

Pythagoreanism and the History of Demonstration

Owen Goldin

According to Aristotle, to have scientific knowledge is to be able to offer a systematic body of demonstrations (*apodeixeis*), accounts which explain necessary and recurrent features of the world by showing how they follow from certain causally basic facts, which do not themselves demand explanation. The details of his account of scientific knowledge, laid out primarily in the *Posterior Analytics*, are, like many other aspects of Aristotle's thought, best understood as refutations and developments of the responses to certain philosophical puzzles that faced Aristotle's predecessors. Most notable among these is of course Plato, and recent work on the *Posterior Analytics* has concentrated on understanding the work as a response to, and development of, Platonic themes. Thus, Ferejohn has clarified how Aristotle fully works through the Socratic enterprise of showing how the answer to a "what is it" question grounds the answer to a "why is it" question.¹ Bronstein has shown how the *Posterior Analytics* can be understood as a series of nested solutions to the learner's paradox of Plato's *Meno*.² Both have explored how Aristotle adopts and adapts Plato's method of division as a crucial aspect of his own scientific methodology. But Aristotle has a number of additional philosophical predecessors, some of whom can be called "Presocratic," others of whom are contemporaries of Socrates or Plato. Aristotle's philosophical responses to what these other predecessors had to say in the fields of physics and metaphysics are well recognized: this is not so much the case with regard to issues in epistemology and the philosophy of science worked through in the *Posterior Analytics*. Perhaps this is because scholars have presumed that, although Presocratic and Pythagorean philosophers were engaged in doing science, they did not reflect on what theoretical or scientific knowledge is. The recognized exception here is Parmenides, as the first part of his poem concerns the "way of inquiry." Here, a goddess tells Parmenides that inquiry seeks an account of the form "it is," and this insight is of central importance to the nature of definition as developed by Plato and Aristotle. But no matter how one interprets Parmenides' poem, it offers little

¹ Ferejohn 2013.

² Bronstein 2016.

precedent for Aristotle's account of demonstration, which, goes beyond statements of the form "it is" insofar as it links such statements together to form a kind of deduction that reveals causal and conceptual relations.³

Nonetheless, Aristotle's theory of demonstration has significant precedent outside of the writings of Plato. This is to be found within the Pythagorean tradition. As Aristotle is aware, certain Pythagoreans thought that understanding, in the strict sense, demanded proofs and explanations. Aristotle himself provides evidence concerning the structure of the earliest such accounts, which appealed to a "table of opposites." The presence of one feature was inferred and explained on the basis of the presence of another feature with which the table associates it. Aristotle respects the table as a source of *endoxa*, the generally held beliefs that provide a starting point for dialectical inquiry,⁴ but rejects the accounts based on them as not truly explanatory, on the grounds that they do not distinguish cause and effect. In this paper I argue that the later Pythagoreans Philolaus and Archytas did recognize that it is crucial to distinguish the principles, which are basic, and items that are derivative, which are made sense of as resultant from the principles. Even in its early stages, Greek mathematics provided examples of this. Aristotle's theory of demonstration can be understood as a way of integrating these Pythagorean and mathematical insights with the Socratic/Platonic account of definition.

In order to trace how Aristotle's theory of demonstration builds on and responds to the thought of his predecessors, we must first lay out the main lines of this theory, which, like Solmsen and Barnes,⁵ I take to be independent of the theory of syllogism which is grafted onto it.

3 Granted, within Parmenides' poem the goddess does present an *argument* for her conclusion that the way of inquiry is that of "it is," and she does refer to her argument as an "elenchus" (DK B 7.5-6). As Lloyd writes: "The method, then, no less than the content, of the Way of Truth is revolutionary. ... He is the first thinker to set up a fundamental opposition between the senses and abstract argument or reason, and to express an unequivocal judgement on the relative trustworthiness of each," Lloyd 1979, 70–1. Nonetheless it is hard to see how the goddess could maintain that inquiry itself must proceed through a deduction. On a standard interpretation of Parmenides, according to which Parmenides is a "numerical monist," arguing that there is only one thing that can exist or be said, this would be impossible. Inquiry could not result in an elenchus insofar as any argument involves multiplicity of parts (unless the argument is a kind of ladder to be dispensed with). On the "numerical monist" interpretation, and for an argument against it, see Curd 1998.

4 See for example: *Ph.* 3.2 201b24–7, *Metaph.* Γ 2 1004b27–1005a5.

5 Solmsen 1928; Barnes 1981, 17–59. Ferejohn does not go so far as to posit a demonstrative theory that had been worked out prior to the development of the syllogistic, but suggests that the term *apodeixis* in *An. post.* 1.2 is being used in a preliminary sense that does not presuppose syllogistic theory. "I interpret the noun *apodeixis* and its derivatives in these early chapters to

- 1) Many regular, necessary features of the world are learned only by virtue of a certain kind of discursive logos, which reveals the relationships that hold among certain features of the world. This logos is called a demonstration (*apodeixis*) (*An. post.* 1 2 71b17-9).
- 2) Demonstration must rest on insights that do not demand demonstration. Aristotle calls these principles (*arkhai*) (*APo.* 1 2 71b20-4).
- 3) A demonstration explains by showing how the derivative features of the world can be inferred from the principles, which are causally basic. For this reason, a demonstration distinguishes causally basic realities from those that are derivative (*APo.* 1 2 71b29-33).
- 4) An adequate demonstration fully works through all of the relevant relations. A demonstration can make no appeal to any demonstrable feature of the world that is not itself demonstrated on the basis of indemonstrables (*APo.* 1 2 72a6-8).

Even independent of the application of syllogistic theory, there is of course much more to Aristotle's theory of demonstration than this. He has much to say on the various kinds of principles, and the method by which one should inquire into what the principles are and what demonstrations are to be based on them. But it is the above four characteristics of demonstration that are anticipated in Pythagorean thought. I take up each of these points in turn.

1 Demonstration

A passage from Iamblicus' *On the General Mathematical Science*, generally accepted as having been derived from Aristotle's lost writings on the Pythagoreans,⁶ distinguishes between the *mathematikoi* and the *akousmatikoi*. The *mathematikoi* take the distinction between the varieties of Pythagoreanism to derive from different ways that Pythagoras himself conveyed his teachings.

Pythagoras came from Ionia, more precisely from Samos, at the time of the tyranny of Polycrates, when Italy was at its height, and the first men of the city-states became his associates. The older of these [men] he addressed in a simple style, since they, who had little leisure on account of their being occupied in political affairs, had trouble when he conversed with them in terms of learning (*mathēmata*) and demonstrations

the generic notion of epistemic justification and not specifically to the syllogistic theory of justification developed later in the treatise." Ferejohn 2013, 75, n. 27.

