

Fabio Acerbi

Composition and Removal of Ratios in Geometric and Logistic Texts from the Hellenistic to the Byzantine Period

Abstract: The historical path of the operations of ‘composition’ and ‘removal’ of ratios from Greek antiquity to Byzantine times is followed by presenting and briefly describing the extant historical evidence.

Keywords: Greek mathematics; Byzantine mathematics; Ratio.

I Introduction

What follows is a long essay on the monolithic inertia of the ancient Greek approach to mathematics. One might speculate about the social or cultural origins of such a formidable inertia, but what I would like to offer in this paper is simply a series of facts and documents attesting to its existence. Of course, the only reasonable way to do this is to pick up a specific conception or tool, and then check whether its pattern of evolution, from Greek antiquity to Byzantine times, is continuous or discontinuous. It so happens that concepts or techniques suitable to serve as examples are quite difficult to find: to the best of my knowledge, the notion discussed in the present paper is the most suitable one—if not the only available one over a long time span; in this sense, my analysis is tendentious. Whether from this example one may infer that *no* revolution *could* take place in the field of logistic (or even in Greek mathematics as a whole), and, if so, why, I leave it up to the reader to decide.

To be more precise, the present study has two particular aims: (a) to follow the historical path of the operations of ‘composition’¹ and ‘removal’ of ratios from Greek antiquity to Byzantine times; (b) to present and briefly describe the extant historical evidence. These operations are both first attested in the context of geometric proofs, since they are two species of a small constellation of standard manipulations of ratios and proportions. In particular, composition of ratios bears a misleading resemblance to the operation named *ex aequali* (ἐκ ἰσότητος), whose theoretical foundations are laid in *Elem.* V.20 and 22. After a puzzling first appearance in the *Elements* and a handful of applications in the Archimedean corpus, compounded ratios were used quite

¹ As it will be shown below, ‘composition’ was never regarded as an operation—with a couple of noteworthy exceptions. I employ the term here for the sake of being concise.

frequently—and apparently with systemic aims—by such authors as Apollonius and Pappus.

The golden age of composition and removal of ratios came along with the proliferation of ‘logistic’² texts that characterized the Late Antiquity and the Byzantine period. These texts have the form of computational primers to the *Almagest*. Expositions of composition and removal of ratios were included in them because the operation of removal is a crucial step in all computations involving the Sector Theorem, which was, in its turn, the key mathematical tool of the entire *Almagest*.

In order to be transferred from proportion theory to logistic, composition and removal of ratios—and, in fact, the notion of ratio itself—were adapted and modified, by introducing the crucial notion of *πηλικότης*; namely, the ‘value’ or ‘size’ of a ratio, i.e. a numerical object.

Two historiographic points of some interest result from an analysis of the sources I am going to present. First (and this is a unique example in ancient Greek mathematics), the evolution of a concept can be followed closely from its original, proportion-theoretical context to a strictly computational framework—in which, most notably, all ratios are numerical objects. Second, a standard procedure of removal of a numerical ratio from another did not exist: on the contrary, several procedures are attested, more or less markedly different from one another. Such a variety originated from three facts: first, all sources insisted on viewing removal as an operation on ratios provided in compounded form; second, the applications of the Sector Theorem introduce the additional constraint that one of the terms of the ratio resulting from the removal is assigned; third, and most importantly, the idea that removal may be reduced to taking a suitable fourth proportional of three assigned numbers can be implemented in several ways. The latter feature shows the extent to which the conception of a ratio as a relation—and not as a numerical object—was deeply entrenched in mathematical practice, despite the introduction of the *πηλικότης*. As a matter of fact, the idea that the *πηλικότητες* can be effectively used in order to shortcut and unify all these procedures—the result of a removal of one ratio from another being simply the ratio of

² According to Eutocius, dividing the unit does not pertain to arithmetic but to logistic (*AOO* III, 120.28–30: ὥστ’ ἐπ’ ἐκείνων [*scil.* superparticular and superpartient ratios] διαφετέον τὴν μονάδα, ὃ εἰ καὶ μὴ κατὰ τὸ προσήκον τῇ ἀριθμητικῇ ἀλλὰ τῇ λογιστικῇ τυγχάνει). An earlier definition of logistic—which almost certainly can be ascribed to Geminus (a first-century BC mathematically-minded philosopher and polymath, maybe a pupil of Posidonius)—does not allow dividing the unit. This definition can be found in pseudo-Hero, *Def.* 135.5–6 (*HOO* IV, 98.12–100.3), and is also preserved, by a different line of tradition, as a scholium to Plato’s *Chrm.* 165E6 (Scholium 27 in Cufalo 2007, 173). It is possible that the domain of logistic was expanded to include fractional parts as a consequence of the adoption of the sexagesimal system in Greek mathematical astronomy. This happened some time around Hipparchus’ life span, which certainly included the interval [147–127] BC. The best introduction to Greek logistic is still Vogel 1936.

the ‘values’ of the two ratios—is stated and proved only in Barlaam’s treatise, whose brilliant exposition remained, however, isolated and unparalleled.

This study is a first attempt to expound in a compact way some results coming from my own editorial work on sources relevant to the issue, in particular the *Almagest* scholia, both *vetera* and Byzantine, and the Ur-text of the Pseudo-Domninus there discovered (cf. Acerbi & Riedlberger 2014); the *Prolegomena to the Almagest* (henceforth ‘*Prolegomena*’; cf. Acerbi *et al.* 2010); the *Isagoge* of George of Trebizond;³ the treatises of Theodoros Metochites and John Chortasmenos; a short text of George Gemistus Plethon (cf. Acerbi *et al.* 2016).

The presentation of the material is divided into two broad sections, each one dealing, respectively, with the geometric and the logistic approach to composition and removal of ratios. The first section contains the evidence found in Euclid (third century BC), Archimedes (287–212 BC), Apollonius (early second century BC), Ptolemy (ca. 150 AD), Pappus (ca. 320 AD), and Theon of Alexandria (ca. 360 AD). I also discuss some relevant features of this evidence. The second section deals in detail with general expositions and worked-out examples, found in late logistic texts. The authors whose contributions are presented and discussed in this section are: Nicomachus (second half of the second century AD); again Theon of Alexandria; Eutocius (early sixth century AD); the scholiast of the *Almagest* whose text on removal of ratios was anonymously reworked and came to be ascribed to Domninus of Larissa (middle fifth century AD, but the scholium comes from the same scholarly circles as the *Prolegomena*); the anonymous author of the *Prolegomena* (early sixth century AD); again an anonymous scholiast of the *Almagest*; Leo the Mathematician (early ninth century AD); some scholiasts to the *Elements*, among whom Maximus Planudes (second half of the thirteenth century AD); George Pachymeres (second half of the thirteenth century AD); Manuel Bryennios (early fourteenth century AD); Theodoros Metochites (early fourteenth century AD); Barlaam of Seminara (middle fourteenth century AD); Demetrios Cydones (second half of the fourteenth century AD); John Chortasmenos (early fifteenth century AD); George Gemistus Plethon (early fifteenth century AD); George of Trebizond (middle fifteenth century AD). Quite surprisingly, the issue of composition and removal of ratios does not at all feature in the computational primer to the *Almagest* written by Theodoros Meliteniotes, a distinguished scholar contemporary with Demetrios Cydones (ed. Leurquin 1990). Less surprisingly, the same omission appears in an anonymous computational primer that proves to be a compilation of extracts from some of the authors just mentioned (ed. Moll 1965), and in Maximus Planudes’ *Great Calculation According to the Indians* (Allard 1981) and in

³ Critical edition, with an introduction and a commentary, of the relevant *Almagest* scholia in Acerbi 2017. I am also preparing a critical edition of the *Isagoge* of George of Trebizond, which is attested in two, quite different, versions in Latin and in Greek.

its anonymous source (Allard 1977): these two treatises bear no connections with the *Almagest*.

The choice of the presented authors was made on the basis of mere linguistic continuity: they all write in ancient Greek. Still, it is worth mentioning the sophisticated general exposition of composition of ratios, characterized by a markedly combinatorial approach, redacted by Thābit ibn Qurra (Lorch 2001).

II Geometric Approach

II.1 Euclid's *Elements* and the *Sectio Canonis*

The standard definition of 'compounded ratio' is *Elem.* VI.df.5: "A ratio is said to be compounded of ratios when the values of the ratios, multiplied by one another, make some <value of a ratio>".⁴ Heiberg, on good grounds, though the definition spurious:⁵ it is never used in the rest of the *Elements*; it is placed after definition 2 in the Theonine manuscripts Bodl. Dorv. 301 and Par. gr. 2466; it is only transcribed, with a sign referring to the last line of definition 4, in the lower margin of f. 86r of the unique non-Theonine manuscript Vat. gr. 190. Some remarks are in order:⁶

- The compounding ratios are not 'multiplied': what is 'multiplied' are their *πηλικότητες*.⁷ For this reason, when writing a compounded ratio in symbolic form, I shall put the standard mathematical sign of composition 'o' between the two compounding ratios, thus: $a:b = (c:d) \circ (e:f)$.⁸ As we shall see below, the formula stands for the expression "the ratio of a to b is compounded both of the ratio of c to d and of that of e to f ".
- A ratio is said to be 'compounded' of two ratios: it neither 'is' the two compounding ratios, nor it is 'equal' or 'identical' to them. The sign '=' in the above formula is thus misleading and I have introduced it *faute de mieux*. I shall call 'left-hand side' and 'right-hand side' what appears on the left or right side of the sign '=', respectively.

⁴ Λόγος ἐκ λόγων συγκεῖσθαι λέγεται, ὅταν αἱ τῶν λόγων *πηλικότητες* ἐφ' ἑαυτὰς πολλαπλασιασθεῖσαι ποιῶσι τινα; *EOO* II, 72.13–15. My translation of *Elem.* VI.df.5 includes a final integration based on Theon, in *Alm.* I.13, iA, 533.1–2, who is the only independent source from the Late Antiquity that completes the final *τινα* of *Elem.* VI.df.5 with *πηλικότητα λόγου*.

⁵ See *EOO* II, 72.7 and 73.13–15 *app.*; see also 73 n. 2.

⁶ See Saito 1986 and Sylla 1984, 18–19 and 22–23, for earlier discussions.

⁷ As we shall see, Greek arithmetic theory employs an 'additive' lexicon to conceptualize and formulate these operations on ratios.

⁸ I shall write a ratio in the form $a:b$ and the corresponding *πηλικότης* in the form a/b .

- Despite the fact that VI.df.5 reduces ‘composition’ to a multiplication, a compounded ratio was never conceived of as the result of an *operation* on ratios. With reference to the formula above, the two ratios $c:d$ and $e:f$ are not ‘compounded’ to produce $a:b$; what is ‘compounded’ is $a:b$ itself, while $c:d$ and $e:f$ are the ‘compounding’ ratios. In fact, if an operation is at issue here, this is not a ‘composition’, but a ‘de-composition’ of $a:b$ into component ratios. This conceptualization is strictly adhered to in all our sources and had momentous consequences from a mathematical point of view. Only in later sources the noun ‘composition’ came to be used, but only as a convenient shortcut, as I have done in the introductory section.

In the *Elements*, compounded ratios only feature in propositions VI.23 and VIII.5, whose enunciations and proofs are strictly parallel. These two propositions, one must stress, are deductive dead ends. Let us read their enunciations:⁹

VI.23. Equiangular parallelograms have to one another the <ratio> compounded of <those of> their sides.

VIII.5. Plane numbers have to one another the <ratio> compounded of <those of> their sides.

Before discussing the proofs of these propositions, it is better to have a look at the standard designation of a compounded ratio; this will also allow us to fix some conventions. Let us read the first occurrence of the formula, that I shall call ‘canonical’, in the *Elements*:¹⁰

ὁ τῆς Κ πρὸς Μ λόγος σύγκειται ἔκ τε τοῦ τῆς Κ πρὸς Λ λόγου καὶ τοῦ τῆς Λ πρὸς Μ
the ratio of K to M is compounded both of the ratio of K to Λ and of that of Λ to M.

This expression has the following, noteworthy features:

- The predicate is a form of ‘to be compounded’ (συγκεῖσθαι), usually used as the passive of συντιθέναι. It is often replaced by the synonym ‘to be conjoined’ (συνάπτεσθαι).
- The correlative structure ‘both ... and’ (τε ... καὶ) links the two compounding ratios more tightly than a simple ‘and’ (καί).
- The term ‘ratio’ (λόγος) is often understood, especially in the designations of the compounding ratios.
- The article that determines λόγος is always present, even if the noun is understood. The double article τοῦ, which would precede any of the compounding ratios when numbers (masc.) are at issue, is frequently simplified. For instance, we read: ἔκ τε τοῦ Κ πρὸς Λ καὶ τοῦ Λ πρὸς Μ, and not ἔκ τε τοῦ τοῦ Κ πρὸς τὸν Λ καὶ τοῦ τοῦ Λ

⁹ *EOO* II, 146.9–10 and 286.2–3, respectively.

¹⁰ *Elem.* VI.23, *EOO* II, 146.24–25.

πρὸς τὸν M. Here, the first τοῦ in both occurrences of τοῦ τοῦ stands for τοῦ λόγου, while the second is an integral part of the designation of the first term of the ratio.¹¹

- Quite unexpectedly if one has *Elem.* VI.df.5 in mind, there is a common term Λ to the two compounding ratios. This common term was called ‘middle’ (μέσος). This name is unfortunate since it obviously interferes with the standard meaning of ‘mean <proportional>’ (μέσος) C of two terms A and B, namely, the term that realizes the continuous proportion $A:C::C:B$. Of course, a mean proportional can always be used as the middle term in a compounded ratio, but this does not need to always be the case: any term homogeneous with the terms of a given ratio can serve as a middle in order to write the ratio in compounded form. I shall say that a compounded ratio is ‘written in standard form’ when there is a common term to the two compounding ratios.

Anticipating on what we shall see in what follows, here are some further terminological points.

- I shall call ‘replacement of identical ratios’ in a compounded ratio $a:b = (c:d)^\circ(e:f)$ any substitution of a ratio with another identical to it: if $a:b = (c:d)^\circ(e:f)$ and $c:d::m:n$, then $a:b = (m:n)^\circ(e:f)$. No proof in ancient Greek sources validates this inference, that amounts to requiring that any definition of ‘compounded ratio’ is well-founded (that is, it only depends on equivalence classes of ratios). Note that we cannot apply V.11 (transitivity of identity of ratios) in order to validate this inference, simply because we do not know what kind of relation is the one (if any) holding between a ratio and any rewriting of it in compounded form—this is the relation I have written ‘=’. Nor can we apply the δι’ ἴσου proposition V.22, for the same reason as for V.11 and for the additional reason that V.22 allows operating on compounded ratios ‘written in standard form’ only.
- If a ratio is in common between two compounded ratios $a:b = (c:d)^\circ(e:f)$ and $m:n = (c:d)^\circ(p:q)$, as here $c:d$, it will be called the ‘shared ratio’. Of course, it is impossible for two *different* ratios $a:b$ and $m:n$ written in compounded standard form to have a shared ratio among their compounding ratios. If two identical compounded ratios admit of a shared ratio, then the other compounding ratios will also be identical: if $(c:d)^\circ(e:f) = a:b::m:n = (c:d)^\circ(p:q)$, then $e:f::p:q$. Operationally, this amounts to ‘removing’ the shared ratio (see sections II.2–4). No proof in ancient Greek sources validates this inference.

The proof of *Elem.* VI.23 runs as follows, where $p(a,b)$ is the parallelogram whose sides are a and b :

- 1) *Setting out and determination*: the equiangular parallelograms $p(a,b)$, $p(x,y)$ are assigned; to prove that $p(a,b):p(x,y) = (a:x)^\circ(b:y)$.

¹¹ Similarly, in the enunciations of *Elem.* VI.23 and VIII.5 we read λόγον ἔχει τὸν συγκεκριμένον ἐκ τῶν πλευρῶν and not λόγον ἔχει τὸν συγκεκριμένον ἐκ τῶν τῶν πλευρῶν; my emphasis.

