another. Let these numbers be EZ and H, and let them be the least numbers in this ratio. Then EZ is not a unit. For if EZ is a unit and has the ratio to H which AC has to AB, and AC is greater than AB, then EZ is greater than H, which is impossible. Thus EZ is not a unit; hence it is a number. And since AC is to AB as EZ is to H, so also the square on AC is to that on AB as the square of EZ is to that of H. The square on AC is double that on AB, so the square of EZ is double that of H. The square of EZ is thus an even number; thus EZ itself is even. For if it were odd, the square on it would also be odd; since, if an odd number of odd terms is summed, the whole is odd. So EZ is even. Let it be divided in half by T. Since EZ and H are the least numbers of those having this ratio, they are relatively prime. And EZ is prime; so H is odd. For if it were even, the dyad would measure EZ and H. For an even number has a half part. Yet they are relatively prime; so this is impossible. Thus H is not even; it is odd. Since EZ is double ET, the square of EZ is four times the square of ET. But the square of EZ is the double of that of H; so the square of H is double that of ET. So the square of H is even, and H is even for the reasons already given. But it is also odd, which is impossible. Hence, AC is incommensurable in length with AB. This was to be proved.

4. A possible reconstruction of the proof in Euclidean style

A superparticular ratio [proportio] cannot be divided into equal parts by the interpolation of a mean proportional number... Let A B be a superparticular ratio and F one mean proportional between them, thus A, F, B are in continued proportion. I say that a mean proportional G fall between the least numbers C, DE which have the same ratio, and hence have the same superparticular ratio of A to B.

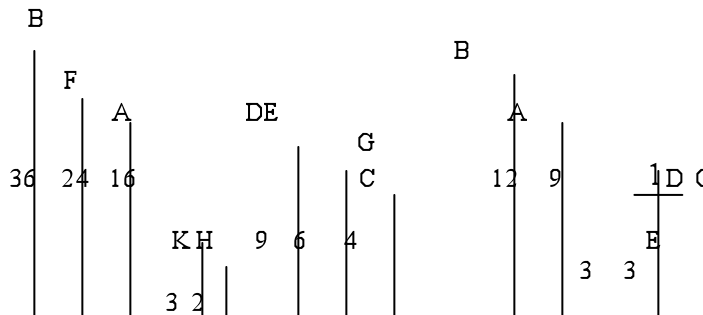
Let the ratio of H to K the least in the ratio of A to F; it is required to find numbers in continued proportion the least that are in the ratio of H to K. Let H by multiplying itself make C and by multiplying K let it make G; let K by multiplying itself make DE.

Now as H is to K, so C is to G. Again, as H is to K, so G is to DE. Therefore C, G, DE are in continued proportion in the given ratio.

I say they are the least numbers that are so. For since H,K are the least of those which have the same ratio with them, and the least of those which have the same ratio are prime to one another, therefore H,K are prime to one another. And the numbers H,K by multiplying themselves respectively have made the numbers C, DE; therefore C,DE are prime to one another. C,DE are the extremes of the continued proportion C, G, DE and are primes, therefore they are the least, which

have the same ratio with them.

Since C DE are the least in the same ratio and are superparticulars, the number DE exceeds the number C by a part of itself. Let this be D. I say that D will not be a number, but a unit. For if the number D is also part of DE, the number D measures DE; whence it measures the number E, so that it also measures C. Thus the number D measures both of the numbers C and DE, which is impossible. For those numbers which are the least in the same ratio as any other numbers are relatively prime, and they have the unit as sole difference. Thus D is the unit. Thus DE exceeds C by a unit. Whence between them no mean number lies, which cuts their ratio equally. Thus, neither between those which have the same ratio can there be found a mean number which cuts the same ratio equally.



5. Epinomis' fragment (Loeb translation, some passages have two translations: <Loeb translation>and [my translation]).

When he has learnt these things, there comes next after these what they call by the very ridiculous name of geometry,

<when it proves to be a manifest likening of numbers not like one another by nature in respect of the province of planes;>

[when it proves of numbers not being similar to one another by nature, similitude can be manifest in respect of the province of planes;]

and this will be clearly seen by him who is able to understand it to be a marvel not of human, but of divine origin.

<And then, after that, the numbers thrice increased and like to the solid nature, and those again which have been made unlike, he likens by another art,>

[And then , after that, the numbers thrice increased similar to the solid nature, those which are not similar made similar by another art]

namely, that which its adepts called stereometry; and a divine and marvelous thing it is to those who envisage it and reflect, how the whole of nature is impressed with species and class according to each <analogy> [proportion], as power and its opposite continually turn upon the double. Thus the first <analogy> [proportion] is of the double in point of number, passing from one to two in order of counting, and that which is according to power is double; that which passes to the solid and tangible is likewise again double, having proceeded from one to eight; but that of the double has a mean, as much more than the less as it is less than the greater, while its other mean exceeds and is exceeded by the same portion of the extremes themselves. Between six and twelve comes the whole-and-a-half ($9=6+3$) and whole-and-a-third ($8=6+2$): turning between these very two, to one side or

the other, this power (9) assigned to men an accordant and <proportioned>[commensurable] use for the purpose of rhythm and harmony in their pastimes, and has been assigned to the blessed dance of the Muses.

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