6 See Burkert 1972, 195–200; Huffman 2015, Horky 2013, 16–7.

(*apodeixeis*). He thought that they would fare no worse if they knew what to do, even if they lacked the reason (*aitia*) for it, just as people under medical care fare no worse when they do not additionally hear the reason why they are to do each thing in their treatment. The younger of these [men], however, who had the ability to endure the education, he conversed with in terms of demonstrations and learning. So, then, these men [i.e. the mathematicians] are descended from the latter group, as are the others [i.e. the akousmatics] from the former group

IAMBlichus, *On the General Mathematical Science* 77.4–8⁷

The Pythagoreans were a kind of brotherhood, bound by a lifestyle of ritual prohibitions and prescriptions. Most of the elements of this lifestyle, as expressed in the *akousmata*,⁸ seem arbitrary to us, and no doubt were so considered by the Pythagoreans' contemporaries. What reasons could there be behind these rules? Why not eat beans? Why not pick up food that fell on the floor (Diogenes Laertius, *Lives of the Philosophers* 8.34)? For each rule, there are three alternatives: it is indeed arbitrary, it is a means to some end (unknown to us), or following it is an intrinsic good (although we cannot recognize that). The Pythagoreans surely did not accept these rules as arbitrary. Aristotle reports that there were causes (*aitiai*) behind them which were sometimes made explicit and sometimes not,⁹ but not everyone was capable of grasping these causes. The behaviors are prescribed for extrinsic purposes (when they are for the sake of something that results from the behaviors) or for intrinsic purposes (when the behaviors are themselves good, regardless of their result).¹⁰ It is not hard to imagine a defense of religious or cultural taboos as habituating one in self-control, a disposition which is a good external to the acts in question.¹¹ But we see no evidence among the Pythagoreans for such a justification.¹² We can

7 Horky 2013, 15–6.

8 On the *akousmata* see especially Burkert 1972, 166–92. Many are collected in DK A 58C, Boehm 1905 and Dumont 1988.

9 See Iamblichus *VP* 86, also derived from Aristotle: “In some cases an account is appended concerning why this must be done (ἐπιλέγεται τί δεῖ), for example, one must bear offspring in order to leave another in one's place, to serve the gods, but for others there is no additional account.”

10 Cf. Plato, *Resp.* 2 357b–d, Aristotle, *Eth. Nic.* 111094a3–5.

11 On this see Maimonides, *Guide for the Perplexed* Book 3, chapter 33.

12 Granted, DL 8.1.34, quoting Iamblichus, in turn quoting Aristotle, does relate that the prohibition against picking up food that has fallen on the floor might be for the sake of habituating oneself against gluttony, but the suggestion that the prescription is for the sake of habituation was tentatively put forward by Aristotle, for whom habituation is key in ethical training, and he does suggest another possibility: the crumbs are “on the side of

infer that the elements of the Pythagorean lifestyle were themselves taken to be good, either for the individual or the community. Nonetheless, the fragment from Aristotle tells us that there was some reasoning behind them, which, except for the most straightforward cases (like the obligation to have children) only the intellectual elite were in a position to learn. What form do these *logoi* take? Aristotle, if not the Pythagoreans themselves, called the accounts that revealed their purpose “demonstrations.” So two major questions arise. First, what light is shed on them from the fact that they were called “demonstrations”? Surely Aristotle does not think that they constitute demonstrations in his strict sense. We are led to consider how the term “demonstration” (*apodeixis*) was used prior to Aristotle, especially in theoretical contexts. Second, to what extent can we reconstruct the form that these demonstrations take?

We begin with the term *apodeixis*. The verb *apodeiknunai* has the basic sense of “showing forth.” Its earliest uses were in regard to cases in which one, via a speech, revealed something that would have been otherwise concealed or in dispute. It is distinguished from an *epideixis*, which, according one of the senses of the term,¹³ is a showing off of the speaker, in order to reveal his or her abilities; rhetoricians would customarily give an *epideixis* to an audience of prospective students, in order to display the speaker’s rhetorical abilities. An *apodeixis*, in contrast, shows something about the world, not the speaker; it shows that something is the case.¹⁴ Especially illustrative here is the use of *apodeixis* in the proemium of Herodotus’ *Histories*:

This is the display of the inquiry (ἱστορίας ἀπόδειξις) of Herodotus of Halicarnassus, so that things done by man not be forgotten in time, and that great and marvelous deeds, some displayed by the Hellenes, some by the barbarians, not lose their glory, including among others what was the cause (δι’ ἣν αἰτίην) of their waging war on each other.

As events recede into the past, it is up to Herodotus to inquire what happened and why. The *apodeixis* that Herodotus offers is an account not only of what happened, but also of why it happened. It reveals the cause (*aitia*) of the war, the deeds that were responsible for it.¹⁵

someone’s death” (ἐπὶ τελευτῇ τινος). I am not sure what this means but the preposition ἐπὶ suggests a tabular association with death, instead of life.

13 See Plato, *Grg.* 447a–c.

14 On this see Huffman 2005 and Thomas 2000, 221–8.

15 On this see especially Nagy 1994, 215–29.

The Hippocratic writings also employ *apodeixis*. Sometimes the term is used not to refer to a kind of logos, but to a state or process that results in a perceptible result or symptom, referred to as the object of ἀποδεικνύναι.¹⁶ So in this case the demonstrator is the cause; what is demonstrated is an effect. But the writings also use the term in the sense familiar from Herodotus, as referring to a spoken or written exposition,¹⁷ sometimes to such an exposition that justifies its conclusions on the basis of argument, especially a discourse that reveals the causes of things.¹⁸ In this case what is demonstrated is the existence of an underlying condition, which is the cause of the symptoms. (In both cases the usage differs from that of Aristotle, according to which, the object of ἀποδεικνύναι is the effect whose cause is revealed in the demonstration.) As Lloyd has emphasized,¹⁹ these medical writings are not so different in form from legal speeches in regard to rhetorical strategy. The medical author, like the orators in the law courts, is offering an account in a forum in which that account competes with others. As in the courtroom, the account given is intended to reveal what happened and why it happened. Insofar as a demonstration shows something, there is something that is shown. Because what becomes manifest through a demonstration is not illusory, a demonstration has a true conclusion, for which reason the term is often rendered as “proof.”²⁰ Thus, the author of *On the Nature of Man* repeatedly indicates that his goal is to prove or demonstrate (ἀποδεικνύναι) what the constituents of the human body are, why their interactions have certain results, and that the conclusions drawn follow of necessity from his account.²¹ There is an emphasis on the necessity of what is said to be demonstrated, suggesting that one mark of demonstration is the necessity of the conclusion drawn, but as Lloyd points out, it is unclear what this necessity is: (The necessity of the inference to the conclusion? The necessity of the fact in question, itself? Something else?). It is accordingly unclear exactly what it means to call an account a demonstration.²²

16 Examples include *VM* 19.41; *Fract.* 19, 10,12; 31.31; *Prorrh.* 2 15,3.

17 *De Arte*, 3,3, 9.

18 *Nat. Hom.*, 2,23; 5,26.

19 Lloyd 1987, 83–108.

20 See for example the Loeb translations of the passages cited in n. 18 above in Jones 1931. As Thomas 2000, 222 points out, “proof” in such contexts is “nearer to the sense of ‘show[ing] decisively’ than formal proof.”

21 See Lloyd 1987, 114–23.

22 Lloyd 1987, 120: “Clearly, logical and physical, conceptual and causal, necessity are not here differentiated. Many instances represent a conflation of one or more ideas that we might distinguish. Often the underlying idea seems merely to be the claim that something is always or usually the case. At the limit, the addition of the term *necessarily* appears to reflect little more than the writer’s desire to assert his point with emphasis.”

The term *apodeixis* is however sometimes used to refer to purported proofs or arguments based on probabilities (of which the conclusion is that something is the case, not that something is probably the case). For example, at *Phaedo* 92d and *Theaetetus* 162e, Plato uses the term to refer to a variety of argument that encompasses proofs based on probabilities, of which the conclusions lack certainty, and for that reason might be false. Similarly, within the *Rhetoric* Aristotle considers enthymeme a variety of demonstration, understood as an inferential variety of *pistis* (a means employed by a speaker to persuade a listener to do something or endorse a view).²³ “Enthymeme” here does not have the sense that it does in later rhetorical theory (a syllogism with a missing premise); rather, the term refers to any speech that derives its appeal from its argumentative structure.²⁴ A conclusion is inferred on the basis of at least one agreed-upon premise, and in this way the conclusion is shown to be the case. Since Aristotle recognizes that successful rhetorical speeches need not argue for a true conclusion, when he says that an enthymeme is an *apodeixis*, he is not using *apodeixis* in the strict sense of the *Posterior Analytics*, or even in the somewhat looser sense of the *Topics* (1.1 100a26-7) according to which demonstration is marked by having principles that are primary in justification but not, as in *Posterior Analytics*, primary in explanation. The term *apodeixis* as applied to an enthymeme is equivocal, like the term “eye” when applied to a formerly living eye that is no longer able to see,²⁵ or to the image of an eye;²⁶ perhaps in English scare quotes would be employed.