- 2) *Construction*: set sides a and x adjacent and on a straight line. As a consequence, sides b and y are also adjacent and on a straight line; complete parallelogram $p(b, x)$.
- 3) *Construction*: introduce auxiliary straight lines K , Λ , M such that $K:\Lambda::a:x$ and $\Lambda:M::b:y$. Since a , b , x , and y are given, this can be done by taking a chance straight line K and by applying twice the construction of fourth proportional of two straight lines (VI.12) to find Λ and M .
- 4) *Proof*: Since $a:x$ and $b:y$ are the ratios of the sides of the assigned parallelograms, and since $K:M = (K:\Lambda) \circ (\Lambda:M)$, it follows that $K:M$ is compounded of the ratios of the sides of the parallelograms. This involves (i) a replacement of identical ratios applied to the two compounding ratios: $K:\Lambda \rightarrow a:x$ and $\Lambda:M \rightarrow b:y$; (ii) a change of designation from denotative letters to a definite description and a related shift in the wording of the formula: $\delta \text{ X λόγος σύγκειται ἔκ τε τοῦ Y καὶ τοῦ Z} \rightarrow \text{X λόγον ἔχει τὸν συγκεῖμενον ἔκ + \{definite description of Y\&Z\}}$. As we have seen, no proof in ancient Greek sources validates either of these inferences. For obvious reasons, I shall call the complex of steps (i)–(ii) ‘transfer of denomination’ on the right-hand side of a compounded ratio.
- 5) Two applications of VI.1 and V.11 give $p(a, b):p(b, x)::a:x::K:\Lambda$ and $p(b, x):p(x, y)::b:y::\Lambda:M$.
- 6) Application of the $\delta\iota'$ ἴσου proposition V.22 to $K:\Lambda::p(a, b):p(b, x)$ and $\Lambda:M::p(b, x):p(x, y)$ gives $K:M::p(a, b):p(x, y)$.
- 7) Transfer of denomination on the left-hand side (it involves the compounded ratio): since $K:M$ is compounded of the ratios of the sides of the parallelograms, also $p(a, b):p(x, y)$ is. Again, nothing validates this inference, that amounts to assuming that, if two ratios are identical, they can be said to be compounded of identical ratios. This is a form of replacement of identical ratios, applied to a discursive identification of the compounded ratio.

Here is a tabular representation of the proof; the numbers correspond to the steps just listed.

	$(a:x) \circ (b:y)$	=	$(K:\Lambda) \circ (\Lambda:M)$	=	$K:M$			$p(a, b):p(x, y)$	
$K:\Lambda::a:x$						::	$p(a, b):p(b, x)$		
$\Lambda:M::b:y$						::	$p(b, x):p(x, y)$		
3		4				5		6	7

Proposition VIII.5 is parallel to VI.23. The corresponding steps in the two propositions are listed in the following table:¹²

¹² *Elem.* VIII.5 also implicitly uses commutativity of multiplication, namely, *Elem.* VII.16. This has no counterpart in VI.23, since the proposition holds for ‘parallelograms’ (+ two-letter designation) and not simply for ‘rectangles contained by’ (+ three-letter designation). In fact, the geometric ‘counterpart’ of commutativity of multiplication simply amounts to a permutation of the letters denoting the parallelogram.

VI.23	completion of a parallelogram	VI.12	transf. denom.	VI.1	V.11	V.22 δτ' ἴσθι	transf. denom.
VIII.5	multiplication of sides	VIII.4	/	VII.17	/	VII.14 δτ' ἴσθι	transf. denom.

The completion of a parallelogram in VI.23 corresponds, in VIII.5, to multiplying one side coming from any of the plane numbers by one side coming from the other. Taking a fourth proportional of two straight lines—VI.12, but note that this step is not explicit in the construction—is made to correspond, in VIII.5, to finding the least three numbers containing two given ratios (VIII.4). The last proposition even deserves an instantiated citation in VIII.5: this shows that VIII.5 fits better the deductive structure of Book VIII than its counterpart VI.23 fits the structure of Book VI. Transitivity of identity of ratios (proposition V.11) is not proved in the arithmetic books of the *Elements*, but its validity is obvious, given the definition of numbers in proportion at VII.df.21.¹³ It is important to note that, in VIII.5, the entities having the same function the straight lines K, Λ, M have in VI.23 come out more ‘naturally’ than in the geometric case, since they are the least three numbers containing the ratios of the sides.

Both propositions were modified by Theon: he completed the setting-out of VI.23, in order to make the reference to the names of the sides explicit;¹⁴ and he completed the proof of VIII.5, by adding the transfer of denomination that corresponds exactly to step 4 of the proof of VI.23.¹⁵

The drawbacks of VI.23 are obvious: first, the introduction of the straight lines K, Λ, M—and hence the application of the δτ' ἴσθι—is unnecessary; second, the replacement of identical ratios, both on the left and right-hand side of the compounded ratio, is, as we have seen, an operation that is left unproven. Add to this that VI.df.5 is not applied at all in the proof.

As for the first point, it is enough to note that the proof works equally well if one directly introduces parallelogram $p(b,x)$ as a middle term in the ratio of parallelograms $p(a,b):p(x,y)$ and then argues by means of VI.1, replacement of identical ratios, and transfer of denomination. It suffices to read the Euclidean proof backwards to realize this.¹⁶ On the other hand, if one wants to keep the deductive arrow going from

¹³ The definition is a disjunction of conditions and only involves the relational operators ‘to be equal to’ and ‘to be identical to’.

¹⁴ See *EOO* II, 146.14 *app.* Theon’s supplement is unnecessary, since the names of the sides are univocally determined by the names of the parallelograms and by the names of their equal angles, and these names are provided in the setting-out. Moreover, Theon’s supplement shows that he misunderstood the logic of the proof, since there are two perfectly equivalent possibilities as to which sides are to be coupled in the compounding ratios, and this symmetry is spontaneously broken only in the construction. Note that the same overspecification is forced by my, mildly symbolic, formulation of the proof.

¹⁵ The deduction at issue in VI.23 is at *EOO* II, 146.21–148.1, to be compared with *EOO* II, 286.13 *app.*

¹⁶ Reasons for introducing the lines K, Λ, M are adduced in Mueller 1981, 154–155 (where the above alternative proof is also proposed), and Saito 1986, 32. The solution of this interpretive problem can only be a matter of guesswork.

sides to parallelograms, and assuming that a compounded ratio must be written in standard form, the introduction of K, Λ, M is necessary. Note that, contrary to what happens in all other geometric authors we shall be dealing with, in the proofs of VI.23 and VIII.5 no compounded ratio features that is not written in standard form:¹⁷ Euclid never operates ‘inside’ a compounded ratio; the transfer of denomination can be taken as a trick to avoid doing so.

The second point is more delicate and shows, either that the concept of a compounded ratio is not well-defined or that using it is a sheer verbalism. The way the subject is handled in the *Elements*, in fact, hides a trivial substratum, that will show up in the purely geometric exposition we shall read in Theon: it is simply obvious that any middle term can make a ratio a compounded ratio, and there is nothing to prove about this. If, however, one wants to perform operations on compounded ratios, like replacing identical ratios or removing shared ratios (as Euclid, Archimedes, or Apollonius do), the concept of compounded ratio must be grounded more firmly compared to how it is treated in the *Elements*. The introduction of $\pi\lambda\iota\kappa\acute{o}\tau\eta\varsigma$ and of VI.df.5 associated to it, apparently had the function of de-trivializing the whole affair. As a by-product—and since two ratios are identical if and only if their $\pi\lambda\iota\kappa\acute{o}\tau\eta\tau\epsilon\varsigma$ are equal—, this move will make any (misleading) use of $\delta\iota'$ ἴσου unnecessary, as we shall see in Barlaam's exposition (see section III.1.8). On the other hand, VI.df.5 will obviously give a decidedly logistic turn to the subject of compounded ratios and of the operations on them. Still, the subject will allow being founded on purely number-theoretical grounds, as Eutocius will do (see section III.1.2).

In the *Sectio Canonis*, a treatise ascribed to Euclid, compounded ratios are handled perfunctorily (cf. **Document 1** in the Appendix). The concept is involved in almost every proposition of the theoretical part of the treatise. It is always assumed that a compounded ratio (which is called ‘interval’ (διάστημα) in the language of harmonic theory adhered to in the *Sectio Canonis*) is written in standard form, and such that the middle term is also a mean proportional. In proposition 7, it is shown that a triple interval is compounded of a double and of a sesquialter interval; namely, $3:1 = (2:1) \circ (3:2)$. In proposition 8, it is shown that, if one removes a sesquitertian interval from a sesquialter, the remainder is a sesquioctave interval, namely, the assumed compounded ratio is $3:2 = (9:8) \circ (4:3)$. Both proofs argue by partition of the antecedent of a ratio once the consequent of the same ratio is taken as the unit, and by counting the ‘units’ resulting from taking a suitable multiple of both; for instance, in a sesquitertian ratio, the antecedent contains the consequent once and a third of it ($\frac{4}{3} = 1 + \frac{1}{3}$), and, therefore, 12 consequents are equal to 9 antecedents. A similar argument is used by Theon in his commentary on the *Almagest* (see section III.1.1).

¹⁷ Actually, in VIII.5, no canonical formula is present at all.

II.2 The Archimedean corpus

Compounded ratios in the Archimedean corpus can be found in a handful of propositions.¹⁸ In *Fluit.* II.10, a compounded ratio is first introduced; after a replacement of identical ratios, it is reduced to the ratio compounded of 2:5 and 5:1, which is stated to be identical to 2:1.¹⁹ In *Con. sph.* 10 (but this simply consists in an enunciation), 23, 24, and 31, a property analogous to one that follows from combining two theorems proved in the *Elements* is formulated in the language of compounded ratios; any of such ratios is subsequently transformed by replacement of identical ratios.²⁰ In *Sph. cyl.* II.4, a compounded ratio written in standard form is introduced, repeatedly transformed by replacement of identical ratios, and finally ‘simplified’ by ‘removal’ of a ratio shared with another compounded ratio.²¹ Only II.8 *aliter*, a proposition that, however, is almost certainly spurious, depends in a decisive way on a clever application of compounded ratios: at the beginning of the proof, a multiple compounded ratio between solids, and written in standard form with two middle terms, is introduced and subsequently transformed by replacement of identical ratios; one of these replacements involves an application of the theorem we read as *Elem.* VI.23.²² In the sequel of the proof, however, in order to express ratios of solids as compounded of the ratios of suitable surfaces and linear elements of the solids themselves, compounded ratios are replaced by expressions involving a ‘multiplication’ of these magnitudes.²³

When Archimedes introduces compounded ratios outside applications of previously stated theorems, he does insert a middle term in *Sph. cyl.* II.4 and II.8 *aliter*.

18 As for the verbal forms employed in the canonical formula, the Archimedean corpus offers 12 occurrences of forms of συγκεῖσθαι (they all are participial forms); 6 of forms of συνάπτεσθαι—of these, 5 are of the form συνήπτται (in *Sph. cyl.* II.4 (*quater*) and in II.8 *aliter*), and 1 of the participle συνημμένος (in *Sph. cyl.* II.8 *aliter*).

19 See *AOO* II, 388.12–14 and 390.1–4, respectively.

20 See *AOO* I, 304.13–18; 364.10–18; 364.22–26 (rightly bracketed by Heiberg) and 366.6–9; 368.25–370.3 and 370.4–8; 430.25–432.4 and 432.10–14, respectively. The property states that cones and their segments have to one another the ratio compounded of that of their bases and of that of their heights; it combines *Elem.* XII.11 and 14. As Pappus also does, and very much in line with a ‘solid’ counterpart of *Elem.* VI.23, in *Con. sph.* 10, 23, 24, and 31, compounded ratios are thus used by Archimedes to express ratios of solids as compounded of ratios of suitable surfaces and linear elements of the solids themselves.

21 See *AOO* I, 190.4–6, 8–10, 14–18.

22 See *AOO* I, 216.14–20 and 216.24–26 (application of what we read as *Elem.* VI.23). To justify the insertion of several middle terms, Eutocius (*AOO* III, 203.2–16) refers back to his own general treatment of compounded ratios, which we shall discuss in sect. III.1.2.

23 On this thorny issue, see Netz 2004 and Acerbi 2005. Eutocius provides a very long commentary on *Sph. cyl.* II.8 *aliter*, in which he completes the attested proof at several points, adding, in particular, the synthesis (*AOO* III, 202.1–216.12). Before this, he proves an interesting lemma, linking the formulation in terms of compounded ratios with that in terms of ‘multiplication’ of magnitudes; see the first footnote in section III.1.2.

He does not do so in *Fluit.* II.10. The involved property, however, requires a lengthy proof, that Archimedes does not provide—the proof suggested by Heiberg²⁴ introduces the compounded ratio written in standard form. On the other hand, in *Con. sph.* 31 Archimedes transforms the original compounded ratio in order to obtain one with a middle term, which is eliminated in the subsequent step to get a simple ratio between the extremes.

In *Sph. cyl.* II.4, shared ratios are ‘removed’ from identical compounded ratios to produce identical ratios. The removal is expressed by the canonical clause κοινὸς ἀφηρήσθω + ‘object to be removed’ in the nominative, a formula that is usually applied to removal of magnitudes common to the two sides of an equality.²⁵ Heiberg bracketed this clause because Eutocius repeats the entire argument by explicitly stating the identity of the compounded ratios and by performing the removal.²⁶

The occurrence in *Sph. cyl.* II.4 triggered Eutocius’ exposition on compounded ratios which will be discussed in section III.1.2.

II.3 Apollonius’ *Conica*

In Apollonius’ *Conica*, compounded ratios appear in 22 propositions, and in 5 instances already in the enunciation (with an asterisk *). These are: *Con.* I.11, 12, 13, 38, 39*, 40* (restatement of 39), 41*, 43 (instantiated quotation of 39), 45 (instantiated quotation of 39), 54, 55, 56, 58; II.11, 20; III.14 (instantiated quotation of I.40), 15 (instantiated quotation of I.41), 24, 53, 54*, 55, 56*.²⁷ Propositions I.11, 12, 13, 54, 55, 56, 58; II.11, 20; III.24, 53, 54, 55, 56 use—but do not quote—*Elem.* VI.23, often in combination with VI.1, in order to convert ratios of surfaces to ratios of straight lines and vice versa.

When Apollonius introduces compounded ratios outside applications of *Elem.* VI.23 or of previous theorems in the *Conica*, he always writes them in standard form. This happens in *Con.* I.55, 58; III.15; IV.54, 56. Inside compounded ratios, Apollonius freely replaces ratios with ratios identical to them.

In I.41, 54, and II.20, shared ratios are ‘removed’ from identical compounded ratios to produce identical ratios. In I.41 and 54, the removal is expressed by the canonical clause κοινὸς ἀφηρήσθω + ‘object to be removed’ in the nominative.

Our data suggest that it is Apollonius who first systematically employed compounded ratios. Apollonius’ use of this tool is so deeply entrenched in crucial

²⁴ In *AOO* II, 389 n. 4.

²⁵ There are 21 occurrences of this formula in the *Elements*.

²⁶ We read Eutocius’ argument in *AOO* III, 128.20–130.2.

²⁷ As for the verbal forms employed in the canonical formula, the *Conica* offers 79 occurrences of συγκεῖσθαι (59 of participial forms, 20 of σύγκειται); 4 of συνάπτεσθαι (1 of the form συνήπτται (in I.38), 3 of the participle συνημμένον (in III.14 and 15 (*bis*)).

features of his approach to the theory of conic sections that we may safely assume that it was not introduced by Eutocius, when the latter produced the only recension of the *Conica* we have access to.

The occurrence in *Con.* I.11 spurred Eutocius himself to reprise the exposition on compounded ratios he had previously inserted in his commentary on *Sph. cyl.* II.4. We shall discuss them in section III.1.2.

II.4 Ptolemy's *Almagest*

The most celebrated result of Greek spherical trigonometry is the Sector Theorem, also known as 'Menelaus' Theorem' because its being attested in the *Sphaerica*—it appears as proposition III.1 in Abū Naṣr's redaction. The Sector Theorem is a powerful mathematical tool, suited to determine arcs of a great circle on the surface of a sphere. It is the keystone of some of the most important technical results in the *Almagest*, where it is applied seventeen times in total.²⁸ It comes as no surprise then that the Sector Theorem is also proved in *Alm.* I.13 and, with many more cases on offer, in Theon, in *Alm.* I.13, *iA*, 535.10–570.12 (see section II.6).²⁹

The configuration adopted by Ptolemy in his proof is the following: From the endpoints B, Γ of two mutually intersecting arcs AB, AΓ of great circles on the surface of a sphere, two arcs BE, ΓΔ are drawn across, meeting at Z and intersecting arcs AΓ, AB at E, Δ, respectively. All these arcs must be less than a semicircle. Then the following relations hold:

$$\begin{aligned} ch(2ΓE):ch(2EA) &= [ch(2ΓZ):ch(2ZΔ)] \circ [ch(2ΔB):ch(2BA)] \\ ch(2ΓA):ch(2AE) &= [ch(2ΓΔ):ch(2ΔZ)] \circ [ch(2ZB):ch(2BE)], \end{aligned}$$

where, for instance, $ch(2ΓE)$ is the chord subtended by twice the arc ΓE. Ptolemy provides a detailed proof that the first relation holds, leaving the proof of the second to the reader. The Sector Theorem is proved by Ptolemy last of a series of seven propositions. The first of these propositions is a version of the Theorem in the plane, involving compounded ratios of line segments.³⁰ The argument runs as follows (see Fig. 1): from the endpoints B, Γ of two mutually intersecting straight lines AB, AΓ, two lines BE, ΓΔ are drawn across, meeting at Z and intersecting straight lines AΓ, AB at E, Δ, respectively. It is required to show that $ΓA:AE = (ΓΔ:ΔZ) \circ (ZB:BE)$. The proof sets out the 'obvious' compounded ratio written in standard form $ΓΔ:HE = (ΓΔ:ΔZ) \circ (ΔZ:HE)$,

²⁸ This happens in *Alm.* I.14, 16, II.2, 3 (*ter*), 7 (*bis*), 10, 11, 12 (*bis*), VIII.5 (*ter*), 6 (*bis*); see *iA*, 555 n. 1.