The term *apodeixis* can be used to refer to an argument with a false conclusion. Because there is no evidence that Aristotle himself accepted the prohibitions and prescriptions, the validity of which was “demonstrated” by the *mathematikoi*, we can conclude that in the passage from Iamblichus in question he is using the term in this derivative sense, to refer to purported

23 Aristotle, *Rh.* 1.1 1355a4-8: ἐπεὶ δὲ φανερόν ἐστιν ὅτι ἡ μὲν ἔντεχνος μέθοδος περὶ τὰς πίστεις ἐστίν, ἡ δὲ πίστις ἀπόδειξις τις (τότε γὰρ πιστεύομεν μάλιστα ὅταν ἀποδεδείχθαι ὑπολάβωμεν), ἔστι δ' ἀπόδειξις ῥητορικὴ ἐνθύμημα, καὶ ἔστι τοῦτο ὡς εἰπεῖν ἀπλῶς κυριώτατον τῶν πίστεων, τὸ δ' ἐνθύμημα συλλογισμὸς τις, “Since it is clear that the inquiry fitting for a *tekhnē* concerns ways of inducing conviction, and a way of inducing conviction is a kind of *apodeixis* (for we have the highest degree of conviction when we take something to have been subject to an *apodeixis*, and since an enthymeme is a rhetorical demonstration, and this is, so to speak, the most important mode of persuasion, and an enthymeme is a kind of deduction ...”.

24 On this see Burnyeat 2016, 3–56.

25 Aristotle, *De an.* 2.1 412b20-1.

26 See Aristotle *Cat.* 1 1a2-3.

proofs that this or the other obligation is binding, by showing how the obligations follow from certain reasons (*aitiai*).

Aristotle is contrasting the *akousmatikoi* and the *mathematikoi* on the grounds that the former accepted certain doctrines and strictures simply because they were asserted. They would be the ones who, purportedly, backed up certain views simply by saying “*ipse dixit*”: “that’s what he said.”²⁷ The *mathematikoi*, in contrast, gave more: an account by which the point at issue is inferred. This signals a significant juncture in the history of epistemology. The early Pythagoreans are the first Greek thinkers to show an explicit awareness of the distinction between two kinds of nonempirical belief: that accepted on the basis of testimony, and that accepted on the basis of some sort of reasoning. The latter is signaled as preferable. The Pythagoreans have taken a step towards the teaching of the *Theaetetus* that knowledge (*epistēmē*) is a belief plus something else, a *logos*, a direct antecedent to the common though much disputed definition of knowledge as justified true belief.

2 Inference

The second key feature of a demonstration, as Aristotle accounts for it, is that it is inferential. To what extent is this anticipated in the Pythagorean tradition?

We need to be careful here, as Aristotle himself is the first to articulate the notion of inference, when he defines a *sullogismos* as “a *logos* in which, from certain things being posited, something different from what was posited necessarily follows, from their being the case” (*An. pr.* 1.1 24b18–20). As has been often noted, this definition encompasses far more than the notion of syllogism, as Aristotle analyzes it within the *Prior Analytics*, for which reason it is often translated as “deduction.” Any argument offered as a proof or demonstration would meet this definition, but nothing is thereby said about the form the argument must take, or of how exactly the conclusion is thought to follow from the premises. The notion of deduction implicit in earlier conceptions of demonstration is inchoate and nontechnical. Further, there is no evidence of anyone prior to Aristotle holding the view that a deduction proper involves more than one premise. But even if drawing a conclusion from a single premise (as in the conversion of a predication) is to count as a deduction, surely a deduction, demonstrative or otherwise, cannot consist in a monadic indication of some being (like the Parmenidean *ἐστὶ*). Nor could it be something said

27 DL 8.1.46.

that leads the mind from one being to another in a single jump – how would such a jump be rational? How would the second feature be “shown”? We need a notion of logical form, some sort of structure or rule that determines how one can infer the existence of one sort of thing from the existence of another, supplemented by the indication of that first kind of thing. We see precisely this sort of logic in Aristotle’s report concerning the early Pythagoreans.

In earlier work I have argued that the “table of opposites” attributed to “the Pythagoreans” in Aristotle’s *Metaphysics* A 5 986a23-4 is an authentic Pythagorean representation of the relationships thought to hold among associated characteristics.²⁸ On the basis of both textual and anthropological evidence I argued that the table is a diagram expressing a world view shared by many prescientific peoples: all features of reality can be classified as having either positive or negative value. The table of opposites can be used to show why certain features of the world, found in the same column, are found together. It can also show why such features of the world ought to go together.²⁹ The relationship between two items in the same column is apparently symmetrical; there is no indication that one is more causally basic than the other.³⁰

It is likely that the conceptual correlations found in the table is the source not only of a number of accounts making sense of a number of features of the world, but also of many of the prescriptions, prohibitions, and ritual practices that constituted the Pythagorean lifestyle. For example, the version of the table that Aristotle preserves for us has the good, and the right, on the right-hand column. This means that the right is good, the good is right. Such an association is the source not only of the widespread European belief that righthandedness is the norm, not only in the sense that most people are right

28 Goldin 2014, 171–193.

29 A common way of expressing the view that one item A ought to be with another item, B, is to say that A ἀρμόττει B (in the dative). *Harmonia* is of course a crucial notion in later Pythagorean thought. Although we do not have direct evidence for this, it is not inconceivable that earlier Pythagoreans would have employed this term in regards to items in the same column of the Table of Opposites. The first clear reference to the notion of *harmonia* among the presocratics is in the fragments of Philolaus, for whom features of the world are to be explained on the basis of the fitting together of opposites, classified as limits and unlimited (DK B 6). This is an advance over the kind of dualism found in the *Doxa* section of Parmenides’ poem, which I take to be a simplification of the ontological scheme implicit in the table of opposites, according to which opposed opposites and their associated features exclude each other. But Pythagorean interest in musical scales and their mathematical basis predated Philolaus, and it is not inconceivable that the general strategy of explaining order and structures through the appropriate fitting together of disparate aspects of the world did so as well.

30 On this see Goldin 2014.

handed, but also in the sense that right-handedness is normative – that people should be right-handed—a belief which results in the once widespread practice of forcing children to write with their right hands, contrary to their inclinations.³¹ I propose that the sort of account that would justify this (the right hand is *χρεῖττον*, which means both “stronger” and “better”) would underlie the Pythagorean prescription that one ought to enter a temple with the right foot, Iamblichus (*VP* 156, 5–9.) Rightness and goodness correspond; entering with the left foot would show that one does not recognize either the special characteristics of where one was going, or the special characteristics of the right side. Similarly, one ought not to injure the good, and, since white is ranked with the good, one ought not to slaughter a white rooster.³² This sort of account, I propose, would be of the sort of proof (*apodeixis*) that, according to the above passage, the *mathematikoi* possessed, and the *akousmatikoi* did not, by which some sense was made of the Pythagorean way of life. That the Pythagoreans employed their table in this way is confirmed by means of Aristotle’s occasional citations of how it was employed in Plato’s Academy. To pick a familiar example, within the *Nicomachean Ethics* Aristotle appeals to how the unlimited and the bad were in the same column (2.6 1106b28–35). He does so to indicate that what deviates from limit is bad, and that bad states of character deviate from limit. Aristotle appeals to the Pythagorean table in dialectical support of his own account of why actions without limit are bad: a mean is a kind of a limit, and it is the action in accordance with that limit that is rational and accordingly partially constitutive of the human good, happiness. Although for Aristotle, *limit* is the cause of goodness, and not vice versa, this is not the case for the Pythagoreans of the table of opposites. As noted above, in the Pythagorean table, the relationship between two items in the same column is symmetrical; there is no priority.³³ The good and the limit, or the good and the right, have equal status as being present on the same column of the table of opposites.