²⁹ For a clear exposition of the mathematics involved, see Neugebauer 1975, 26–30.

³⁰ The plane case is proved in *POO* I.1, 68.23–70.16, and the Sector Theorem at 74.9–76.9.

draws a suitable parallel HE to one of the assigned straight lines, and readily argues by similar triangles and replacements of identical ratios in a compounded ratio.

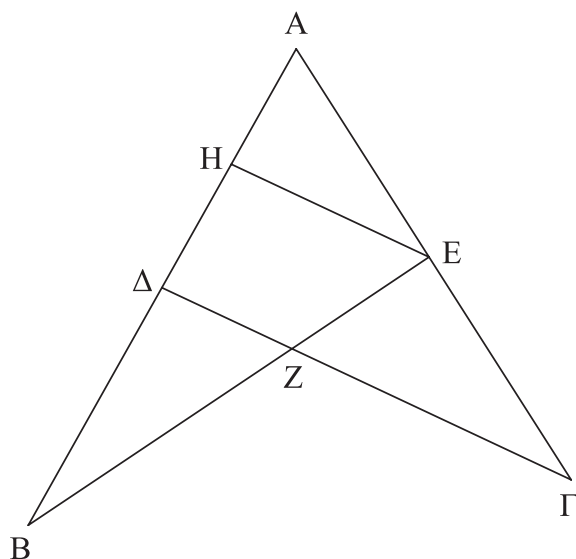


Fig. 1

In the proof of the plane case of the Theorem, the compounded ratio is written in standard form when introduced, and then transformed by replacement of identical ratios; the middle term is said to be taken ‘from outside’ (ἐξωθεν). In the proof of the spherical case, the compounded ratio arising in a suitably related plane case is simply transformed by replacement of identical ratios.

In all applications of the Sector Theorem in the *Almagest*, the relevant compounded ratio features only once, and a relation of the kind written above is simply stated to hold; five of the six arcs are numerically given, and one must find the sixth. The procedure, which was never worked out in detail by Ptolemy, amounts to using the Table of Chords to calculate chords from arcs and vice versa, and to performing the operation of ἀφαίρεσις ‘removal’ of a ratio from a ratio.³¹

Since Ptolemy never explains how to remove a ratio from a ratio, the gap was filled in Pappus’ and Theon’s commentaries on the *Almagest*, and by all subsequent generations of (Late-Antique and Byzantine) scholiasts. Filling the gap entailed either producing general expositions that tried to unify the several cases of the operation of

³¹ As for the verbal forms employed in the canonical formula, the *Almagest* has 6 occurrences of συγκεῖσθαι (1 of the form συγκεῖμενος, 1 of συγκεῖσθαι, 4 of σύκειται); 24 occurrences of συνάπτεσθαι (6 of συνημμένος, 16 of συνήπται, 1 each of συνήφθαι and συναφθήσεται).

removal under a unitary treatment, or working out the calculations involved in a specific application of the Sector Theorem. The major role played by the Sector Theorem in the *Almagest* explains even the mere existence of most of the general expositions I will summarize in the following pages.

II.5 Pappus' *Collectio*

Pappus, in his *Collectio*, is the first mathematician who uses compounded ratios with explicit metamathematical goals: he offers several alternative proofs that hinge on compounded ratios, clearly suggesting that the idea of a grand-scale exploitation of the method was his own and that he was proud of it (with an asterisk * the applications of VI.23): see *Coll.* VII.37–40*, 42*, 68*, 74*, 75*, 84*, 86*, 194, 197*, 210, 246*, 253*, 255*, 272*, to which one must also add VII.240 (which will be discussed later in this section) and VIII.9–10.³²

When Pappus introduces compounded ratios (this may happen several times within the same proposition), he rarely writes them in standard form. The exceptions are in *Coll.* VII.68, 84, 86, 197, 210, 240, VIII.9–10. In VIII.10, the middle term is twice said to be taken ‘from outside’ (ἐξωθεν). Inside compounded ratios, Pappus freely replaces ratios with ratios identical to them. In VII.194 there is no application of VI.23, nor are the compounded ratios written in standard form when introduced. However, the passage is obviously corrupt. In VII.210, one finds a unique instance of a replacement, within a compounded ratio identical to another, of a compounding ratio with a compounded form of it, and the subsequent removal of shared ratios is applied, on the one side, to a compounded ratio with one middle term and, on the other side, to a multiple compounded ratio with two middle terms.

In VII.194, 210 (*bis*), shared ratios are ‘removed’ from identical compounded ratios to produce identical ratios. The removal is expressed by the clause κοινὸς ἐκκεκρύσθω + ‘object to be removed’ in the nominative, a totally un-canonical *variation* of the standard formula that is usually applied to removal of magnitudes common to the two sides of an equality.

A striking application of the method can be found in *Coll.* VII.37–40,³³ where the problem is to provide a generalization of the enunciation of the ‘locus on three and four lines’. Pappus first remarks that the enunciation immediately generalizes to five or six lines: it is enough to replace the rectangles with parallelepipeds. The higher loci of this type raise problems, “since there is nothing contained by more than three

³² As for the verbal forms employed in the canonical formula, the *Collectio* has 20 occurrences of συγκεῖσθαι (10 of participial forms, 10 of σύγκειται); 67 of συνάπτεσθαι (20 of the form συνήπται, 47 of forms of the participle συνημμένος).

³³ See also the enunciation of what is known as ‘Gouldin’s theorem’, in *Coll.* VII.42.

dimensions”.³⁴ Pappus claims that his predecessors did not employ a language adequate to the formulation of such general enunciations, since “it was possible to enunciate and generally to prove these things by means of compounded ratios”.³⁵ Pappus provides an example of an adequate formulation:

ἐὰν ἀπὸ τινος σημείου ἐπὶ θέσει δεδομένης εὐθείας καταχθῶσιν εὐθεῖαι ἐν δεδομέναις γωνίαις, καὶ δεδομένος ἢ λόγος ὁ συνημμένος ἐξ οὗ ἔχει μία κατηγμένη πρὸς μίαν καὶ ἑτέρα πρὸς ἑτέραν καὶ ἄλλη πρὸς ἄλλην καὶ ἡ λοιπὴ πρὸς δοθεῖσαν (ἐὰν ὦσιν ζ· ἐὰν δὲ η·, καὶ ἡ λοιπὴ πρὸς λοιπὴν), τὸ σημεῖον ἄψεται θέσει δεδομένης γραμμῆς.

If straight lines are drawn from some point at given angles onto straight lines given in position, and there is given the ratio compounded of that which one drawn line has to one, and another to another, and a different one to a different one, and the remaining one to a given (if there are seven—but if eight, the remaining to the remaining one), the point will touch a curve given in position.

From a theoretical point of view, the most relevant passage is *Coll.* VII.240, a lemma to Apollonius, *Con.* I.39 or 41 (cf. **Document 2** in the Appendix). The proof does not make use of the *πηλικότης*, and it proves that, if $a:b = (c:d) \circ (e:f)$, then $c:d = (a:b) \circ (f:e)$. The proof runs as follows:

- Let h be such that $d:h::e:f$.
- Since $a:b = (c:d) \circ (e:f) = (c:d) \circ (d:h)$ by assumption, and $(c:d) \circ (d:h) = c:h$ by inserting the middle term d in ratio $c:h$, then $a:b::c:h$. This amounts to assuming that, if two ratios are compounded of identical ratios, they are identical.
- Since $c:d = (c:h) \circ (h:d)$ by inserting the middle term h in ratio $c:d$, and $c:h::a:b$ by the previous deduction, and $h:d::f:e$ by inversion of the construction for h , then $c:d = (a:b) \circ (f:e)$.

Thus, the proof argues by replacement of ratios both in the assigned compounded ratio and in suitable compounded ratios written in standard form.

II.6 Theon’s *Commentary on the Almagest*

In this section, I shall only deal with the long proof of the Sector Theorem in *in Alm.* I.13, *iA*, 535.10–570.12 (see sections III.1.1 for Theon’s general expositions on compounded ratios and removal, and III.2.1 for the procedures of removal he employs in the applications).

Theon’s long proof of the Sector Theorem is just an expansion of that provided by Ptolemy in *Alm.* I.13. The well-known habit of mathematical commentators to imitate

³⁴ Here and henceforth, I follow Jones’ 1986 translation.

³⁵ The language deemed inadequate by Pappus resorted to ‘multiplication’ in Heronian style of squares and rectangles; this is the approach of *Sph. cyl.* II.8 *aliter*, as we have seen.

the style and lexicon of the model makes Theon's text a poorly significant witness of linguistic changes. At any rate, in the proof of the plane case, the compounded ratio is written in standard form when introduced, and then transformed by replacement of identical ratios; the middle term is said to be taken 'from outside' (ἐξωθεν).³⁶ In the several partial proofs of the spherical case, either the compounded ratio arising in a suitable plane case is simply transformed by replacement of identical ratios, or compounded ratios arising in other spherical cases are transformed by replacement of identical ratios. Theon also provides complete enunciations of all the theorems that Ptolemy had instead enunciated in instantiated form. In the first of these enunciations, Theon implicitly introduces the reader, by means of the disjunction "is conjoined, that is, is compounded" (συνήπται ἤτοι σύγκειται), to the major verbal *variatio* in the canonical formula of a compounded ratio.³⁷

III Logistic Approach

The data collected in section II have already given prominence to the two actors that will play a prime role in what follows: the Sector Theorem and *Elem.* VI.df.5. In principle, logistic is only involved in explanations of the former, but we shall see that extensive register-crossing took place. For this reason, and because of their context, I have included in this section the expositions 'in geometric style' (γραμμακῶς)—that is, in deductive style—by Theon and by Eutocius. Also notice—since it belongs to the same category—the very short argument formulated in the 'language of the givens' which appears in section III.1.3.

Before proceeding, I recall the main terminological conventions pertaining to our subject and typical of the logistic domain.

- A 'ratio' (λόγος) is a relation between two 'terms' (ὅροι), called in the order 'antecedent' (πρόλογος) and 'consequent' (ὑπόλογος), and much less frequently ἡγούμενος and ἐπόμενος. To avoid ambiguities, I shall often use the noun 'extreme' as a synonym of 'term'. In specific numerical examples, recall that the terms of a ratio are usually given in the form 'greater-to-lesser'.
- In the operation of 'removal' (ἀφαίρεσις) of a ratio from a συγκείμενος λόγος 'compounded ratio', the basic elements are the ratio 'from which the removal is going to occur' (ἀφ' οὗ ἡ ἀφαίρεσις γίνεται), the ratio 'to be removed' (ἀφαιρούμενος),

³⁶ In *iA*, 543.6, ἐξωθεν, in the expression ἐξωθεν λαμβανομένης, is replaced by μέσης.

³⁷ As for verbal forms employed in the canonical formula, books I–IV of Theon's commentary on the *Almagest* have 62 occurrences of συγκείσθαι (8 of forms of συγκείμενος, 1 of συγκείσθαι and of συγκείσθω, 52 of σύγκειται); 18 occurrences of συνάπτεσθαι (17 of συνήπται, 1 of συνήφθαι). I have excluded the occurrences in quotations from the *Almagest*. In *iA*, 542.6–7 and 545.2–3, we read citations of *Elem.* VI.23.

and ‘the remainder’ (ὁ λοιπός). I shall translate the verb καταλείπειν by ‘to leave out’, even in the case in which participial forms are involved. We shall also see that a type of the procedure of removal will require ‘to fit’ (ἐναρμόζειν) a ratio to another.

- The adjective ‘paronymous’ (παρώνυμος) is applied to the number that expresses the πηλικότης ‘value’ of a ratio: ‘six’ is the πηλικότης of the sextuple ratio and is paronymous to it; there cannot be strict homonymy.³⁸ Likewise, a part is paronymous to a number if it bears the same name as the number, only transformed into an ordinal, like ‘three’ and ‘<one> third’.

We have already seen that there is no operation of ‘composition’ of ratios. This is quite paradoxical, since *Elem.* VI.df.5 defines a compounded ratio in terms of an operation performed on the values of the ratios themselves. Even more paradoxical is the fact that the removal of a ratio from a compounded ratio was indeed conceived of as an operation. Moreover, the implicit assumption that one most appropriately remove a ratio from a compounded ratio, and not simply from a ratio *tout court*, had as a consequence that the operation of removal more or less entirely consisted in a prescription for writing an assigned ratio in a suitable compounded form. It is such a prescription that quite naturally—that is, according to the ‘suitable’ compounded form required by the context—branches in a variety of cases.³⁹ Accordingly, the operation of removal was almost always performed on ratios that were already provided in compounded form. Therefore, if from $a:b = (a:d) \circ (d:b)$ we want to remove ratio $d:b$, it suffices to literally ‘remove’ (ἀφαιρεῖν) $d:b$ from the right-hand side; ratio $a:d$ is, in a most concrete sense, the ‘remainder’.

The translation of the verb ἀφαιρεῖν by ‘to remove’ and of the associated *nomen actionis* ἀφαίρεσις by ‘removal’ calls for a few words of justification.⁴⁰ On the one hand, ἀφαιρεῖν (and the substantives related to it) have a well-defined technical meaning in the Greek mathematical lexicon: ‘to subtract’, either a magnitude from a magnitude or a number from a number. Moreover, as we have already seen, the Greek

³⁸ Still, ‘homonymous’ (ὁμώνυμος) is the adjective we read, with identical meaning, in *Elem.* VII.37–9 and in some of the sources we shall meet in the following pages.

³⁹ In some sources, this division into cases is also motivated by the fact—*a priori* irrelevant if one only sets out to remove ratios from compounded ratios—that no applications of the Sector Theorem simply require to remove a ratio from another, but also to calculate one of the terms of the remainder, the other term being given.

⁴⁰ Heath (1921 II, 538), and Bulmer-Thomas (1971, 160) following him, translate the verb as ‘to take out’; Knorr (1989, 202–204) has ‘to subtract’; Ruelle (1883) is more inconsistent by using ‘retrancher’ in the introduction, and ‘ôter’ or ‘enlever’ in the translation; Tannery (1885, 221 of the reprint) ‘supprimer’; Riedlberger (2013) ‘to remove’. The problem of this translation is discussed by O. Riemann in note 1 of Ruelle 1883, 85; cf. Riedlberger, 2013, 201–202; Acerbi & Riedlberger 2014, 406; and Acerbi & Pérez Martín 2015, 123 (the last two articles use ‘rimuovere’).

mathematical tradition formulates within the lexicon of adding the operations on ratios we express within the lexicon of multiplying. Both conceptually and operationally, an ἀφαίρεσις λόγων does not correspond to the modern conceptualization of the ‘same’ operation as a division of fractions. On the other hand, if Greek mathematical style has two lexical families corresponding to what we would call ‘adding’ (namely, the verbs προστιθέναι and συντιθέναι and the substantives related to them),⁴¹ it has just one lexical family corresponding to what we would call ‘subtracting’ (namely, ἀφαιρεῖν and the substantives related to it). Such considerations lead to regard as *a priori* perfectly legitimate to translate ἀφαιρεῖν by ‘to subtract’. Still, employing ‘to subtract’ would call for ‘to add’ as a natural correlative, and not ‘to compound’, which is currently used for συντιθέναι. Further, such renderings would clash with well-entrenched modern mathematical habits, since the clause ‘to subtract a ratio from another’ has a well-defined and technical meaning also in modern mathematics, and this meaning is at variance with the meaning of the Greek technical expression the above clause is intended to translate.

III.1 General Expositions

III.1.0 Nicomachus’ *Introductio Arithmetica*

In *Ar.* II.2.5, Nicomachus briefly deals with compounded ratios. He introduces and perfunctorily employs the lexicon of ‘compounding’ and ‘composition’ (substantive σύνθεσις, adjective σύνθετος ‘compound’, compounding ratios that are said to be ‘compounded’ to give a ratio), but does not explain what such an ‘operation’ might amount to. Moreover, his lexicon is far from fixed, as he also uses the verbs συλλαμβάνεσθαι ‘to be taken together’, ἀναλύεσθαι and διαλύεσθαι ‘to be analyzed’, συνίστασθαι ‘to be constructed’, and the noun σύστημα ‘complex’ to denote the same operation or its results, either from the viewpoint of the compounded ratio, or from the viewpoint of the compounding ratios. In particular, the lexicon of ‘analysis’ is employed when a given ratio is shown, by inserting a middle term, to be compounded of two ratios. In this way—and these are Nicomachus’ examples—, a double ratio is shown to be analyzed into a sesquialter and a sesquitertian ratio (terms 4 3 2 and 6 4 3). Conversely, from a sesquialter and a sesquitertian ratio, if compounded, a double ratio is in any case constructed. Nicomachus then applies the lexicon of composition to reformulate a property expounded in *Ar.* II.2.3–4; namely, that any generic multiple ratio, compounded with a suitable superparticular ratio, gives the immediately

⁴¹ The former verb as a prevalent arithmetical connotation, while the latter as a prevalent geometrical connotation, even if register-crossing may occur. As we have seen, only συντιθέναι—and most frequently its passive συγκεῖσθαι—occurs in the sectorial lexicon of compounded ratios.

subsequent multiple ratio (or, conversely, that the latter can be analyzed into the former two). For instance, a double ratio, compounded with a sesquialter ratio, gives a triple ratio (terms 18 12 6 and 18 9 6, and again 6 3 2); a triple ratio, compounded with a sesquitercian ratio, gives a quadruple ratio, and so on.

III.1.1 Theon on Compounded Ratios in his *Commentary on the Almagest*

Theon has two general expositions that can be summarized as follows (cf. **Document 3** in the Appendix):⁴²

- 1) The opening passage of *in Alm. I.13*, *iA*, 532.1–535.9, which appears just before Theon’s extended proof of the Sector Theorem.

The argument is of the kind later deemed ‘inductive’ by Eutocius (see section III.1.2). After a transcription of *Elem.* VI.df.5 (notice that Theon is the only independent source in the Late Antiquity who completes the final *τινα* with *πηλικότητα λόγου*), Theon claims that a ratio can be written in a compounded form using any possible middle term. This means that the middle terms can be either (a) lesser than the greater extreme of the ratio but greater than the lesser extreme, or (b) greater than the two extremes, or (c) lesser than the two extremes. Theon gives two different ‘proofs’ of his claim. The first proof operates by multiplication of the denominations of the ratios. It is a circular argument, as Theon himself implicitly admits by resorting to the ‘composition’ of multiple ratios in order to justify the assertion that double times triple gives sextuple.⁴³ The second proof argues by partition of the antecedent of a ratio once the consequent of the same ratio is taken as unit, and by counting the ‘units’ resulting from taking a suitable multiple of both. The argument is similar to the one used in the proof of *Sectio can.* 7 and 8 (see section II.1). Theon’s examples do not identify the ratios by setting out their terms, as Eutocius does, but their denominations: double, triple, sextuple; triple, half, sesquialter; subdouble (*sic*), sesquitercian, subsesquialter. The presence of the middle term in Theon’s exposition—and, as we shall see, in Eutocius’—shows that the goal of his arguments, that seemingly deal with compounded ratios, is, in fact, to set the stage for operating a removal. Theon ends his exposition by pointing out that (*iA*, 535.7–9) “it is obvious that, if from a compounded ratio one whatever of the compounding <ratios> is removed, one of the extremes being made to disappear, the remaining compounding <ratio> will be left out”.

⁴² Cf. *iA*, 532 n. 1, for a discussion on the authenticity of the first passage and of its location in the manuscripts.

⁴³ Cf. *iA*, 534.9.

2) In *Alm. II.11*, iA, 759.8–762.2.

This is a sketchy justification of the operation of removal in deductive style, only applied to a case of the Sector Theorem that can be deduced, by simple inversion of a ratio, from a case that had been treated in the commentary on Book I of the *Almagest*: if $a:c = (a:b) \circ (b:c)$, then $a:b = (a:c) \circ (c:b)$. The proof is said to be immediately evident and simply amounts to point out that it is enough to ‘deplace’ the term b .⁴⁴ Theon’s trivial proof is, in fact, all that is needed if one limits itself to compounded ratios written in standard form. We have seen a proof in deductive style of the same statement, but applied to compounded ratios not written in standard form, in Pappus, *Coll.* VII.240 (cf. section II.5 and **Document 2** in the Appendix). It is thus possible that Theon compiles here an analogous argument in Pappus in *Alm. II.11*, now lost.

III.1.2 Eutocius on Compounded Ratios in his Commentaries on Archimedes’ *De Sphaera et Cylindro* and on Apollonius’ *Conica*.

The early sixth-century Neoplatonic philosopher Eutocius redacted two expositions on compounded ratios.⁴⁵ These are, as Eutocius himself says and as the following parallel examination will confirm, the one a (summary) reprise of the other. I shall refer to the two expositions in their order of redaction: in *Sph. cyl.* first, in *Con.* second; references to the texts are separated by the sign ‘/’.

Eutocius starts with an historico-doctrinal *excursus*, which justifies the introduction of his digression, and boasts about his own contribution; namely, that this is the first abstract exposition ever written (in *Sph. cyl.*, AOO III, 120.3–11: he complains about the fact that only ‘inductive’ (ἐπαγωγῆ) expositions can be found, among others, in Pappus, Theon, Arcadius / in *Con.*, AGE II, 218.6–15: he mentions his own previous exposition in the commentary in *Sph. cyl.*, and another in σχόλια to the first book of the *Almagest*).⁴⁶

⁴⁴ The text is probably corrupt; see Rome’s discussion at iA, 762 n. 2. In the enunciation of this theorem, one must athetize ἀνάπαλιν at 761.1, *contra* Rome.

⁴⁵ Cf. In *Sph. cyl. II.4*, AOO III, 120.3–126.20 and in *Con.*, AGE II, 218.6–220.25. The texts that triggered Eutocius’ digressions are Archimedes, *Sph. cyl. II.4*, AOO I, 190.4–6, and Apollonius, *Con. I.11*, AGE I, 40.8–10, respectively. Add to these the proof that, if there are four terms A, Γ, Δ, B, “the ratio compounded of that of the <rectangle contained> by A, B to the <square described> on Γ with the ratio of B to Δ is identical to that of the <rectangle contained> by A, B times B to the <square described> on Γ times Δ” (in *Sph. cyl. II.8*, AOO III, 198.19–200.31). The proof, that applies the δι’ ἴσου proposition, is translated as **Document 6** in the Appendix.

⁴⁶ The text reads as follows (in *Con.*, AGE II, 218.6–15): “since this was treated by the commentators in a most inductive way, and not according to cogent methods, I had some thoughts on that and I have written them down in my own published <notes> to the fourth theorem of the second book of Archimedes’ *On the sphere and the cylinder*, and in the annotations to the first book of Ptolemy’s *Syntax*. It will do no harm that this be also written here, because it is by no means obvious that

The definition of compounded ratio as appears in *Elem.* VI.df.5 is then transcribed, accompanied by remarks on the *πηλικότης* of a ratio, the key notion in the abstract proof Eutocius is going to offer (the text at 120.12–122.9 is richer: it has in addition a definition of *πηλικότης*, according to Nicomachus and his commentator Heronas, as “the number to which the given ratio is paronymous [*παρώνυμος*]”; this is stated to be equivalent to the definition “the number that, multiplied by the consequent term of a ratio, also gives the antecedent”; note also the characterization, at 120.28–30, of logistic as the discipline in which one divides the unit / 218.16–26).

After this, Eutocius provides the non-inductive proof that ratio $a:c$ is compounded of $a:b$ and $b:c$. He does this by showing that, if d and e are the *πηλικότητες* of $a:b$ and of $b:c$, respectively, and if $de = z$, then z is the *πηλικότης* of $a:c$ (122.10–124.7 / 218.27–220.16; they are nearly identical, the latter being more concise; cf. **Document 7** in the Appendix).

Only in the commentary in *Sph. cyl.* (124.8–126.3), we read an ‘inductive’ proof that a ratio can be written in compounded form using any possible middle term, namely, whether this is (a) lesser than the greater extreme of the ratio but greater than the lesser extreme, (b) greater than the two extremes, or (c) lesser than the two extremes. This is the same argument as that outlined in the first passage in Theon, summarized in section III.1.1. In his examples, Eutocius identifies the ratios by setting out their terms and by providing a middle term that satisfies each of the stated conditions: the middle term lies between the two extremes (12, 4, 2), is greater than both (9, 12, 6), or is lesser than both (9, 4, 6).

Again in *in Sph. cyl.* only (126.4–20), we read an instantiated enunciation that generalizes the main result to several ratios and middle terms: $a:b = (a:c) \circ (c:d) \circ (d:b)$. The proof amounts to taking for granted that, within a compounded ratio, one can replace ratios with compounded ratios: since $a:b = (a:d) \circ (d:b)$ and $a:d = (a:c) \circ (c:d)$, then $a:b = (a:c) \circ (c:d) \circ (d:b)$.

the present readers will also read those expositions, and since almost the entire treatise of the *Conics* makes use of it” (ἐπεὶ δὲ ἐπακτικώτερον μάλλον καὶ οὐ κατὰ τὸν ἀναγκαῖον τρόπον ὑπὸ τῶν ὑπομνηματιστῶν ἐλέγετο, ἐζητήσαμεν αὐτὸ καὶ γέγραπται ἐν τοῖς ἐκδεδομένοις ἡμῖν εἰς τὸ τέταρτον θεώρημα τοῦ δευτέρου βιβλίου τῶν Ἀρχιμήδους περὶ σφαίρας καὶ κυλίνδρου καὶ ἐν τοῖς σχολίοις τοῦ πρώτου βιβλίου τῆς Πτολεμαίου συντάξεως· οὐ χεῖρον δὲ καὶ ἐνταῦθα τοῦτο γραφῆναι διὰ τὸ μὴ πάντως τοὺς ἀναγινώσκοντας κάκεινους ἐντυγχάνειν, καὶ ὅτι σχεδὸν τὸ ὅλον σύνταγμα τῶν κωνικῶν κέχρηται αὐτῷ). Thus, quite clearly Eutocius asserts that what we are going to read is more or less identical to two previous expositions of his. One must infer that the now lost ‘annotations to the first book of Ptolemy’s *Syntaxis*’ contained the same material as we find in the two surviving commentaries. Moreover, Arcadius cannot be the author of the part of the *Prolegomena* we shall deal with in section III.1.4: on the one hand, he is mentioned by Eutocius because he adhered to the ‘inductive’ mode of presentation; on the other hand, the *Prolegomena* does not resort to this mode but expressly refers to Pappus for a general exposition and otherwise only provides specific examples (which *do not* amount to an ‘inductive’ proof). These remarks disprove both Mogenet’s contention (1956, 13–44) that Eutocius was the author of the *Prolegomena* and Knorr’s (wrong) inference from his own (correct) rebuttal of Mogenet’s thesis that Arcadius himself, a character we simply know nothing about, authored the *Prolegomena* (1989, 155–177).

Only in the commentary *in Con.* (220.17–25), Eutocius adds a final remark about the commonalities between logistic and arithmetic (that is, number theory, characterized by proofs γραμμικῶς); in this way, he intends to justify his own choice of using language and methods of arithmetic in order to prove a result belonging to logistic.

III.1.3 A Scholium on Ptolemy's *Almagest* and the Treatise on Removal of Ratios Ascribed to Domninus of Larissa

As said in section II.4, in any application of the Sector Theorem in the *Almagest*, a relation involving a compounded ratio whose terms are chords of suitable arcs is stated to hold, five of the six arcs are numerically given, and one must find the sixth. The procedure, whose details are never worked out by Ptolemy and which admits of a very rich combinatorics of cases and subcases, requires performing the operation of removal of a ratio from a ratio. Since Ptolemy never explains how to perform such a removal, the gap was filled by commentators on the entire *Almagest*, such as Pappus and Theon, as well as by a number of well-known or anonymous scholiasts in the Late Antiquity and in the Byzantine period. The unitary and general exposition, just enriched with a few numerical examples, we find in one of the *Almagest* scholia (cf. **Document 4** in the Appendix; see below in this section for a summary of the actual content of the scholium) can be expounded in a most 'inductive' way as follows.

The problem that has bewildered all commentators is that, in the applications, no given compounded ratio is ever written in standard form. Two ratios are instead assigned, one of which must be removed from the other, the latter being assigned as a compounded ratio not written in standard form. As a consequence, the ratio from which the removal is going to occur must preliminary be written as a compounded ratio in standard form. This is done by operating a 'fitting' inside it (the related verb is ἐναρμόζειν), that automatically takes into account the form of the ratio to be removed. A first ratio is 'fitted' to a second ratio by transforming it into a ratio which is equivalent to it and having the antecedent or the consequent equal to the antecedent or to the consequent of the second; therefore, one has only to calculate the remaining term of the first ratio, and this can be done by taking a suitable fourth proportional.

For instance, let it be required to remove 4:3 from 12:6. Now, 'to fit' 4:3 to 12:6 amounts to find a ratio equivalent to 4:3 with the following, alternative, constraints:

- a. either 12 is the new antecedent; thus one must calculate the new consequent as a fourth proportional 4:3::12: x ,⁴⁷ yielding as a result $x = (12 \times 3)/4 = 9$;
- b. or 6 is the new consequent; thus one must calculate the new antecedent as a fourth proportional 4:3:: x :6, yielding as a result $x = (4 \times 6)/3 = 8$.

⁴⁷ Contrary to what is suggested by the qualifier 'fourth', the position in the proportion of the number to be determined is immaterial, if we allow for a rearrangement of the given terms.

In case (a), 4:3 ‘fitted’ to 12:6 gives 12:9, which is a ratio equivalent to 4:3 and whose antecedent is identical to the antecedent of 12:6. In case (b), instead, the ratio obtained by ‘fitting’ 4:3 to 12:6 is 8:6, whose consequent is identical to that of 12:6.

Conversely, ‘to fit’ 12:6 to 4:3 amounts to find a ratio equivalent to 12:6 with the following, alternative, constraints:

- c. either 4 is the new antecedent; thus one must calculate the new consequent as a fourth proportional $12:6::4:x$, yielding as a result $x = (6 \times 4)/12 = 2$;
- d. or 3 is the new consequent; thus one must calculate the new antecedent as a fourth proportional $12:6::x:3$, yielding as a result $x = (12 \times 3)/6 = 6$.

In case (c), 12:6 ‘fitted’ to 4:3 gives 4:2; in case (d), it gives 6:3.

If we want to remove 4:3 from 12:6, let us choose for instance procedure (a): ‘fit’ 4:3 to 12:6 to give 12:9; insert the middle term 9 between the terms of ratio 12:6 in order to write it as the compounded ratio $(12:9) \circ (9:6)$;⁴⁸ literally ‘remove’ 12:9 and the remainder is 9:6. Since the ratio has been removed that contains the antecedent of the compounded ratio 12:6, the operation of removal is said to be performed ‘with respect to the antecedent’ (πρὸς τὸν πρόλογον); otherwise, it is said to be performed ‘with respect to the consequent’ (πρὸς τὸν ὑπόλογον). Procedures (a) and (c) can only give rise to the first type of removal, (b) and (d) to the second.

Finally, it always happens that one is interested in calculating one of the terms of the remainder, the other being given. This is the reason why a ‘fitting’ has four cases and not two (a, b or c, d): the position of the term to be calculated in the remainder can make it necessary to ‘fit’ either the compounded ratio to the ratio to be removed, or vice versa. If the given term does not match the homologous term of the remainder calculated by any of the above procedures, a further taking of a fourth proportional must be performed. Suppose, as above, that we want to remove 4:3 from 12:6 in order to calculate the antecedent of the remainder, its consequent being given as 8. The operation of removal according to procedure (a) yields 9:6, that must be reduced to a ratio with consequent 8. This is done by setting out the proportion $9:6::x:8$, yielding as a result $x = (9 \times 8)/6 = 12$.

This scholium was cut, pasted, and enriched with a complete series of examples by some anonymous revisor. The accidents of textual transmission made the result to be attributed to Domninus of Larissa, a fifth-century Neoplatonic philosopher contemporary of Proclus.⁴⁹ The text ascribed to Domninus is organized as follows (the section numbers are canonical since the first edition).

⁴⁸ And, in fact, passing to $\pi\eta\lambda\iota\kappa\acute{o}\tau\eta\tau\epsilon\varsigma$ *iuxta Elem.* VI.df.5 and inserting number 9, we get $\frac{12}{6} = (\frac{12}{9})(\frac{9}{6})$, where we are entitled to write the $\pi\eta\lambda\iota\kappa\acute{o}\tau\eta\tau\epsilon\varsigma$ of the ratios as fractions.

⁴⁹ The standard edition of Domninus’ works is now Riedlberger 2013, which also contains the best available discussion of his life and works. The scholium is edited and thoroughly discussed in Acerbi & Riedlberger 2014.