Given both a feature that is listed on the table, and the table, one can infer, as it were, the presence of another feature listed in the same column. We see here a kind of protologic making possible a kind of deduction, which Aristotle

31 On this see Lloyd 1962, 56–66; repr. with additional material in 1991, 27–48.

32 The logic here is explicitly given in *DL* 8.34.10. A number of other *akousmata*, such as “do not urinate turning toward the sun” (Iamblichus, *Protrep.* 115.19–20) are open to such interpretations.

33 It seems to have been Speusippus who understood one of the columns as that of the good, and the other column that of the bad, prioritizing good and bad as the chief items in the right and left columns, accordingly. See Goldin 2015, n. 3, 7, and 41.

would have regarded as a “demonstration,” that reveals the reasons behind Pythagorean commands and prohibitions.

3 Causation

As we have seen, the protologic of the Pythagorean table of opposites allows the mind to move from the existence of one feature of the world to the other, but neither one column of the table nor the other is taken to be causally basic. Consider how “right” and “male” are both in the same column. From “right” one can infer “male” and vice versa. This obviously cannot be intended to hold in all cases: even if someone’s father had been to the right, and one’s mother to the left, the correlation will be disproven as soon as one’s point of view pivots 180 degrees. (That is not of course to deny that the correlation might be understood as having normative force, according to which the man should be on the right from some determinate point of view.) But it does suggest that there will be certain key situations in which the correlation will be observed. Suggestive here is the assertion of Parmenides, whom ancient historiographers associated with the Pythagoreans (DL 9.3.21), that males develop on the right side of the uterus; females on the left (DK B 17). The issue is not even broached as to whether the side of the uterus determines gender or whether a baby of a certain gender moves to one side of the uterus; the two simply go together, just as throughout Parmenides’ account of the way of seeming, certain characteristics are identified with light and others with darkness.³⁴

Although the table likely predates Pythagorean mathematical researches, it is interesting to note that there is mutual entailment in many mathematical contexts as well. So, in geometry, if one can often infer a conclusion from a premise, one can often infer a premise from a conclusion. For example, one can infer that two lines on a plane are parallel on the basis of the fact that a line that intersects both results in opposite interior angles, and one can infer that the intersecting line results in opposite interior angles on the basis of the initial lines being parallel. Aristotle points out that this is what makes geometrical analysis possible (*An. post.* 1.12 78a6-13). But Aristotle also recognizes causal hierarchy. This is why he insists that the upward and the downward paths of inference must be sharply distinguished (*Eth. Nic.* 1.4 1095a31-b1): in a proper demonstration in geometry or any other science, the premises express the

34 In DK B 9, Parmenides goes so far as to say that all things have been named light and darkness, and thereby identifies (or says that mortals identify) all correlative characteristics, an uncompromising dualism that is alien to earlier Pythagorean thought.

cause by which one can explain why the conclusion holds (*An. post.* 1.2 71b20-2). Those propositions that express the basic truths, on the basis of which one can explain derivative truths, are principles (*archai*) (*Metaph.* Δ 1 1013a14-6).

As we have already noted, this distinction between principles and derivative truths is absent from the early Pythagorean scheme represented in the table of opposites. If the “demonstrations” of the validity of Pythagorean precepts did indeed employ the table, Aristotle’s report that the *mathematikoi* offered demonstrations that revealed their causes must be interpreted as quasi-inferential (with the table, and the presence of one opposite as “posited,” and a correlative opposite as quasi-deduced) but falling short of his notion of demonstration insofar as it does not reveal an *aitia* that stands in an asymmetrical relation with what is demonstrated. According to a world view that “explains” by means of symmetrical associations and linkages, neither of the associated items are the cause of each other.

At what point does an asymmetric notion of causation become clear in Greek philosophy? All human beings, of course, make appeal to some things being responsible for other things in contexts where this is not reversible. The most obvious case is that of the responsibility of human beings for their actions. (It is no accident that the primary sense of *aitia* is responsibility or blame, especially in a legal context.) The question before us however is the point at which people become explicitly aware of the asymmetric character of causation. Plato is clearly aware of the asymmetry of cause and effect, even if the varieties of causation are not distinguished in as clear and clean a matter as they are in Aristotle. Forms are responsible for certain features of the things that participate in them, while the participants are not responsible for the character of the Forms. Soul, regarded by Plato as a self-mover, is responsible for motion of moved things but the moved is not responsible for the motion of soul. Plato’s argument for the necessity of a self-mover at *Phaedrus* 245c-e would make no sense without this asymmetry.

But does the recognition of the asymmetry of cause and effect date from before Plato? The fragmentary state of the evidence, and the fact that most of it is testimony, freely employing later vocabulary, makes this hard to say. If, as Aristotle indicates,³⁵ the Milesians did indeed employ the term *arkhē*, it was in the manner of mythological accounts like that of Hesiod, referring to the origin or source from which things come to be.³⁶ Every origin is indeed asymmetric with that which originates from it, but, although Aristotle takes the term *arkhē*

35 *Metaph.* A 5 983b6ff.

36 On this see Graham 2016, 58.

to be synonymous with *aition* (as both cause and explanation, see *Metaph.* Δ 1 1013a16-17) it is not clear what relation the notion of as origin, as employed by early Greek philosophers, might have to that of a cause, that is, that which is responsible or is indicated as the answer to a “why” question.³⁷

There is evidence that this realization has a source within the Pythagorean tradition itself, in the thought of Philolaus. We begin with another passage from Iamblichus, also generally agreed to have been largely derived from Aristotle:

The Pythagoreans devoted themselves to mathematics. They both admired the accuracy of its reasonings, because it alone among things that humans practice contains demonstrations, and they saw that general agreement is given in equal measure to theorems concerning attunement (τὰ περὶ τὴν ἁρμονίαν), because³⁸ they are [established] through numbers, and to mathematical studies that deal with vision, because they are [established] through diagrams. This led them to think that these things and their principles are quite generally the causes of existing things. Consequently, these are what anyone who wishes to comprehend things in existence – how they are – should turn their attention to, namely numbers and geometrical forms of existing things and proportions, because everything is made clear through them. So, then, by attaching the powers of each thing to the causes and primaries – only things that were less opportune or less honorable than them – they defined other things too in nearly the same manner. Therefore, their education in numbers and the objects of mathematics seemed to come through these subjects and in this general sketch. Such was also the method of demonstrations among them, which both began from such principles and thus attained fidelity and security in their arguments.³⁹

Here we are told that the *logoi* of the *mathematikoi*, identified at *VP* 77.9–10 as demonstrations, gain their “fidelity and security” by virtue of identifying

37 Significantly, another source reports that Hipparchus employed the notion of a *arkhē* to refer to a geometrical premise, albeit not an ultimate one. On this see Huffman 2006, 83–4: “What is important about this is that it indicates that in the latter part of the fifth century mathematics had advanced far enough to distinguish between more and less fundamental propositions and was concerned to try to determine a set of first principles.” Hipparchus may well have taken a geometrical principle to not only be that out of which a proof arises, but a kind of epistemological cause, that which, when appropriately grasped, in the context of a proof, is the cause of one’s cognitive grasp of the proven proposition.