- Introduction to the problem; definition of compounded ratio (always written in standard form) according to *Elem.* VI.df.5; two examples: 4 and 40 as middle terms of 2 and 8 (sections 1–5).
- Explanation of the principles of removal of ratios, the removed ratio being always less than the one from which the removal is going to occur: (i) ‘fitting’ of the lesser ratio to the greater ratio or vice versa; (ii) removal made with respect to the antecedent or with respect to the consequent of the ratio to which the fitting of point (i) is made; as a consequence, there are four possible cases (section 6).
- Examples, always with 12:3 and 6:4 (these correspond to my set of examples above). *a*) The first ratio is written as compounded of 12:8 and 8:3 (removal performed with respect to the antecedent of the greater ratio: 12:8 is identical to the ratio to be removed 6:4; the remainder is 8:3) or as compounded of 12:4½ and 4½:3 (removal with respect to the consequent of the greater ratio: 4½:3 is identical to the ratio to be removed 6:4; the remainder is 12:4½) (sections 7–15). *b*) Either ratio 6:1½ (that is, 12:3) is written as compounded of 6:4 and 4:1½ (removal with respect to the antecedent of the lesser ratio; the remainder is 4:1½), or ratio 16:4 (that is, again 12:3) is written as compounded of 16:6 and 6:4 (removal with respect to the consequent of the lesser ratio; the remainder is 16:6) (sections 16–18).
- Things become simpler if the two ratios have the same antecedent or the same consequent. Example: to remove 12:4 from 12:3; the remainder is 4:3, taken by simply forming the ratio of the extremes that are not identical (sections 19–21).
- Again, the definition of compounded ratio (section 22).
- Validation of the operation of removal of ratios using the ‘language of the givens’: if a given ratio is removed from a given ratio, the remainder is also given (section 23).

The original scholium contains only sections 1–5, 22–23, 6, 19–21, in this order. In particular, the presence of sections 22–23 is fully justified only in the original position; the revisor that produced the text ascribed to Domninus also added the long, central portion featuring the numerical examples of removal. Both the reworked text that came to be ascribed to Domninus and the original scholium repeatedly make the conceptual mistake of multiplying ratios and not the related $\pi\eta\lambda\iota\kappa\acute{o}\tau\eta\tau\epsilon\varsigma$.

III.1.4 The *Prolegomena ad Almagestum*

The *Prolegomena* is a computational primer to the *Almagest*: it contains a tightly organized ‘handbook of logistic’, featuring as its main themes: an introduction to the sexagesimal system; a description of computational algorithms for multiplication, division, and extraction of approximate square root; a presentation of interpolation techniques; an exposition about compounded ratios and removal of a ratio from an assigned ratio. According to the author, no comprehensive previous exposition of this kind existed.

The *Prolegomena* deals with compounded ratios as a preliminary to removal.⁵⁰ The author starts by quoting *Elem.* VI.df.5. He then gives a numerical example: the ratio of 100 to 5 makes the twentyfold ratio. This can be seen as a compounded ratio by interpolating one or several middle terms. No general exposition is provided, since “the geometer Pappus has shown that, if there are two magnitudes, and <magnitudes> are set between them, the ratio <contained> by the extreme magnitudes is equal to that <compounded> of all intermediate ratios”. A numerical example follows: 20 as a middle term of 100 and 5; therefore, the twentyfold ratio is compounded of the quintuple and of the quadruple ratio. The case of three ratios with two middle terms is also treated on the basis of the same example, inserting 10 as an additional term. That’s all. The exposition is not inductive (as, for instance, Theon’s is), for no general result is stated and allegedly proved by an example.

Removal of ratios is treated in the *Prolegomena* by means of paradigmatic examples; in order to summarize the procedure underlying them I resort to a general exposition with dummy letters. The similarity of this exposition and of the one seen in the previous section is obvious.

One is required to remove $c:d$ from $a:b$. To this end, one must (i) write the ratio $a:b$ in compounded form in such a way that one of the compounding ratios is $c:d$, say $a:b = (x:y) \circ (c:d)$, where $x:y$ will obviously be the remainder, but (ii) have the compounded ratio written in standard form: $a:b = (a:m) \circ (m:b)$, where m is a middle term. One, then, has to find such a middle term m . This can be done in a general way according to two procedures:

- *With respect to the antecedent.* In $a:b = (a:m) \circ (m:b)$, one identifies $a:m$ as the ratio to be removed (it has the same *antecedent* as the ratio from which the removal is going to occur), and therefore $m:b$ as the remainder. But the ratio to be removed was assigned as $c:d$. Therefore, m can be calculated as a fourth proportional: $c:d::a:m$. By *Elem.* VII.19, $m = ad/c$ is the required middle term.
- *With respect to the consequent.* In $a:b = (a:m) \circ (m:b)$, one identifies $m:b$ as the ratio to be removed (it has the same *consequent* as the ratio from which the removal is going to occur), and therefore $a:m$ as the remainder. But the ratio to be removed was assigned as $c:d$. Therefore, m can be calculated as a fourth proportional: $c:d::m:b$. By *Elem.* VII.19, $m = bc/d$ is the required middle term.

It is obvious that the conditions identifying the two procedures are exhaustive and exclusive. These can be summarized by means of one single prescription: ‘find the fourth proportional of the two extremes of the ratio to be removed and of the extreme of the ratio from which the removal is going to occur not featuring in the name of the

⁵⁰ I use the critical text of the *Prolegomena* that I have established in collaboration with N. Vinet and B. Vitrac. A debased edition (and a bad translation of the section of the *Prolegomena* on compounded ratios and removal) appears in Knorr 1989, 185–201.

procedure, each keeping its original position as antecedent or as consequent; this fourth proportional replaces, in the ratio from which the removal is going to occur, the extreme not featuring in the name of the procedure'. The *Prolegomena*, and other sources as we shall see, make the game more complicated by introducing a further parameter. This comes from the fact, in principle without relevance, that one of the extremes of the ratio to be removed might be identical to one of the extremes of the ratio from which the removal is going to occur. If one excludes, as the author of the *Prolegomena* does, the case in which each of the terms of the ratio to be removed is identical to one or the other of the extremes of the ratio from which the removal is going to occur, five possibilities remain, whose full list is provided: 1) $c = a$ (and therefore $d \neq b$); 2) $c = b$ (and therefore $d \neq a$); 3) $d = a$ (and therefore $c \neq b$); 4) $d = b$ (and therefore $c \neq a$); 5) $c \neq a$ and $d \neq b$. For no apparent reasons, in the sequel the author only works out the details of cases 1, 3 and 5, always in the order 'with respect to the antecedent' / 'with respect to the consequent'. He adds at the end of the exposition, as a 'postponed case', the one in which the ratio to be removed is the inverse of the ratio from which the removal is going to occur ($c = b$ and $d = a$)—a case that had just been excluded at the beginning.

Finally, if the consequent or the antecedent of the remainder do not coincide with the given consequent or the given antecedent, the author of the *Prolegomena* shows, on the basis of an example, how to transform the remainder to a ratio having the given term; this is done, as we have seen in the previous section, by means of a further taking of a fourth proportional. We shall see that a number of commentators will consider this final transformation of the remainder as an integral part of the operation of removal.

III.1.5 Leo the Mathematician on Composition and Removal of Ratios

A long scholium on *Elem.* VI, entitled ὑπόμνημα σχόλιον εἰς τὰς τῶν λόγων σύνθεσιν τε καὶ ἀφαίρεσιν, Λέοντος, is ascribed to the early ninth-century scholar Leo the Mathematician. The scholium is transcribed by the hand of the main copyist at the end of Book VI, and together with other exegetic material, at ff. 120–121 of Bodl. Dorv. 301.⁵¹

As the title suggests, the scholium deals with 'composition and removal of ratios'. As we have seen, this subject is only marginally dealt with in *Elem.* VI, even if df.5 provides a definition of compounded ratio. It is, therefore, misleading to assert, as it is commonly done, that the scholium is related to *Elem.* VI.df.5.⁵² The structure of this text

⁵¹ The scholium is edited in *EOO* V, 714.17–718.22, as a part of sch. 1 of the *Appendix Scholiorum* III.

⁵² On the basis of a misinterpretation of the alphabetic letters employed there as ordinals, the scholium was also absurdly held to contain germs of algebraic notation; cf. Vogel 1960, 661. To employ ordinals as dummy letters was a common and widespread practice in Greek mathematics; it is enough to get a look at *Elem.* V. These alphabetic signs are treated in the scholium as real denotative letters, since they are marked by a superimposed bar and not by an apex, as ordinal numbers usually are.

is as follows: a series of four arithmetic ‘lemmas’, followed by the proof of the main result on composition of ratios. Lemma 2 is followed by an ‘arithmetical clarification’ of both lemmas 1 and 2; the proof of the main result is followed by an ‘example’ (numerical $\acute{\upsilon}\pi\acute{o}\delta\epsilon\iota\gamma\mu\alpha$), in which it is shown that ratio 7:5 is compounded of ratios 7:11 and 11:5, by resorting to the $\pi\eta\lambda\iota\kappa\acute{o}\tau\eta\tau\epsilon\varsigma$ of the ratios. The scholium concludes with a short prescription on how to remove a ratio from another. Let us read the enunciations of the lemmas and of the main result, and see the numerical examples following them:

- <Lemma 1>: “Let there be numbers A, B, Γ , and let Δ be the <number contained> by A, B, and E by B, Γ , and again Z by A, Γ , and once more let H be the <number contained> by A, E, and Θ by B, Z and again K by Γ , Δ . I say that numbers H, Θ , K are equal to one another”. In short, the product of three numbers is the same, whichever is the order of the factors (recall that multiplication is a binary operator, and that its commutativity is proved in *Elem.* VII.16; the proof of the lemma is a straightforward adaptation of that of VII.16).
- Lemma 2: “Let a number A be multiple of B according to Γ . I say that B is also multiple of A according to the part homonymous to Γ ”. Arithmetical clarification of lemma 1: $4 \times 5 \times 6$ is equal to $5 \times 6 \times 4$, that is, to 120. Clarification of lemma 2: 100 is multiple of 20 according to 5, while 20 is multiple of 100 according to $\frac{1}{5}$.
- Lemma 3: “Let A be superparticular of B according to Γ . I say that, alternately, B is also superparticular of A according to the part homonymous to Γ ”. Example: the superparticular ratio 4:3 and the superparticular (in fact, subparticular) ratio 3:4.
- Lemma 4: “The same also happens with epimeric <ratios> [...] one must also conceive the same for compound ratios, such as multiple-superparticular and multiple-superpartient”.⁵³ Examples: 7:5 and 5:7; 7:3 and 3:7; 13:5 and 5:13.
- Main result: “Let there be a magnitude A having a ratio to B, let the value of this ratio be Γ , and between A, B let a chance magnitude Δ fall. I say that the ratio of A to B is conjoined both of that which A has to Δ and Δ to B”. The proof presents the mistake of compounding $\pi\eta\lambda\iota\kappa\acute{o}\tau\eta\tau\epsilon\varsigma$, that by definition can only be multiplied. (For this and the subsequent example, cf. **Document 8** in the Appendix.)
- Example of removal of ratios by means the procedure ‘with respect to the antecedent’ (see sections III.1.3 and III.1.4; the connection is not made explicit by Leo).

Only the proofs of lemma 1 and of the main result are fully-fledged demonstrations (the main result ends with a sentence that can be read as a hint as to how to iterate the proof in order to include nested compounded ratios). The proofs of lemmas 2–4 and of the final prescription are nothing but explanations carried out on paradigmatic examples; in particular, the procedure of removal of ratios remains quite obscure. Only lemmas 2–4 are identified as such in the text, which begins directly with the

⁵³ A superparticular ratio has the form $(n + 1):n$; a superpartient ratio has the form $(n + k):n$, with $n > k$.

enunciation of lemma 1. Lemma 1 is expressly referred to (expression ‘by the first lemma’ (διὰ τὸ α’ λήμμα)) within the arithmetical clarification of lemma 2 and within the main proof; lemma 2 is expressly referred to again within its own arithmetical clarification.

As is clear from the enunciations listed above, the scholium displays a deductive problem: lemmas 2–4 have nothing to do with the rest of the text, and in fact they are not used in the proof of the main result. If we were presented with such a state of affairs in a more structured exposition, there is no doubt that we would regard lemmas 2–4 as interpolated.

If we skip lemmas 2–4, the combination ‘lemma 1 + main result’ adopts a deductive progression that is found, exactly in this form, nowhere in ancient sources, even if it can be seen as a rewriting (longer and more cumbersome) of the proof presented by Eutocius in his two expositions (see section III.1.2). The terminology of ratios adopted in lemmas 2–4 is at best misleading, owing to the fact that the standard prefix ὑπο– for ratios lesser than unit is systematically omitted: see the example above, where it is said that ‘20 is multiple of 100’ instead of ‘20 is submultiple of 100’.

There is no doubt, then, that the scholium is badly organized, and sometimes inappropriately worded. My impression is that it is the result of the stratification of at least two layers: a core text comprising lemma 1 and the proof of the main result—maybe extracted from an ancient exposition to which we have no longer access—and the rest: lemmas 2–4, all numerical examples and the final prescription. Add to this that lemmas 1 and 2 are in their turn annotated (recall that the text is written on the full page in the manuscript) by three references to appropriate propositions of *Elem.* VII, which amounts to a logical *hysteron proteron* once the scholium is appended to *Elem.* VI; a short comment scholium to the effect of clarifying the notion of ‘homonymous part’ is instead contained in an indentation of the text at the beginning of lemma 2. The two- or multi-layered structure of the resulting text could explain its seemingly bizarre designation as ὑπόμνημα σχολίων.⁵⁴

The hypothesis about the composition of Leo’s scholium is confirmed by what precedes it at ff. 118v–119v of Bodl. Dorv. 301, and edited by Heiberg at *EOO* V, 711.6–714.16. Here is a synopsis.

- At 711.6–12. Abbreviated quotation of the definition of compounded ratio; example: the twelvefold ratio is compound both of the triple ratio and the quadruple, or of the double and the sextuple.
- At 711.12–18. Quotation of the enunciation of *Elem.* VI.19, with numerical example (if sides are in a triple ratio, triangles are in a ninefold ratio).
- At 711.19–24. List of the standard names of numerical ratios: multiple, superparticular, superpartient, etc.

⁵⁴ The second word is given in an abbreviated form. Heiberg, tentatively—and unnecessarily in my opinion—proposed the reading σχολικόν.

- At 712.1–713.7. (Wrong) examples of *πηλικότητες*: 3:1 is the *πηλικότης* of the triple ratio, 4:1 that of the quadruple ratio, 3:2 that of the sesquialter ratio (712.1–3; these are ratios, not values, indeed). Examples of composition, with associated diagrams: the text appears to be a description of the diagrams (712.4–9). Example of removal, with two associated diagrams: the remainder of the removal of a triple ratio from a double is sesquialter (712.9–713.7).
- At 713.8–11. Statement that “whenever the three terms of the proportion are not in identity of ratios, then we do not say that the first has to the third a duplicate ratio than to the second”.
- At 713.12–714.16. Texts that coincide almost verbatim with Leo’s lemma 1 and main proof. The main proof of this scholium contains the reference to the previous result in the form ‘by the lemma before itself’ (διὰ τὸ πρὸ ἑαυτοῦ λήμμα). A quick glance at the texts is enough to show that Leo’s scholium is beyond doubt a reworking of the anonymous scholium. In particular, the references to appropriate propositions or principles of *Elem.* VII one read in the margin of Leo’s lemma 1 are a subset of those annotated in the margin of the first proof in the anonymous scholium.

The typical diagrams associated with compounded ratios consist of three numbers arranged in a row, connected by arcs to which the denominations of the ratios are attached; we shall see an example of them in section III.2.2.

III.1.6 Some *Elements* Scholia on Compounded Ratios

Scanty information comes from the scholia n. 2–11 on Book VI of the *Elements* (EOO V, 320.6–331.4), all referring to df.5. Scholium 4 (*ibid.*, 324.10–326.7) is identical to the first exposition by Theon, summarized in section III.1.1. The other scholia mainly insist on providing long series of numerical examples, sometimes within the sexagesimal system in order to ease calculations (n. 3), thereby showing the dependency on a logistic context. One also finds short clarifications of the concept of *πηλικότης* (n. 2—with mention of Diophantus and a lexicon coming directly from the Platonic *Apologia Socratis* (verb *ταλαιπωρεῖν*)—3, 6 and 10), including a short discussion of the extension of this concept to non-expressible ratios (again n. 2). The most articulated exposition is ascribed to Maximus Planudes (n. 6, cf. **Document 10** in the Appendix). It contains: examples; (implicit) distinction between the number paronymous to a ratio and the ratio itself; assertion that “once the <numbers> paronymous to the compounded ratios (*συντιθεμένοις λόγοις*) are taken and multiplied by one another, it results a number paronymous to the compounded ratio”; examples. Finally, scholium n. 9 (*ibid.*, 330.6–8) is worth a transcription: σημείωσαι τὸ λόγος ἐκ λόγων· ἐν τῷ πέμπτῳ τοῦ ὀγδοῦ ἢ σύνθεσις εὗρηται καὶ ἡ διαίρεσις ἐν τῇ ἀρχῇ τοῦ θ’ (“*nota bene* ‘ratio from ratios’: the composition can be found in the fifth of the eight, the division at the beginning of the ninth”). These indications are exact, even if the subject

of interpolation of middle proportionals is one of the *leitmotif* of the entire books VIII–IX of the *Elements*.

III.1.7 George Pachymeres' *Quadrivium*

The first Byzantine treatise containing some elements of logistic is George Pachymeres' *Quadrivium*. These elements can be found in the first six chapters of Book IV, devoted to astronomy. Compounded ratios and removal of a ratio from another are expounded in chapters IV.3 and 4, respectively (cf. Tannery 1940, 352.14–357.30).