38 Retaining ὄτι.

39 Iamblichus, *On the General Mathematical Science* 78.8–26, tr. Horky 2013, 31.

necessary relationships that hold among certain “powers.” Pythagorean demonstration restricts itself to working through the relationships that hold among the characteristics (or “powers”) that in turn are “attached to” those primaries that are their ontological ground. Insofar as we are not told what the primary entities or powers are, we might suspect that we have an appeal to the Pythagorean table of opposites as a universal explanatory scheme. What is new here is the assertion that these *mathematikoi* were mathematicians in our sense. They dealt with numbers, which allowed them to account for the fittingness (*harmonia*) of things,⁴⁰ and with geometrical features of things, which ground accounts of optics. The Pythagorean table of opposites, as it is reported by Aristotle, included odd and even, which are likely what is referred to here as principles of number. But the table included much more. Odd and even did not have a privileged role. Aristotle is here talking about a different group of Pythagoreans, who do give number a special role.

The question of when certain Pythagoreans took special interest in, and did original research in mathematics is highly contested. But it is clear that Philolaus plays an important role in the development of Pythagorean mathematics, and that it was due to his thought that Pythagoreanism came to be closely associated with the study, not just of mathematics in general, but the study of *number*.

Again, much of the evidence derives from Aristotle, who says that some Pythagoreans, identified only as different from the ones who posited the table of opposites, took the principles of all things to be numbers, which are said to be causes in the sense of the *matter* out of which other things are constituted (*Metaph.* A 5 985b23–986a21). Following Burkert, most scholars agree Aristotle has in mind Philolaus, or Pythagoreans closely associated with him.⁴¹ Philolaus’ Fr.4 survives as partial collaboration: “And indeed all things that are known have number. For it is not possible that anything whatsoever be understood or known without this.”⁴² It is a difficult and complex issue as to whether Aristotle is right to interpret Philolaus as ascribing to number the key ontological role as that out of which things are composed, or whether, as Huffman has argued (on the basis of a minimalistic reading of this fragment), Philolaus grants to number only an epistemological role (insofar as all things are known through number).⁴³ On the more expansive reading, Philolaus can be credited

40 There is here a possible reference to the sort of numerical fittingness responsible for musical modes.

41 Burkert 1972, 218–38.

42 Trans. Huffman 2016.

43 Huffman 2006, 172–6.

with the insight that the principles of explanation and intelligibility are also causal principles of reality. I am sympathetic to this traditional interpretation; after all, Parmenides had already declared that “the same thing is for being and thinking (*noein*)” (fr. 3), and his influence was already pervasive at the time of Philolaus’ writing. The matter is however uncertain, and even if we accept Huffman’s more conservative account of Philolaus, we can see the latter playing a crucial role in the development of the theory of demonstration.

Historians of science are agreed that even if the numerical basis of the musical scale was not known to Pythagoras, it was to Philolaus, and is presumably a paramount example of what is known by number. The octave, for example, is known by the numerical ratio of 2 to 1. Is the “demonstration” that a musical octave corresponds to this ratio a demonstration in a stricter sense than that we have been able to attribute to the earlier *mathematikoi*, according to which the demonstration makes some feature of the world intelligible by showing how it belongs to a kind of thing that accompanies another thing? Does the demonstration isolate the ratio as the cause of the octave? That is not clear. If in fact Aristotle is right and these Pythagoreans took number to be the basic entities and other things to be derivative entities, the latter would be caused by the former, and the demonstrations at issue would, like Aristotelian demonstrations, explain an effect as arising necessarily from a cause. But Fr. 4 alone does not enable us to draw this conclusion, especially if interpreted minimalistically, as Huffman does. For inferences on the basis of the association between numbers and qualitative features can go in both directions. Cut a monochord string in half; one knows one will get an octave. Play an octave interval on a monochord; one knows one has reached the half-way point. If this is as far as Philolaus’ demonstrations go, he would have been oblivious to the question of *why* certain ratios are or are not concordant.

In his recent *Plato and Pythagoreanism*,⁴⁴ Horky points out that this is precisely the criticism that Socrates levels against Pythagorean acoustics, in the *Republic*: “They don’t investigate, for example, which numbers are consonant and which aren’t or what the explanation is of each (διὰ τί ἑκάτεροι)” (531c).⁴⁵ Certain physical ratios result in sounds with a certain pleasing phenomenological character, and whenever certain sounds when sounded together have this character, certain ratios are to be found. Left unexplored is which is responsible for which, and what it is about the concordant ratios, or sounds, that lies behind them having the character by which they are grouped together. Horky

44 Horky 2013.

45 Tr. Grube and Reeve, 1992.

associates this with Socrates' dissatisfaction with the "mystic secrecy" of the *akousmata*,⁴⁶ and offers as evidence of Plato's dissatisfaction with Philolaus' more exoteric accounts a passage in the *Phaedo* in which Cebes relates to Socrates that in the *logoi* of Philolaus and his associates he heard "nothing clear" (σαφές οὐδέν) (61d-e). Horky writes that, "by this he seems to mean that Philolaus' arguments were not *demonstrated precisely* in accordance with a rigorous philosophical methodology."⁴⁷

Perhaps Philolaus did not work through lines of causal dependence with as much perspicuity as Plato did, in identifying Forms as explanatory of various phenomena, or as Aristotle thought that demonstrations in the sciences do. But that is not to say that Philolaus, and, by implication, other *mathematikoi* who followed him, were oblivious to the relation of causal and explanatory priority. They took note of this, and they recognized that the sort of accounts by which we make the world intelligible must distinguish what is causally prior from that which is caused – even if there is no clear evidence that they used the term *apodeixis* (demonstration) to refer to such accounts. In order to show this, I begin by briefly considering the generally accepted lines of interpretation of Philolaus' philosophy as developed by Huffman: Philolaus is a natural philosopher in the tradition of the Milesians. He aims at rationalistic accounts of the origin of familiar, experienced features of the world. While each of the Milesians identified one or more kinds as *arkhai*, principles from which other things emerged, Philolaus refrains from identifying any particular kind of stuff as an *arkhē*, because he thinks that there is no such stuff. Rather, there is a plurality of kinds of stuff, unlimiteds, that are such as to take on determinate characteristics, by which there arise things, and characteristics of things.⁴⁸ He accordingly identifies as principles "unlimiteds" in general, supplementing them with "limiters," principles of determination. The two alone do not suffice; there is also required a *harmonia*, a fitting together, of the limiter and the unlimited. This harmony is not itself called a principle; it is what arises when a limiter actually limits an unlimited. This harmony leads to, or, perhaps is,⁴⁹ the being of the explanandum. Both a limiter and an unlimited are principles, as starting points, those items that must preexist that for which they are principles. But, as Huffman has argued, "principle" must have another sense here too, one familiar from the Hippocratic corpus.⁵⁰ The limiter and unlimited are principles,

46 Horky 2013, 173.

47 *Ibid.*, 171.

48 See Barnes 1983, 387ff.

49 This is the suggestion of Horky 2013, 146.

50 Huffman 2006, 78–92.

as causes. They are not simply necessary conditions for what emerges; they are themselves responsible for the characteristics of that which emerges.