Pachymeres' rule for compounding ratios is simple: take the *πηλικότητες* of the ratios to be compounded and multiply them by one another; the result is a number, that can always be taken to be the *πηλικότης* of a ratio in lowest terms; this is the ratio compounded of the given ratios. As a consequence, to Pachymeres 'composition' is an operation. A series of numerical examples follows.

Things become more complicated with removal. Pachymeres first gives a series of instructions for removing a given ratio from a given ratio, the lesser from the greater, by means of a diagrammatic set-up. The removal may take place either with respect to the antecedent or with respect to the consequent of the ratio from which the removal is going to occur. Pachymeres introduces three unnecessary complications: First, the choice of removing with respect to the antecedent or to the consequent is made to depend from the 'position', in the compounded ratio from which the removal is going to occur, of the ratio to be removed (that is, a double ratio can be compounded either from a sesquitercian and a sequialter or from a sesquialter and a sesquitercian, in this order; numerical examples: sequialter 3:2 to be removed from double 8:4 or 12:6, which give the series of terms 8 6 4 and 12 8 6, respectively). Second, Pachymeres gives his instructions twice, the first exposition requiring in fact that the removal occurs from a ratio already provided in compounded form, the second expounding "by means of a certain rule which ratio is left out after the removal of the lesser ratio from the greater", so that the remainder is not known *a priori* but must be calculated (numerical examples in the second exposition: again sequialter 3:2 to be removed from double 8:4; sesquiquintum 6:5 to be removed from double 20:10; Pachymeres also provides a rule for calculating the *πηλικότης* of the remainder). Third, he requires, as many authors did before him and will do after him, that the ratio to be removed is less than the ratio from which the removal is going to occur.

When formulating his general rules, Pachymeres always uses the denominations of the ratios (that is, he refers to whole equivalence classes of ratios); he sets out specific terms of them only in the examples. For this reason, his exposition could not be devised to fit the practice of the *Almagest*, in which specific ratios are always set out—and in fact, Pachymeres offers no applications to this effect of the rules he states: his treatise is a compilation intended to remain on a theoretical level, not a computational primer to the *Almagest*. For Pachymeres' rule and instructions, cf. **Document 9** in the Appendix.

The distinguished Byzantine scholar Theodoros Metochites wrote a computational primer on the *Almagest* in the lines of the *Prolegomena*. He included three chapters on compounded ratios and removal of a ratio from another (chapters 8–10 of Book II of his *Abridged Astronomical Elements*; we read them in Vat. gr. 181, ff. 26r–35r): this is nothing but a plagiarism, with enormous expansions (Metochites' verbosity is legendary), of what we read in Pachymeres' *Quadrivium*.

III.1.8 Barlaam of Seminara's *Treatise of Logistic*

The distinguished Middle-Palaiologan theologian and polemicist Barlaam of Seminara was actively engaged in the study of astronomy and harmonics, and redacted a high-level compendium of number theory, the *Treatise of Logistic*.

Here is a list of the definitions and of the enunciations of Book V (see Carelos 1996, 69–93).

1. A value of a ratio is a number that, multiplied by the consequent term of the ratio, makes the antecedent.
 2. A ratio is said to be compounded of ratios when the values of the ratios, multiplied by one another, make the value of the ratio.
 3. When I remove a ratio from a ratio, I call 'antecedent [ἡγούμενον] ratio' that from which I make the removal, 'consequent' [ἐπόμενον] that <ratio> which I remove. The <ratio> that with the consequent <ratio> makes the antecedent <ratio>, this is said 'remainder' or 'left out' ratio.
 4. When I prescribe to remove a ratio from a ratio, then I prescribe to find the remainder.
 5. When someone, being prescribed to remove a ratio from a ratio, finds the <ratio> left out, the prescription turns out to be accomplished.
-
1. To find the value of a given ratio.
 2. To find the mean proportional of two given numbers.
 3. To find the third proportional of two given numbers.
 4. To find the fourth proportional of three given numbers.
 5. If there be three numbers in continued <proportion>, the ratio of the first to the third is compounded both of the ratio which the first has to the second and of the ratio which the second has to the third.
 6. If there be several numbers in continued <proportion>, the ratio of the first to the last is compounded of the ratios of the numbers lying in continued <proportion>, from the first as far as the last.
 7. If a ratio be compounded of several ratios, the inverse <ratio> is also compounded of the inverse ratios.
 8. If a ratio be compounded of two ratios and their values be taken, the same ratio is also compounded of the two ratios which its [*scil.* of the compounded ratio] value has to each of the remaining values.
 9. The values of identical ratios are equal, and the ratios whose values are equal are identical.
 10. If a ratio be compounded of two ratios, this same <ratio> is also compounded of two ratios, both the one that the antecedent of either ratio has to the consequent of the other, and the one that the antecedent of the latter has to the consequent of the former.

11. If a ratio be compounded of two ratios, any of these is compounded of two ratios, both the one that the antecedent of the compounded <ratio> has to the antecedent of the other ratio, and the one that the consequent of the latter has to the consequent of the compounded <ratio>.
12. If a ratio be compounded of two ratios and again of two other <ratios>, and either of the former two be identical to either of the latter two, the remainder will also be identical to the remainder.
13. If a ratio be compounded of several ratios, each of the compounding ratios is compounded both of the ratio that turned out to be compounded and of the inverse of all the remaining ones.
14. If the values of two ratios are taken, any of them is compounded both of the other and of the ratio which its value has to the value of the former.
15. As many ratios as one pleases and a number being given, to find which is the ratio compounded of them whose antecedent will be the given number.
16. As many ratios as one pleases and a number being given, to find which is the ratio compounded of them whose consequent will be the given number.
17. A given <ratio> being removed from a given ratio, to find which is the ratio left out.
18. To remove from a given ratio the inverse ratio in the same terms, so that the <ratio> left out is left out with respect to a given term.
19. To remove from a given ratio a given <ratio> having the same antecedent, so that the <ratio> left out is left out with respect to a given term.
20. To remove from a given ratio a given <ratio> having the same consequent, so that the <ratio> left out is left out with respect to a given term.
21. To remove from a given ratio a given <ratio> having the antecedent of the antecedent ratio as consequent, so that the <ratio> left out is left out with respect to a given term.
22. To remove from a given ratio a given <ratio> having the consequent of the antecedent ratio as antecedent, so that the <ratio> left out is left out with respect to a given term.
23. To remove from a given ratio a given <ratio> in specific terms, so that the <ratio> left out is left out with respect to a given term.

A few remarks are in order.

- Barlaam regards the compounded ratios written in standard form just as a species of the genus ‘compounded ratio’. He never speaks of an operation of ‘composition’ of ratios. He only deals with numerical ratios. He never uses the numerical $\delta\iota'$ ἴσου *Elem.* VII.14 in his proofs. He introduces a lexicon which is unique with his exposition (the ‘antecedent’ and ‘consequent’ ratio).
- Proposition 1 is a problem and explains how to find the $\pi\eta\lambda\iota\kappa\acute{o}\tau\eta\varsigma$ of a given ratio: just divide the antecedent by the consequent.
- The proof of proposition 5 is identical to Eutocius’ corresponding proposition. Barlaam also provides an alternative proof.
- In proposition 6, and very much like Eutocius in his exposition at *in Sph. cyl.* II.4, Barlaam takes for granted that, within a compounded ratio, one can replace ratios with compounded ratios: if $a:b = (a:d)\circ(d:b)$ and $a:d = (a:c)\circ(c:d)$, then $a:b = (a:c)\circ(c:d)\circ(d:b)$.
- Proposition 8 shows that a compounded ratio can be written in a canonical form that involves only the $\pi\eta\lambda\iota\kappa\acute{o}\tau\eta\tau\epsilon\varsigma$ of the compounded and of the compounding

ratios, and hence puts the operation of ‘replacement of identical ratios’ on safe grounds.

- Proposition 9 shows that ratios with the same $\pi\lambda\iota\kappa\acute{o}\tau\eta\varsigma$ form an equivalence class.
- Proposition 10 shows that the antecedents or the consequents of the compounding ratios can be interchanged.
- Propositions 11 and 13 show that any ratio compounding a compounded ratio can be written as a ratio compounded of the original compounded ratio and of the inverses of the other compounding ratios, possibly with antecedents or consequents of different ratios interchanged.
- Proposition 12 shows that the remainder of a removal of a ratio from a compounded ratio only depends on the $\pi\lambda\iota\kappa\acute{o}\tau\eta\varsigma$ of the removed ratio, that is, in modern parlance, on its equivalence class.
- Proposition 14 shows that either of any two ratios can be written as the ratio compounded of the other and of the ratio of their $\pi\lambda\iota\kappa\acute{o}\tau\eta\tau\epsilon\varsigma$.
- Propositions 15–23 are problems: 15–16 explains how to find the ratio compounded of a number of given ratios, whose antecedent or consequent is a given number; 17 shows, as an immediate consequence of proposition 14, that the result of a removal of one ratio from another is simply the ratio of the $\pi\lambda\iota\kappa\acute{o}\tau\eta\tau\epsilon\varsigma$ of the two ratios. Propositions 18–23 require to find, in a plurality of cases of removal, a remainder having a given extreme; they operate first the removal and then, by taking a suitable fourth proportional, transform the remainder to a ratio having the given extreme.

For Propositions 1, 8, 9, 14, and 17 cf. **Document 12** in the Appendix.

The distinguished Byzantine scholar John Chortasmenos wrote a tract on compounded ratios and removal of a ratio from another (we read it in the autograph Vindob. suppl. gr. 75, ff. 234r-256v, a copy of which is Ambros. C 263 inf., ff. 195r-212r): this is nothing but a plagiarism of a part of Book V of Barlaam’s treatise, in particular propositions 18–23.

III.1.9 Demetrios Cydones on Removal of Ratios

The Middle-Palaiologan scholar Demetrios Cydones was actively engaged in the study of astronomy. He personally owned an important manuscript of Ptolemy’s *Almagest*, Laur. Plut. 28.1, in which we read many scholia penned by his own hand,⁵⁵

⁵⁵ For a first assessment, see Acerbi & Pérez Martín 2015. A complete edition of Cydones’ scholia—which have never been studied—will follow.

as well as Manuel Bryennios' scholia we shall deal with in section III.2.3. In one of his own scholia, Cydones asserts that Ptolemy did not prove all cases of removal he actually uses in his treatise, and sets out to prove all of them. He does so by considering the compounded ratios arising in a plane case of the Sector Theorem actually proved by Ptolemy (see sections II.4 and III.1.3). He refers to the actual configuration of such a plane case, that provides many instances of identical ratios between line segments, in order to replace identical ratios in compounded ratios written in standard form. Cydones treats in this way most of the combinatorics allowed by the Sector Theorem.

Cydones states some of his results in general terms, and calls them 'porisms' (πορίσματα); cf. **Document 13** in the Appendix. Since his arguments refer to an actual configuration of the plane case of the Sector Theorem, they cannot be considered as valid proofs of the general statements he sets out. For instance, the proof leading to his first result has the following argument scheme.

- In a suitable plane case of the Sector Theorem, Ptolemy proves that $a:b = (c:d) \circ (e:f)$.
- To show that $b:a = (d:c) \circ (f:e)$.
- In fact, the actual configuration of the plane case envisaged by Ptolemy gives that, for a suitable line segment h , $b:a::h:c$ and $h:d::f:e$.
- Therefore, $b:a::h:c = (h:d) \circ (d:c) = (f:e) \circ (d:c)$.
- Therefore, $b:a = (d:c) \circ (f:e)$.
- Therefore, if a ratio be compounded of two ratios, the inverse is also compounded of the inverse ratios.

III.1.10 George Gemistus Plethon on Composition and Removal of Ratios and on Taking a Fourth Proportional

The short text of the distinguished Renaissance philosopher George Gemistus Plethon (cf. **Document 14** in the Appendix) was edited by Heiberg as an anonymous scholium to the *Elements*.⁵⁶ Only very recently the autograph, allowing the ascription to Plethon, has been discovered and published (Acerbi *et al.* 2016). The text contains concise descriptions of the operations of composition of ratios, of removal of a ratio from another, and of taking of a fourth proportional in the most generic case. The most interesting feature of Plethon's approach is that he introduces composition as an operation that removes the middle term between two extremes, all of them as a whole being conceived as a triad of terms: $(a,m,b) \equiv (a:m) \circ (m:b) \rightarrow a:b$. Removal is described along the same lines, the aim to keep the parallel between the two descriptions being evident: $(a,b,c) \equiv a:b \wedge c:b \rightarrow (a:c) \circ (c:b) \rightarrow a:c$. Finally, the operation of

⁵⁶ EOO V, 720.1–721.2, as sch. 2 of the *Appendix scholiorum* III.

taking of a fourth proportional is described as follows (note that Plethon's formulation in natural language is far clearer):

$$\begin{aligned} (a,b,c) \rightarrow [ac/b \wedge (a:b::m:c \vee a:m::b:c \vee c:b::m:a \vee c:m::b:a)] \vee \\ \vee [ab/c \wedge (c:a::b:m \vee m:a::b:c \vee c:b::a:m \vee m:b::a:c)] \vee \\ \vee [bc/a \wedge (a:b::c:m \vee m:b::c:a \vee a:c::b:m \vee m:c::b:a)]. \end{aligned}$$

III.2 Worked-out Examples of Removal of Ratios: Tabular Set-ups

III.2.1 Pappus' and Theon's *Commentaries* on the *Almagest*

Pappus, in the portion of his commentary on the *Almagest* that has been transmitted (namely, Books V–VI), applies the Sector Theorem and performs a removal of ratio from a ratio a number of times. This happens in *iA*, 81.15–83.3 and 83.4–19 (*Alm.* V.13), 101.16–102.15 (*Alm.* V.14), 185.7–16 (*Alm.* VI.5), 228.5–21 (*Alm.* VI.6), 243.14–244.5 (*Alm.* VI.7) (I will refer to these passages as numbers from 1 to 6). Surprisingly enough, an application of the Sector Theorem does not correspond in the *Almagest* to any of these passages. Pappus applies three different procedures: it is always required to remove $c:d$ from $a:b$, where $a:b$ is a compounded ratio not written in standard form, and to find one of the terms of the remainder the other being given (cf. section III.1.4). Of course, a , b , c , d are dummy letters standing for the specific numbers set out by Pappus.

- a) Passages 3 and 4. Case $d \neq a$ and $c \neq b$. In passage 3, Pappus computes $m = (bc)/d$ —that is, he takes a fourth proportional m according to the proportion $m:b::c:d$ —stating that “we reduce [ὑποβάλλωμεν] ratio $c:d$ to term b by multiplying the second by the third and taking one d^{th} of the result” (*iA*, 102.12–14). He then directly gives $a:m$ as the remainder. The intended procedure (see Theon's item i just below) inserts m as a middle term between a and b : $a:b = (a:m) \circ (m:b)$, removes $m:b::c:d$ and gets the remainder $a:m$. No explanatory calculation at all is provided in passage 4: Pappus just gives $a:m$ as the remainder.
- b) Passages 1, 2, and 6. Case $d \neq a$ and $c \neq b$, with the exception of 2, in which $c = b$ and $d \neq a$. Pappus computes $m = (ad)/b$ —that is, he takes a fourth proportional m according to the proportion $a:b::m:d$. He then sets (the verb is a passive participial form of ῥάσσειν) c as a middle term between m and d : $a:b::m:d = (m:c) \circ (c:d)$, so that $m:c$ obviously is the result of the removal of $c:d$ from $a:b$. In passage 1, the ratio $m:c$ is immediately stated to be identical to a suitable one having the given consequent of 120 degrees; in passage 2, $c = 120$ degrees and no such additional step is required. No explanatory calculation at all is provided in passage 6: Pappus just gives the ratio transformed of $m:c$ to the given consequent of 120 degrees as the remainder.
- c) Passage 5. Case $d = b$ and $c \neq a$. Without stating it as a general rule, Pappus directly gives $a:c$ (namely, the ratio of the homologous terms which are not identical) as the remainder.

In the portion of his commentary on the *Almagest* that has been edited by Rome, namely, Books I–IV, Theon repeatedly has the occasion of working out the details of the removals of ratio from a ratio that Ptolemy only alludes to in his treatise. This happens in *iA*, 575.8–578.5 (*Alm.* I.14), 578.17–579.10 (*Alm.* I.14), 591.5–594.7 (*Alm.* I.16), 595.18–596.4 (*Alm.* I.16), 619.14–620.7 (*Alm.* II.2), 622.5–623.7 (*Alm.* II.3), 624.3–13 (*Alm.* II.3) (I will refer to these passages as numbers from 1 to 7). Theon applies three different procedures; it is always required to remove $c:d$ from $a:b$, where $a:b$ is a compounded ratio not written in standard form, and to find one of the terms of the remainder the other being given (cf. again section III.1.4).