The knowledge of the arithmetical foundation of the musical scale exhibited by Fragment 6a indicates that among Pythagoreans, Philolaus, at the very least, took number as an example of such limiter, at least in the context of sound. According to Aristotle, what Philolaus elsewhere identifies as the principles of number, even and odd, are to be understood as principles of unlimited and limiter, respectively (*Metaph.* A 5 986a23-4); numbers when harmonized (as when musical notes are arranged in a scale) will also be a being of some sort. So, we have a hierarchy. Regardless of the ontological status of these principles, odd and even are principles of number, number is a principle of harmony, and harmony is in turn a principle of musical structures. We have progressed far from accounts that simply identify the symmetrical relationships between items in the same column of opposites. The relationship between principle and that of which it is a principle is asymmetric. The derivative entity is understood only as resultant from its principles, but not vice versa. Like Aristotelian demonstrations, Philolaus' accounts do more than justify their conclusions, they explain them. If Horky is right, and Plato condemns Philolaus' accounts on the grounds that they are "not clear," it is not because the fundamental distinction between causal principle and explanandum has not been made. In distinguishing between principle and derivative entity, Philolaus is signaling a recognition of the distinction between the sort of the thing that is a cause and the sort of thing that is an effect.

Who first employed the term *aitia*, or a variant, in a way that recognized the asymmetrical distinction? We are not sure, but it may well have been another thinker associated with the Pythagorean tradition, Plato's friend Archytas.⁵¹ The story, again, begins with the Pythagorean table of opposites.

In his *Physics* commentary, Simplicius comments on Aristotle's refutation of unnamed predecessors who identified motion with items belonging to the "second column": difference, inequality and non-being. Simplicius tells us that this is a reference to the second column of the Pythagorean table of opposites. He reports that Alexander faults Plato with having made this same mistake. Simplicius proceeds to clear Plato of this charge by showing how Plato himself takes the motion of the elements to result from their disequilibrium, not the other way around. Accordingly, Plato did not identify motion and inequality. According to Simplicius, Plato's realization that inequality stands in an asymmetrical causal relationship with motion follows Archytas. In support, he cites

51 Plato, *Seventh Letter*, 350a.

testimony from Eudemus, who, although mistaken in taking Plato to identify motion and its cause, reports that Archytas did make the appropriate distinction. According to Eudemus,

Plato says that motion is the great and the small and not-being and the uneven and as many things as have the same force as these (ὅσα τούτοις ἐπὶ ταὐτὸ φέρει). But it seems paradoxical to say that motion is just this. For, when motion is present, that in which it is present seems to be moved, but, when something is unequal or uneven, it is ridiculous to require in addition that it is moved. It is better to say that these are causes [of motion] just as Archytas does. (431.4-12)⁵²

As Huffman has argued against Burkert,⁵³ Eudemus is distinguishing between Archytas and those Pythagoreans who posited the table of opposites. Although Archytas recognizes that inequality is associated with motion, he is said to differ from these Pythagoreans insofar he recognizes it is unequal distribution that is the origin of motion, and not vice versa. The evidence from Eudemus via Simplicius shows us that Archytas explicitly indicates that the table of opposites is deficient as a theoretical account of the sorts of regular associations found in the world; even if the opposites that the table identifies are principles of a sort, the table fails to distinguish which of these are more causally basic than others.

4 Unmediated Steps in Demonstration

The fourth feature of Aristotle's account demonstration that can be traced to Pythagorean thought is the need for demonstration to be made up of atomic inferential steps: No step in a demonstration should be such that it is itself subject to demonstration. Again, the evidence is less certain than one would hope, but in my view it indicates that it is likely that the source of this insight is again, the Pythagorean Archytas.

The first extant instances of the term *apodeixis* being used in an epistemological or metatheoretical context, in which the author does not simply refer to proof but reflects on their epistemological force, are found in passages by Plato and Archytas. The two were friends and contemporaries, and the dating of the

⁵² The translation is from Huffman 2005, 508.

⁵³ Huffman 2005, 509–12, responding to Burkert 1972, 47, n. 106.

respective texts is uncertain. It is Plato, however, who apparently employs the term in a less rigorous fashion. As noted above, he sometimes employed it in regard to a nonprobative argument. DK B4 4 of Archytas, in contrast, employs the term as a way of pointing to the clarity, completeness, and revelatory nature of inferential mathematical lines of thought. Archytas offers important clues concerning thoughts on what features a demonstration must have to accomplish its cognitive goals. DK B4 4 reads

καὶ δοκεῖ ἄ λογιστικά ποτὶ τὰν σοφίαν τῶν μὲν ἄλλαν τεχνῶν καὶ πολὺ διαφέρειν, ἀτὰρ καὶ τὰς γεωμετρικὰς ἐναργεστέρῳ πραγματεύεσθαι ἃ θέλει. *** καὶ ἃ ἐκλείπει αὐτὴ ἄ γεωμετρία, καὶ ἀποδείξιας ἄ λογιστικά ἐπιτελεῖ καὶ ὁμῶς, εἰ μὲν εἰδέων τεὰ πραγματεία, καὶ τὰ περὶ τοῖς εἵδεσιν

I offer a provisional translation:

And it seems that logistic is indeed far superior to the other crafts in regard to wisdom, and deals with it intends to deal with in a clearer manner than does even geometry ... And again, in those respects in which geometry is deficient, logistic completely works through demonstrations, and likewise, if there is something that deals with shapes, (logistic completely works through) matters dealing with shapes, as well.

Much is unclear here. First, we have a reference to mathematical demonstrations. There is an explicit reference to the demonstrations of “logistic,” which is said to give more adequate accounts than does geometry. The implication is that geometry gives demonstrations too, but that those of logistic are superior. What these demonstrations of logistic are is a contentious issue; as we shall see, there is still unclarity concerning exactly what logistic is. There is a bit more evidence concerning the history of geometry, but the evidence is indirect and incomplete. The earliest geometrical proof may have simply consisted in diagrams that show certain relations.⁵⁴ We are however on sure ground in attributing inferential proofs to geometers contemporary with Archytas, for they were aware of the incommensurability of the side of a square with its diagonal, and there is no way for this to have been determined except through a one of a number of possible proofs that this must be so.⁵⁵ Further, we are on sure ground in attributing some of the first steps of the axiomatization of geometry

54 For a speculative reconstruction of the sorts of proof (δείξεις) that the Pythagoreans may have employed prior to the first stages of the axiomatization of geometry, see McKirahan 2013, 180–2. Units, when arrayed in certain ways, show (δεικνύναι) certain regular relationships; one can “see” that these relationships can be generalized.

55 On the first proofs of the incommensurability of the diagonal, see Mueller 2006, 703–4.

to Archytas' contemporary Hippocrates of Chios, who is reported to be the first geometer to compile an *Elements* (Proclus, in *Eucl.* 66.7). There is however no evidence that geometers at the time of Archytas would have attempted to ground such proofs on a determinate set of basic premises that were not themselves subject to proof.