- i) Passages 1 and 2. Case $d = a$ and $c \neq b$. Theon takes a fourth proportional m according to the proportion $d:c::b:m$. He then takes (in all passages, the verb is a passive participial form of λαμβάνειν) m as a middle term between a and b : $a:b = (a:m) \circ (m:b)$, removes $m:b::c:d$ and gets the remainder $a:m$. In passage 1, he also provides a recipe to identify which is the remainder and which is the ratio to be removed between the two compounding ratios that result from inserting m as a middle term: one must always remove the second compounding ratio, that is, the one containing the consequent of the ratio from which the removal is going to occur (*iA*, 577.1–4).
- ii) Passages 3 to 6. Case $d \neq a$ and $c \neq b$, with the exception of 5, in which $c = b$ and $d \neq a$. Theon computes a fourth proportional m according to the proportion $a:b::m:d$, but takes c as a middle term between x and d : $a:b::m:d = (m:c) \circ (c:d)$, so that $m:c$ obviously is the result of the removal of $c:d$ from $a:b$. Passage 3 is enriched by a remark to the effect of introducing an alternative procedure, identical to that which we have called ‘with respect to the antecedent’ (cf. again section III.1.4). Theon also points out that, in this case, the middle term does not yield the result with respect to the antecedent of the ratio from which the removal is going to occur, but with respect to the consequent. Therefore one must always remove the first compounding ratio, and keep the second as the remainder. At the end of this very passage, he comments on the variety of cases involved in the operation of removal of ratios (*iA*, 592.19–594.6).
- iii) Passage 7. Again case $d \neq a$ and $c \neq b$. Theon takes a fourth proportional m according to the proportion $c:d::a:m$. He then takes m as a middle term between a and b : $a:b = (a:m) \circ (m:b)$, removes $a:m::c:d$ and gets the remainder $m:b$. Without stating it as a general rule, he removes the first compounding ratio, that is, the one containing the antecedent of the ratio from which the removal is going to occur.

If the consequent or the antecedent of the remainder do not coincide with the given consequent or the given antecedent (passages 3, 4, 6, 7: the given value is always 120 degrees), Theon shows how to transform the remainder to a ratio having the given term; this is done by means of a further taking of a fourth proportional. We have seen (and we shall see again) that a number of commentators will consider this final ‘fitting’ as an integral part of the operation of removal.

III.2.2 Two Ancient Scholia to Ptolemy's *Almagest*

The context is the same as that outlined in section III.1.3. The scholia described here (cf. **Document 5** in the Appendix) almost immediately follow, in the margins of our manuscripts of the *Almagest*, the scholium described in section III.1.3 (cf. **Document 4** in the Appendix).

The first scholium presented in this section works out the details of the procedure of removal of ratios.⁵⁷ It does so by outlining first a general prescription. A tabular set-up allowing to perform the removal in an orderly way is then described. The tabular set-up so described coincides with a second scholium, to be seen below. In the general prescription, the case in which the antecedent (consequent) of the compounded ratio coincides with the antecedent (consequent) of the removed ratio is treated first: the terms different from those which coincide will form the ratio resulting from the removal. This case is exemplified by setting out two triples of numbers, 12 6 4 and 12 4 3, joined by suitable arcs, so as to make clear that the compounded ratio coincides with 12:4 or 12:3, respectively, the removed ratio with 12:6 or 4:3, respectively (that is: no arc joins 6 and 4 in the sequence 12 6 4, no arc joins 12 and 4 in the sequence 12 4 3). The triple 12 4 3 coincides with the one set out to the same effect in the scholium described in section III.1.3, but there the removed ratio is 12:4.

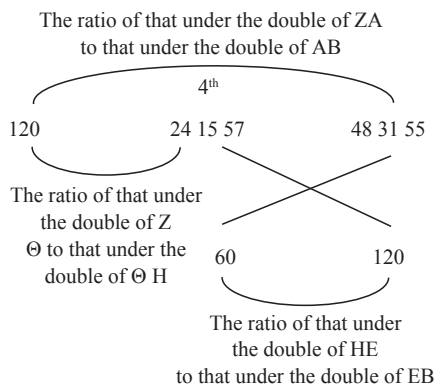
The general case is dealt with by taking a suitable fourth proportional; the numerical examples, set out in a standard *chi*-shaped scheme of calculation, are heavily corrupt and cannot be reconstructed without introducing arbitrary corrections. The general prescription runs as follows, taking again $a:b = (c:d) \circ (e:f)$ as a reference: (i) if you want to remove the first compounding ratio, make $c:d::a:m$ and the remainder is $m:b$; (ii) if you want to remove the second compounding ratio, make $d:c::b:m$ and the remainder is $a:m$.

After this, the scholium describes the tabular set-up associated to the removal alluded to in the *Almagest* passage to which the scholium itself refers. This amounts to a modification of the standard *chi*-shaped scheme of calculation of a fourth proportional; the gist of the description resides in the general indications about the places to be assigned, in the tabular set-up, to the relevant terms of the given ratios, namely, the compounded ratio and the removed ratio. After performing the calculation and identifying its result with the relevant chord, the scholiast points out that the case at issue is particular, since the antecedent of the remainder (that is, of the compounded ratio: the removal is made according to procedure (ii) above) is equal to the given

⁵⁷ Its *relatum* in Ptolemy's treatise is *Alm.* I.14, *POO* I.1, 78.3–7: “therefore, if from the ratio of 120 to 48;31,55 we remove that of 60 to 120, the ratio is left out of the <chord> under the double of <arc> ΖΘ to that under the double of ΘΗ, the one of 120 to 24;15,57” (ἐὰν ἄρα ἀπὸ τοῦ τῶν ρκ πρὸς τὰ μη λα΄νε’’ λόγου ἀφέλωμεν τὸν τῶν ξ πρὸς τὰ ρκ, καταλείπεται ὁ λόγος τῆς ὑπὸ τὴν διπλὴν τῆς ΖΘ πρὸς τὴν ὑπὸ τὴν διπλὴν τῆς ΘΗ ὁ τῶν ρκ πρὸς τὰ κδ ιε΄νζ’’).

antecedent of the sought ratio. If they were different, one simply would have to take again a fourth proportional between the terms of the remainder and the given antecedent of the sought ratio (note that the scholiast does not state this in general terms). On the other hand, the scholiast rightly omits the case in which the removed ratio comes first in order: as he himself claims, one has to “operate in a similar way”, the only difference being that one has to put the first number in order of the removed ratio under the first number in order of the compounded ratio. All in all, this is the simplest and most concise exposition of the operation of removal of a ratio.

The tabular set-up described by the main scholium, and attested as a further scholium, can be summarized as follows: one has to remove the compounding ratio 60:120 from the compounded ratio 120:48;31,55. One sets first in a row the terms of the compounded ratio, then (in case the assigned compounding ratio comes second in order) the consequent 120 of the ratio to be removed under the consequent of the compounded ratio, and its antecedent on the side of the former consequent, ‘in the intermediate space’. A standard *chi*-shaped scheme of calculation of a fourth proportional gives the result 24;15,57, to be written between the terms of the compounded ratio, with ‘4th’ marked above. Now there are three terms in a row: 120 and 24;15,57 and 48;31,55; in order to find the remainder of the removal of ratio 24;15,57:48;31,55 (which is identical to ratio 60:120) from ratio 120:48;31,55, it is enough to look at the first two terms in the row. Here is the tabular scholium, where arcs also connect the terms of the compounded ratio and the terms of the remainder.



III.2.3 Manuel Bryennios' Scholia to Ptolemy's *Almagest*

Manuel Bryennios' autograph scholia on Ptolemy's *Almagest* have been edited and studied only very recently (Acerbi & Pérez Martín 2015). Four of the five scholia perform in all details some removals only alluded to by Ptolemy. The fifth scholium (cf. **Document 11** in the Appendix) describes a tabular set-up of the calculations

involved in the related removal. The procedures at work in the first four scholia can be summarized as follows.

The compounded ratio $a:b = (c:d) \circ (e:f)$ is given and it is required to find e . Take a fourth proportional m such that $a:b::m:d$. Write a ‘fictitious’ compounded ratio $(m:c) \circ (c:d) = m:d::a:b = (c:d) \circ (e:f)$. This entails that $e:f::m:c$. Now, if $c = f$, one immediately has $m = e$ (first and third scholium). If, however, $c \neq f$, by taking again a fourth proportional e of antecedent and consequent of the remainder $m:c$ of the ‘fictitious’ compounded ratio and of term f —that is, by identifying the two remainders above—namely, $m:c::e:f$, one gets the required term as e (second and fourth scholium). Bryennios’ procedure, which is an elaboration (with obvious lexical loans) of Theon’s prescriptions seen in section III.2.1, can be viewed as an application of the method of simple false position.

The fifth scholium runs as follows.⁵⁸ The Sector Theorem gives rise to the relation $a:b = (c:d) \circ (e:f)$, where $a = 70;32,3$, $b = 97;4,56$, $c = 117;31,15$, $d = 24;15,57$, $f = 120$; to find e . Ratio $a:b$ is given and we have first to find $e:f$ by removal of $c:d$. Let us trace a diagram in form of letter *chi* and place (i) at its two upper extremes from left to right, the consequent and the antecedent of the compounded ratio, respectively; (ii) at its lower left extreme, the consequent of the removed ratio. Write at the lower right extreme of the *chi* the result $m = 17;37,45$ of taking the fourth proportional of the terms set out according to this very diagram, namely, by multiplying the two terms coupled by a stroke of the *chi* and by dividing the result by the uncoupled term: $b:a::d:m$. By the side of this result, between the two lower extremes of the *chi*, place the antecedent of the removed ratio. As explained above, if one writes a ‘fictitious’ compounded ratio $(c:d) \circ (e:f) = a:b::m:d = (m:c) \circ (c:d)$, and if one has that $c = f$, one immediately gets $e = m = 17;37,45$. But $c \neq f$, and by taking again a fourth proportional by means of a *chi*-diagram, applied to antecedent and consequent of the remainder $m:c$ of the ‘fictitious’ compounded ratio and to the term $f = 120$ that had not yet been employed—that is, by identifying the two remainders above—namely, by setting $m:c::e:f$, one gets $e = 18;0,15$.

III.2.4 George of Trebizond’s *Isagoge* to the *Almagest*

The renowned Cretan scholar and translator George of Trebizond operated in early-Renaissance Italy. Among his several activities, in 1451 he both translated Ptolemy’s *Almagest* from Greek into Latin and redacted a monumental commentary thereon, still lying unedited. He had a complex personality which eventually drove him to quarrel with almost all scholars of his time. His controversy with cardinal

⁵⁸ Its *relatum* in Ptolemy’s treatise is *Alm.* II.7, *POO* I.1, 121.21–122.21.

Bessarion, whom he also thought to be the Antichrist, had as a consequence that Regiomontanus, an associate of Bessarion, wrote an even more monumental counter-commentary on the *Almagest*, explaining in detail where his rival was wrong, and where Theon of Alexandria, in his turn attacked by George of Trebizond, was right.⁵⁹ Since George of Trebizond's skills as a translator of scientific texts were also questioned by Jacopus of Cremona, a translator of Archimedes from Greek into Latin, the fate of George of Trebizond's scientific translations was decided already at the end of the fifteenth century.

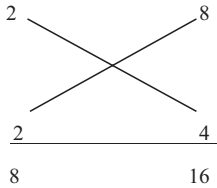
The first section of George of Trebizond's commentary is the so-called *Isagoge*, a computational primer on the *Almagest*, very much in the style of the *Prolegomena* and of its Byzantine avatars. I have studied this text in detail in view of a forthcoming edition. Here is a summary of the sections on compounded ratios and removal.

The operation of composition of ratios is first introduced: "a composition of ratios is that obtained from two ratios, and a compound ratio is that composed of two ratios, either rational or irrational" (σύνθεσις λόγων ἐστὶν ἢ ἐκ δύο λόγων συναγομένη, καὶ σύνθετος λόγος ὁ ἐκ δύο συνιστάμενος λόγων ῥητῶν ἢ ἀρρήτων).⁶⁰ After a long series of examples, partly intended to clarify the difficulties involved in compounding irrational ratios, George of Trebizond shows, again by means of examples, that "the denomination of compound ratios can be found [...] once the numbers denominating the compounding ratios are multiplied" (ἡ δὲ παρονομασία τῶν συνθέτων λόγων εὕρηται [...] πολλαπλασιαζόμενων τῶν παρονομαζόντων τοὺς συντιθέντας λόγους ἀριθμῶν). After a further series of examples, a final remark points out that any numerical ratio can be thought of as compounded of the ratios of the square roots of its terms.

As for removal, it is defined as follows: "a removal of a ratio from a ratio is that by means of which, one of the compounding <ratios> being thrown away, the other is left over" (ἀφαίρεσις λόγου ἀπὸ λόγου ἐστὶ δι' ἧς τοῦ ἐτέρου τῶν συντιθέντων ἀπορριφθέντος ὁ ἕτερος ἀπολείπεται). The computational tool of 'chiasmus' (χιασμός), or 'crosswise arrangement' (cf. sections III.2.2 and III.2.3) is introduced, in order to perform the removal (cf. **Document 15** in the Appendix). If you are given a compounded ratio written in standard form, for instance $2:8 = (2:4) \circ (4:8)$, whose terms are usually written as the sequence 2 4 8, put the terms of the compounded ratio 2 and 8 at the upper extremes of a *chi*-shaped scheme, the terms of one of the compounding ratios, say 2 and 4, at the lower extremes of the same *chi*-shaped scheme, perform crosswise multiplication and write the results below the corresponding term of the compounded ratio, thus:

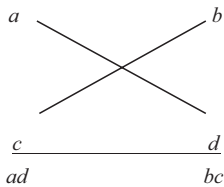
⁵⁹ For George of Trebizond's translation of the *Almagest*, see <http://www.wilbourhall.org/index.html#ptolemy>; Regiomontanus' *Defensio Theonis* is transcribed and studied in <http://regio.dartmouth.edu/about/about-project.html>. On George of Trebizond, see Monfasani 1976 and 1984.

⁶⁰ An 'irrational' ratio is such that its terms contain fractional parts; a 'rational' ratio has only integer numbers as its terms. This was standard terminology in Byzantine authors.



The ratio of the results is the remainder, as is easy to check: if $a:b = (a:c) \circ (c:b)$, the operation performed on the *chi*-shaped scheme amounts to taking the ratio of ac to ab , which is obviously identical to $c:b$. If we put the terms of the other compounding ratio, namely, c and b , below the terms a and b of the compounded ratio, we get ab to cb , which give rise to the other compounding ratio $a:c$.

If the given compounded ratio is not written in standard form, the same *chi*-shaped scheme provides again the two terms of the reminder, according to the following set-up:



Since $ad:bc$ is the remainder of the removal of ratio $c:d$ from ratio $a:b$, George of Trebizond's prescription is correct and a most general one. He backs up his prescription by a further rule, allegedly conforming to Ptolemy's practice: put the four terms of the compounded and of the removed ratio in a row, in such a way that those of the compounded ratio are the first and the third term, thus: $a\ c\ b\ d$. Then the remainder can be found in four different ways. Form the product of the middle terms c, b or that of the extremes a, d ; divide by either of the extremes or of the middle terms, respectively; form the ratio which has the result of the division as the consequent (resp. antecedent) and the term not yet employed among $a\ c\ b\ d$ as the antecedent (resp. consequent): this ratio is the remainder. This amounts to write the remainder $ad:bc$ in one of the following four ways: $d:(a/bc)$, $a:(d/bc)$, $(ad/b):c$, $(ad/c):b$.

IV Conclusion

The evidence collected above shows that compounded ratios and removal, as a theoretical theme and not as a tool, were exegetical *topoi*, related to two very specific *loci* of the Greek mathematical corpus: the Sector Theorem in the *Almagest* (more specifically related to removal) and VI.df.5 of the *Elements* (more specifically

related to compounded ratios). Still, the expositions of the topic did not organize in exegetic chains. On the contrary, the slight variations in the actual procedures adopted to implement an algorithm that involves just a couple of basic arithmetic operations show that, with some exceptions, our sources worked independently. One of the reasons must have been that, on the logistic side, no general recipe for removing ratios was provided in Theon's commentary on the *Almagest*, and that, on the number-theoretic side, no general exposition was provided before the Late Antiquity—at least if we are to believe what Eutocius says, and if we concede that he had access to all relevant sources.

As for the exceptions, we may safely assert: that the exposition of removal in the *Almagest* scholium that eventually came to be ascribed to Domninus (section III.1.3) and the one in the *Prolegomena* (section III.1.4) have the same origin, whereas a different, and simpler approach is suggested in another *Almagest* scholium (section III.2.2); that Leo (section III.1.5) simply appropriated an anonymous scholium that provided a proof, different from the one we read in Eutocius (section III.1.2), of the basic result about compounded ratios written in standard form; that Bryennios (section III.2.3) obviously followed Theon (sections III.1.1 and III.2.1). Two authors stand in isolation: Barlaam (section III.1.8), whose treatise on the number-theoretical foundations of logistic I regard as the solitary achievement of brilliant mind; George of Trebizond (section III.2.4), whose approach may however depend on medieval Latin sources to an extent that future researches will determine. Other authors either contributed trivial material to the issue (this is the case of Maximus Planudes: section III.1.6), stated general rules supported however by non-general proofs (Demetrios Cydones: section III.1.9) or by no proofs at all (George Pachymeres: section III.1.7), or just jotted down a few lines trying to catch the gist of the operations of composition and removal (George Gemistus Plethon: section III.1.10).