The next question to face in interpreting this fragment concerns craft *logistikē*. Two interpretations have emerged in the literature. The first, championed by Neubager, identifies *logistikē* with the techniques of Babylonian algebra which, on the basis of how closely much of Book 2 of Euclid's *Elements* parallels Babylonian modes of calculation, must have been taken over by the Greeks at an early period. Neugebauer explicitly cites Archytas' fragment in this regard. On his view, Archytas' meaning is that the sort of algebraic techniques that had been borrowed from the Babylonians are able to solve problems that purely geometrical techniques cannot.⁵⁶ As Rowe explains, "His point was that rigorous axiomatic reasoning in the style of Euclid arose rather late, and that Archytas, a contemporary of Plato, was bearing witness to the primacy of algebraic content over the geometrical form in which the Greeks dressed their mathematics."⁵⁷ The second, developed by Klein, on the basis of an analysis of Plato's *Gorgias*, which contrasts *logistikē* and *arithmetikē*, takes *logistikē* to be the study of numbers in relation to one another, as opposed to the study of the internal character of a countable assemblage.⁵⁸ As such, it would encompass most of what is now referred to as "arithmetic," as well as ratio theory. Neubager's account has the advantage of having some basis in the evidence, such as it is, concerning how the ancient Greeks actually figured out mathematical problems. But its direct support is tenuous at best, and the view that any inferential structure was incidental to the nature of Greek geometrical thinking is gainsaid by Aristotle's *Posterior Analytics* (written only a generation after Archytas), the first book of which takes geometry to be paradigmatic for the inferential structure of science. That geometrical proofs existed and were recognized as such shows that it would have been a commonplace that geometry is not a matter of the application of recognized tricks in the determination of relationships, but is rather based on inferences from accepted premises. For this reason, I, following Huffman,⁵⁹ accept Klein's view that *logistikē* here refers to the study of numbers in relation to one another, and is here being contrasted with the study of magnitudes as arrayed in space, in relation to one another.

⁵⁶ Neugebauer 1936, 245–259.

⁵⁷ Rowe 2010, 133.

⁵⁸ Klein 1968, 17–25.

⁵⁹ Huffman 2005, 240–4.

This science is not a mere technique that allows a certain answer to be generated; it is, as Plato puts it, a study of realities, as is geometry. Although we do not have definitive support for it, the most natural reading of Archytas Fr. 4 is to take *logistikē* to be an inferential variety of mathematical geometry, since, as we have seen, demonstrations in the contexts of rhetoric and medicine were understood as inferential proofs. Archytas is surely not making the point that a non-inferential mode of mathematics is superior than an inferential mode. He is rather saying that the demonstrations of *logistikē* are more adequate than those of geometry. In what sense?

There are two major possibilities of interpretation. According to the first, the proofs of *logistikē* are more complete or perfect on formal grounds. According to the second, Archytas' remarks have nothing to do with formal consideration; his point is that the proofs of *logistikē* are deficient in some other respect.

The first line of interpretation is the traditional one, although scholars have been at a loss in saying what exactly Archytas thought was structurally lacking in the proofs of geometry. It has recently been argued against by Huffman. According to Huffman, Archytas' point is that numerical calculation accomplishes more than geometry in regard to the practical enterprises faced by human beings. On this account, when Archytas unpacks his claim that the demonstrations of *logistikē* are more fully carried out than those of geometry on the basis that they are more clear or evident (ἐναργής) the point is not that it is the conclusion that is more evident or clearly shown, but that it deals with its objects in a more vivid or concrete way. It brings the matter home, as it were.

Huffman's argument, like all arguments for a certain interpretation of Archytas, is circumstantial. He offers the following as evidence for his interpretation. First, in DK B 3, Archytas praises *logistikē* for its use in matters of equity. "Once calculation was discovered. It stopped discord and increased concord. For people do not want more than their share, and equality exists, once this has come into being. For by means of calculation we will seek reconciliation in our dealings with others ..." ⁶⁰ Archytas goes on to explain how those with a command of *logistikē* will be able to detect the injustices of others, and the awareness on the part of the unjust of their knowledge will act as a deterrent against unjust deeds. ⁶¹ This understanding of the value of *logistikē* is consonant with Huffman's interpretation of DK B 3. But it certainly does not clinch the case.

60 Huffman 2005, 183.

61 *Ibid.*, 235–6.

Stronger evidence is found in Huffman's survey of the use of the term ἐπιτελεῖν, which most commonly means "carry out" or "to put into effect" something that exists only "in thought or in word."⁶² Examples are the carrying out of commands, the fulfilling of oracles, the fulfilling of a promise, or the putting into effect of a design. Interpreting the fragment along these lines, it tells us that the proofs of *logistikē* are put into effect in a manner different from, or to a lesser extent than, geometrical proofs.⁶³ It is not beyond the bounds of semantic possibility for Archytas' meaning to be that what *logistikē* reveals is only truly revealed when put into effect. After all, a good example is sometimes what is needed in order to bring a general point home; after hearing that example one might say "now I see what you mean." But if this were Archytas' meaning, he would be making two separate points in regard to *logistikē*. The first is that it finds its completion in attaining its intended use within practical human affairs, and second, that it has maximal cognitive value when seen as applied to human affairs. Presumably on this interpretation Archytas supports each of these points in other passages.

One problem with this argument is that in all of the cases cited by Huffman, in which the verb ἐπιτελεῖν means "to put into effect" the putting into effect in question is part of the semantic content of the object of the verb. A command is the sort of thing that is uttered with the intention of its being followed. An oracle is the sort of thing that is uttered with the intention of having it be seen to be fulfilled. A promise is meant to be kept—or is at least understood as having that intention behind its utterance. A design is a paradigm that is intended to be followed in making something. This is not so in regard to *apodeixis*, the core meaning of which is a showing forth. Following the cues of the other uses of the verb, to ἐπιτελεῖν a demonstration is to *really* show something, to have its being manifest really, fully accomplished. The accomplishment in question is cognitive, not practical.

Within the *Prior Analytics* Aristotle talks of what it takes to ἐπιτελεῖν a deduction (*sullogismos*). It is not to apply it in everyday affairs; it is to *fully* deduce, which means to express the deduction in question via a "perfect" syllogism, or a sequence of them (*An. pr.* 1 24 26b29–30). The term "perfect" is τέλειος, which shares the root of ἐπιτελεῖν. Aristotle specifies a perfect syllogism by virtue of its being cast in a certain logical form, one according to which one "needs no external term in order to show the necessary result" (24b23–24). Scholars disagree about what it is about a perfect syllogism, as specified by Aristotle,

62 Ibid. 237–8.

63 Ibid. 249–50.

that he thinks allows it to satisfy this criterion, but the important point for our purposes is that a deduction is perfect on account of its having two features. First, its structure is such that it fully reveals why the conclusion follows as a necessary result from its premises. Second, no relevant premise is left out; in terms of Aristotelian syllogistic, no middle term is missing. Although neither the terms τέλειος nor ἐπιτελεῖν are employed in the *Posterior Analytics*, which is specifically concerned with the formal and epistemological character of those deductions that qualify as demonstrations, within that work the idea of perfect deductions is present and is fully worked through. Not only are demonstrations to be cast as first figure syllogisms (which are perfect deductions), their premises are to be unmediated. That is to say that no premises are implicit. In order for a demonstration to *really* show that its conclusion holds, and to show why it holds, it must be fully worked through.

I suggest that Archytas too is making the same point. To say that a demonstration is ἐναργής is to say that it is an account that *really* makes things manifest and shows things as they are. The demonstrations of geometry do not reveal geometrical truths with as much clarity as is desired because they have not been worked through as fully as they could have been. Steps are missing; the form of the exposition is somewhat obscure.

This interpretation has two main advantages. While on Huffman's reading Archytas is making two independent points, on the present, more traditional interpretation, his first point directly supports the second. Further, this line of interpretation is in accordance with Aristotle's usage of the term ἐπιτελεῖν when an inference (such as a demonstration) is its object.

But what could Archytas mean by saying that *logistikē*, unlike geometry, fully carries out its demonstrations? In order to answer this, I turn to the thought of predecessor and fellow Pythagorean Philolaus, and surviving Greek geometrical texts.

Recall Philolaus DK B 4: "And indeed all things that are known have number. For it is not possible that anything whatsoever be understood or known without this." As noted above, this fragment has traditionally been given a metaphysical reading, on the basis of Aristotle's declaration that the Pythagoreans took number to be the principles of things, that, ultimately, all things are number.⁶⁴ Huffman has recently argued for a more minimalistic interpretation, limiting itself to what is said in what can be reliably identified as Philolaus' words. On his account, Philolaus' point is epistemological. Things are known by virtue of their numerical relations. Restricting ourselves to this more conservative

64 Examples can be found in Guthrie 1979, 181–305; Barnes 1983, 378–83.

reading, Philolaus' point would be that those aspects of a thing that are intelligible to us are the ones that are grasped through the study of numbers. As we have seen, those studies are arithmetic and *logistikē*. The spatial aspects of things, studied by number, are missing—unless it is the case that those spatial aspects are at bottom numerical aspects. The only reason we might have to attribute to Philolaus this possibility is the metaphysical reading of DK B 4: if all things are number, the spatial aspect of reality, too, is to be understood on the basis of number, and nothing besides number (as there is no such thing). In this case, there is a gap between the kind of accounts that arithmetic and *logistikē* give us and what it is that we aim to know. On the more restrictive reading of Philolaus, if we take Archytas to be building on Philolaus, he would be saying the geometry tries to discuss aspects of the world, such as spatiality, that ultimately cannot be fully understood. The demonstrations of geometry are necessarily incomplete as they try to understand what is not numerical, and accordingly cannot be understood. If we take Archytas to be building on Philolaus, understood in the metaphysical sense, his point would be that in principle the spatial aspects of things can be understood through the sciences of number. But, as they are, they do not do so. The only aspects of number that they do fully make intelligible are their internal character (as odd, even, prime and the like) and the relationships that they have to one another, as made clear through addition, subtraction, multiplication, division, and theory of ratios. Once one's study of spatial figures goes beyond these attributes, one's demonstrations will have gaps.

Archytas likely did not specify the respects in which he thought geometrical demonstrations fall short, for we have no evidence of what he thought was required to *really show* a mathematical proposition, that is, what it takes for there to be a mathematical demonstration. Insofar as he worked at a time in which the incommensurability of the diagonal would have been common knowledge, he would have been aware of what we now recognize as genuine mathematical proofs, and there is evidence that he himself demonstrated how to duplicate a cube.⁶⁵ It fell to Aristotle to lay out with precision the

65 DK A 14 and 15, on which see Huffman 2005, 342–401. The evidence is uncertain here. Eratosthenes reports that Archytas offered a demonstration, but that this was not a sufficient answer to the architectural challenge posed by the Delian oracle that his account was meant to address: to come up with a practical way of building a cubical space of double the volume of another cube. On the other hand, Plutarch tells us that Archytas was able to solve the practical problem, but that his solution was deficient insofar as it lacked the theoretical rigor of a demonstrative proof, suggesting that Plato had Archytas in his sights at *Resp.* 7 527c–528b when he suggests that Pythagorean mathematicians had an undue focus on practical application and neglected theorization concerning intelligibles.

conditions that must be met for the proposition in question to be shown in a strong sense, that is, proven, but that is not to say that such an account is required for one to argue that a certain argument does not in fact constitute a proof. Jurors and attorneys with no training in logic often are able to make apt criticisms of the arguments of others, declaring that the case is not proven. Typically, such an objection takes the form of challenging a stated or implicit premise: “how do you know that such and such might not be the case”? As Aristotle sees in *An. post.* 1.3, such objections are forestalled only if all premises are either such that this sort of question simply cannot arise (that is to say, they are principle), or follow of necessity from that sort of premise. Archytas’ point, I suggest, is that premises concerning spatial relations give rise to “how do you know that?” sorts of objections, in a way that premises concerning numbers do not, at least for those who have mastered the arts of arithmetic and calculation.

In making such an observation, Archytas would have been anticipating the sorts of objections made against antiquity’s most famous fully worked out body of proof, Euclid’s *Elements*. Bertrand Russell famously condemned the book for its inferential gaps;⁶⁶ it took Hilbert to more adequately work through the ultimate premises of geometry so that its demonstrations were in fact fully completed, to the point where there are no gaps. Mueller has given a partial defense of the *Elements* by arguing that the diagrams Euclid appeals to play a crucial role; the proofs are incomplete without appeal to what intuition reveals, when one makes use of diagrams.⁶⁷ If fully worked through mathematical proofs fulfill their function of cognitively revealing quantitative relations, even for those not in possession of logical theory, it stands to reason that one can be aware that there has not been maximal cognitive revelation, even in the absence of such a theory. Archytas, I suggest, was aware that this is the case in regard to geometrical proofs, in a manner, or to an extent, that is not the case in regard to demonstrations concerning numbers. But the account of

Huffman speculates that the two reports can be reconciled by the thesis that Archytas did indeed offer a legitimate mathematical demonstration, but that it was motivated by practical concerns; it was not part of a body of proof that would constitute a science of solid geometry. This interpretation is in accordance with Huffman’s interpretation of DK B 4: for Archytas, science is primarily oriented towards making sense of the particulars before one. I omit consideration of these texts in my account of Archytas’ role in the history of demonstration, on account of the unclear conflicting evidence they offer, but they do offer some collaborating testimony that Archytas was aware of the sort of geometrical account that later philosophers recognized as demonstrative.

66 Russell 1902, 165–167.

67 Mueller 1969, 289–309.

demonstration that would allow for a more precise account of what exactly is missing waits for Plato and Aristotle.⁶⁸

Three key elements of Aristotle's theory of demonstration have Pythagorean antecedents. Demonstration is a revelatory discourse that is 1) inferential, 2) explicitly based on premises that are not themselves demonstrated on the basis of more basic premises, and 3) explanatory, insofar as the premises express those basic facts that are explanatory of the conclusion. Missing is the requirements that inferential steps be explicit and complete, and an account of the sorts of premises that ground demonstration; according to Aristotle, the primary role is played by definition. The centrality of definition to explanation is an insight that derives not from the Pythagoreans but from Plato.⁶⁹

As in other aspects of his philosophical thinking, the great achievement of Aristotle is a systematic synthesis of the insights of his predecessors, of such richness and complexity that it opens up new philosophical puzzles and problems that these predecessors could never have pondered.⁷⁰

68 As far as we know, Plato was the first to explicitly indicate that demonstrative proofs must rest on undemonstrated casual principles. We see this in the Divided Line passage of the *Republic*, in which we are told that mathematical reasoning rests on starting points or principles that are "hypotheses," to be distinguished from the beings or truths that are deduced on their basis, and that mathematicians err in taking their starting points to be such hypotheses (*Resp.* 6 510b-511d). Plato presumes that we are familiar with "hypothesis" as a term that the mathematicians themselves used. The hypotheses of the mathematicians are said to be opposed to the unhypothetical first principle attained by dialectic. This unhypothetical principle is usually identified with the Form of the Good, which is said to be the cause of the being of things. The difficulties of this passage are many, but it is clear that Plato has here anticipated the third and fourth aspects of Aristotle's theory of demonstration—mathematical accounts, in order to be fully adequate, are inferences on the basis of determinate causal and epistemological starting points. (What Plato did not recognize, according to Aristotle, was that different sciences, concerned with different genera, must have different starting points.) Likewise, in the *Theaetetus*, we are told that a science involves belief (*doxa*) in the context of a certain *logos*, which must be grounded on direct apprehension (*αἴσθησις*) of elements (*στοιχεῖα*) (202b); in at least one example of how this works, that in which the elements are the material constituents of a things, as the wheels and so forth are in respect to a wagon, (207a-c), these elements are in some sense responsible for the object known (even if Plato does not here use the word "cause" to refer to them). Plato is again discussing themes present in the epistemology of Pythagoreans prior to or contemporary to him, although the chronology and matter of who influenced whom cannot be determined.

69 On this see especially Ferejohn 2013, 2013.

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