Compounded ratios were used perfunctorily in the geometric tradition (sections II.1–6), from the *Elements* to Pappus; they allowed Apollonius to treat a number of properties of conic sections in a compact way; they allowed to formulate mathematical problems that would otherwise have been beyond reach of the Greek mathematical lexicon (the loci on more than four lines).

The logistic tradition dealt with the issues of compounded ratios and, more specifically, of removal of ratios in the context of the applications of the Sector Theorem—that is, one must stress, in a *numerical* context. In any such application, a compounded ratio is provided not written in standard form and it is required to find one of the terms, say e in $a:b = (c:d) \circ (e:f)$, the others being given. Is this trivial?—After all, it is enough to calculate $e = adf/bc$: three multiplications and one division. It was not.

There are two facets to the issue. First, the basic operational unit of what we formulate as a sequence of two multiplications and one division was almost invariably conceptualized as a taking of a suitable fourth proportional: three numbers a, b, c being given in this order, the associated fourth proportional $m(abc)$ satisfies the proportion $a:b::c:m$ and, as a consequence, can be calculated as $m = bc/a$. A first

complication came from the fact that, on the one hand, in order for m to be a fourth proportional, its position in the above proportion is immaterial: it is not a fourth proportional because it is placed in the *fourth* position—but, on the other hand, changing the position of the fourth proportional in the above proportion amounts to calculating a different number. Was this problem settled once and for all by adopting a convention?—It was not.

Second facet of the issue, the calculations leading to the value of e in $a:b = (c:d) \circ (e:f)$ were almost invariably split in two operations: first, to remove ratio $c:d$ from the compounded ratio $a:b$ in order to single out a ratio $x:y$ corresponding to $e:f$; second, since both f and the ratio corresponding to $e:f$ are given, to determine e , which is given by *Data 2*. The point is that the ratio corresponding to $e:f$ is given as the remainder of a removal, so that in general its consequent y is not equal to f : again, one has to take a fourth proportional $m(yx)$ in order to finally find e . Since removal of ratios almost uniquely occurred in the applications of the Sector Theorem, were these two operations unified under one single standard prescription, possibly admitting branchings in its formulation?—They were not.

Still, the general procedure for solving the above compounded ratio for e would simply entail no more than two nested takings of a fourth proportional: $e = m(cm(bad)f)$ —‘Simply’?

Appendix

Document 1: Extracts from Euclid, *Sectio Canonis*⁶¹

1. If a multiple interval compounded twice [δὶς συντεθέν] make some interval, it will also be multiple.

Let there be an interval BF , and let B be a multiple of Γ , and let, as Γ is to B , B be made to Δ . Then, I assert that Δ is a multiple of Γ .

In fact, since B is a multiple of Γ , therefore Γ measures B ; and, as Γ is to B , B was made to Δ , so that Γ also measures Δ ; therefore Δ is a multiple of Γ .

3. If an interval compounded twice make the whole multiple, it will also be multiple.
4. If a non-multiple interval be compounded twice, the whole will neither be multiple nor superparticular.
5. If an interval compounded do not make the whole multiple, it will not be multiple either.
6. A double interval turns out to be composed [συνέστηκεν] of the two greatest superparticular intervals, both the sesquialter and the sesquitercian.

⁶¹ Translated from *EOO* VIII, 160.5–13 and 14–16; 164.1–3 and 15–17; 166.1–3; 168.1–26. Proposition 3 is excluded since it does not deal with compounded ratios.

7. From a double interval and a sesquialter, a triple interval results.

In fact, let A be double of B and B sesquialter of Γ . I say that A is triple of Γ .

In fact, since A is double of B, therefore A is equal to two B. Again, since B is sesquialter of Γ , therefore B contains Γ and half of it; therefore two B are equal to three Γ ; and two B are equal to A; therefore A is also equal to three Γ : therefore A is triple of Γ .

8. If from a sesquialter interval a sesquitertian interval be removed, the remainder is left out as sesquioctave.

In fact, let A be sesquialter of B and let Γ be sesquitertian of B. I say that A is sesquioctave of Γ .

In fact, since A is sesquialter of B, therefore A contains B and half of it; therefore eight A are equal to twelve B. Again, since Γ is sesquitertian of B, therefore Γ contains B and a third of it; therefore nine Γ are equal to twelve B; and twelve B are equal to eight A; therefore eight A are equal to nine Γ ; therefore A is equal to Γ and an eighth of it; therefore A is sesquioctave of Γ .

Document 2: Pappus, *Collectio* VII.240⁶²

Let A to B have the ratio conjoined both of that which Γ has to Δ and of that which E has to Z. That Γ also has to Δ the ratio conjoined both of that which A has to B and Z to E.

In fact, let the <ratio> of Δ to H be made identical to that of E to Z.

Now, since <the ratio> of A to B is conjoined both of that of Γ to Δ and of that of E to Z, that is, of Δ to H, but the <ratio> conjoined both of that which Γ has to Δ and of that which Δ has to H is that of Γ to H, therefore, <as> A is to B, so Γ is to H. And since Γ has to Δ the ratio conjoined both of that which Γ has to H and of that which H has to Δ , but the <ratio> of Γ to H has been proved identical to that of A to B, and by inversion the <ratio> of H to Δ is identical to that of Z to E, therefore Γ also has to Δ the ratio conjoined both of that which A has to B and of that which Z has to E.

Document 3: Theon's Main Arguments⁶³

In fact, let AB to $\Gamma\Delta$ and $\Gamma\Delta$ to EZ have a given ratio. I say that the ratio of AB to EZ is compounded both of AB to $\Gamma\Delta$ and of $\Gamma\Delta$ to EZ, that is, that, if the value of the ratio AB to $\Gamma\Delta$ be multiplied by that of $\Gamma\Delta$ to EZ, it makes that of AB to EZ as a value of ratio.

In fact, let first AB be greater than $\Delta\Gamma$ and $\Gamma\Delta$ than EZ, and let AB be double of $\Gamma\Delta$ and $\Gamma\Delta$ triple of EZ. Now, since $\Gamma\Delta$ is triple of EZ and AB double of $\Gamma\Delta$, therefore AB is sextuple of EZ—since, if we double the triple of something, the sextuple of it also results, for this is a ‘composition’ in strict sense.

⁶² Translated from Jones 1986, 301.30–303.12.

⁶³ Translated from *in Alm.* I.13, edited in *iA*, 533.3–535.7; also, *in Alm.* II.11, edited in *iA*, 761.1–762.2.

Or in this way here. Since AB is double of $\Gamma\Delta$, let AB be divided into AH, HB equal to $\Gamma\Delta$. And since $\Gamma\Delta$ is triple of EZ, exactly for the same reasons HB is also triple of EZ; therefore AB as a whole is sextuple of EZ; therefore, through the middle term $\Gamma\Delta$, the ratio of AB to EZ turns out to be deduced to be compounded both of that of AB to $\Gamma\Delta$ and of $\Gamma\Delta$ to EZ.

And similarly, even if $\Gamma\Delta$ be less than any of AB, EZ, the same will be deduced. In fact, let again AB be triple of $\Gamma\Delta$ and $\Gamma\Delta$ half of EZ. And since $\Gamma\Delta$ is half of EZ and AB triple of $\Gamma\Delta$, therefore AB is sesquialter of EZ—for, if we triple the half of something, it will contain it once and one half time.

Or also in this way. Since AB is triple of $\Gamma\Delta$ and $\Gamma\Delta$ half of EZ, EZ is so many two of which AB is three equal to $\Gamma\Delta$, so that AB will be sesquialter of EZ; therefore, through the middle term $\Gamma\Delta$, the ratio of AB to EZ turns out to be deduced to be compounded both of that of AB to $\Gamma\Delta$ and of $\Gamma\Delta$ to EZ.

But now, again, let $\Gamma\Delta$ be greater than any of AB, EZ, and let AB be half of $\Gamma\Delta$ and $\Gamma\Delta$ sesquitertian of EZ. Now, since $\Gamma\Delta$ is so many four of which AB is two and EZ is so many three of which $\Gamma\Delta$ is four, therefore EZ is also so many three of which AB is two; therefore, through the middle term $\Gamma\Delta$, the ratio of AB to EZ—the one of two to three, subsesquialter—turns out to be deduced again to be compounded both of the subdouble and of the sesquitertian. Exactly in a similar way also for several <terms> and in the other cases. [...]

We shall prove that if[, inversely,] a ratio be compounded of two ratios, one of the compounding <ratios> is compounded both of the remaining compounding <ratio> inversed and of the compounded one.

In fact, let the ratio of A to Γ , once B is taken as a middle, be compounded both of that of A to B and of B to Γ . I say that the ratio of A to B is also compounded of that of A to Γ and of Γ to B.

And it is immediately evident: for, if we place B after Γ , the ratio of A to B, once Γ is taken as a middle, will be compounded both of that of A to Γ and of Γ to B.

Document 4: The *Almagest* Scholium at the Origin of the Treatise Ascribed to Domninus of Larissa⁶⁴

When we undertake to remove a ratio from a ratio, it is clear that this is nothing other than to resolve [διαλῦσαι] the ratio from which the removal is going to occur into the removed <ratio> and the remaining <ratio> after this. And in fact that from which the removal is going to occur is compounded of the removed <ratio> and the remainder after this—for the lesser <ratio> is removed from the greater. Now, how will the resolution come to occur? Or is it clear that <it will> conversely to composition. A ratio is said to be compounded of ratios when the values of the ratios, multiplied by one another, make some. If, in fact, another term is set between two terms,

⁶⁴ Translated from Acerbi 2017, 228–229.

the ratio of the first to the second, multiplied by that of the second to the third, makes the ratio of the extremes, whatever the middle <term> might be. For instance, let 4 be a middle of 2 and 8. Now, since 2 has to 4 the ratio of a half, and likewise 4 to 8, I multiply a half by a half and I make $\frac{1}{4}$, and I find that 2 is a fourth part of 8. Yet even if I set 40 instead of 4 as a middle term, the <ratio> of 2 to 8 is, again, compounded both of that of 2 to 40 and of 40 to 8, for 2 is a 20th part of 40, 40 is quintuple of 8, and $\frac{1}{20}$ times 5 makes $\frac{1}{4}$.

Now, since a ratio is said to be compounded of ratios exactly when the values of the ratios, multiplied by one another, make some, it is clear that, if a given ratio be removed from a given ratio, the remainder will also be given. For if we have both the value of the ratio from which the removal is going to occur and the <value> of the removed <ratio>, we will also have the remaining value, which, when it is multiplied by the value of the removed <ratio>, makes the one of the compound <ratio>, $\frac{1}{4}$.

Now, if one removes the value of the lesser of the given ratios from the one of the greater, this will come about in two ways, either by the lesser ratio being transposed to the greater and fitted to it, or by the greater being transformed into the lesser and made to contain it; and each of these cases will be accomplished in two ways. For, each of the ratios having an antecedent and a consequent, if the least is going to be fitted into the greater ratio, the removal will occur either with respect to the antecedent (as for 12 4 8) or with respect to the consequent of the greater ratio (as for 12 $4\frac{1}{2}$ 3) from which the removal is going to occur; if the greater <ratio> has been transformed into the lesser, again, the removal will occur either with respect to the antecedent or with respect to the consequent of the lesser.

Now, this is the way it should be done in general. But if we should happen to find in the greater and the lesser ratio the same term in either the antecedent or the consequent, finding <the ratio> will be easier for us, for these <terms> other than the identical terms will contain the remaining ratio. For instance, from the ratio of twelve to 3, let it be required to remove that of 12 to 4; 12 being fitted to 12 and 4 being set as a middle <term>, the ratio of 12 to 4 turns out to be removed, and that of 4 to 3 remains, which is sesquitercian, for sesquitercian multiplied by triple makes quadruple. Here, too, it is similarly possible to carry out the transition from the greater to the lesser or from the lesser to the greater, and the removal can occur with respect to either the antecedent or the consequent.

Document 5: The Tabular Set-up Described in an *Almagest* Scholium⁶⁵

If, removing a given ratio from a given ratio, we want to find the remaining <ratio>, if the antecedent of the removed <ratio> is found to be identical to the antecedent of

⁶⁵ Translated from Acerbi 2017, 233.

the ratio from which one removes, or the consequent identical to the consequent, the terms in the ratios that are different immediately contain the remaining ratio (as for 12 6 4 or 12 4 3), for the remaining term of the removed ratio will come to be a middle <term> of the <terms> containing the ratio from which one removes. If, instead, there is no such relationship among the <terms>, one must make, either (namely, when the removed ratio is first among the compounding <ratios>) as the antecedent of the removed <ratio> to its consequent, so the antecedent of the ratio from which the removal is going to occur to some other <number>, or (namely, when the removed ratio is second among the compounding <ratios>) as the consequent to the antecedent <of the removed ratio>, so the consequent of the ratio from which the removal <is going to occur> to some other <number>—for the fourth proportional, together with the remaining <term> that contains the ratio from which the removal is going to occur, will make the remaining ratio.

16 8^{4th} 4 16 8^{4th} 4
6 10 6 3

In which way must these things be carried out, and in which way must we arrange the numbers, we shall learn as follows. We first arrange those of the compounded ratio, and we take first the first in the ratio, and second the second (namely, first 120 and second 48;31,55). Next, since we have 3 numbers given and one single ratio given among the compounding <ratios>, we arrange next underneath the ratio which is given among the 2 compounding ratios. And, if the given ratio comes last in order (as in the theorem at issue), we arrange the last number in order of the given ratio under the the last number in order of the compounded ratio (namely, 120 under 48;31,55), and we arrange the first <number> of the given ratio (namely, 60) in between, and so to speak in the intermediate space, and multiplying 60 by 48;31,55 and dividing by 120 we find 24;15,57, and we arrange it above 60, and, clearly, we have removed from the ratio of 120 to 48;31,55 the given ratio of 24;15,57 to 48;31,55 (for it is identical to that of 60 to 120) and we have the ratio of 120 to 24;15,57—which we also seek—given as a remainder.

Now, since we have found the sought ratio as that of 120 to 24;15,57, and the sought ratio was that of the <chord> under the double of <arc> ZΘ to that under the double of ΘH, therefore we have the ratio of the <chord> under the double of <arc> ZΘ to that under the double of ΘH, namely, that of 120 to 24;15,57; and that under the double of ZΘ is 120 (which is also that under the double of ZA); therefore that under the double of ΘH is 24;15,57. And in this way we may find how much is it that under the double of ΘH. If instead that under the double of ZΘ had not been 120, but by hypothesis 40, we would have made, as 120 is to 24;15,57, so 40 to some other <number>, and in this way we may find how much is it that under the double of ΘH. We shall operate in a similar way even if the sought ratio happens to be the last in order.

Document 6: Eutocius on *Sph. Cyl. II.8*⁶⁶

Let there be four terms A, Γ, Δ, B . I say that the ratio compounded of that of the <rectangle contained> by A, B to the <square> on Γ with [μετά] the ratio of B to Δ is identical to that of the <rectangle contained> by A, B times [ἐπί] B to the <square> on Γ times Δ .

In fact, let K be equal to the <rectangle contained> by A, B , and Λ to the <square> on Γ , and let, as B is to Δ , so Λ come to be to M ; therefore the ratio of K to M is compounded of that of K to Λ , that is, of that of the <rectangle contained> by A, B to the <square> on Γ , and of Λ to M , that is, of B to Δ .

Then, let K multiplying B make N , and Λ multiplying B make Ξ , multiplying Δ make O .

Now, since the <rectangle contained> by A, B is K , and K multiplying B turns out to make N , therefore N is that by A, B times B . Again, since the <square> on Γ is Λ , and Λ multiplying Δ turns out to make O , therefore O is that on Γ times Δ , so that the ratio of that by A, B times B to that on Γ times Δ is identical to that of N to O . Therefore one has to prove that the ratio of K to M is identical to that of N to O .

Now, since each of K, Λ multiplying B turns out to make each of N, Ξ , therefore, as K is to Λ , so N is to Ξ . Again, since Λ multiplying each of B, Δ turns out to make each of Ξ, O , therefore, as B is to Δ , Ξ is to O ; but as B is to Δ , Λ is to M ; therefore, also, as Λ is to M , Ξ is to O ; therefore K, Λ, M are in the same ratio as N, Ξ, O taken two and two; therefore, also, ex aequali [δι' ἴσου], as K is to M , so N is to O .

Document 7: Eutocius' Main Proof⁶⁷

Let there be two given numbers A, B and let a certain middle Γ be taken of them. Then one must prove that the ratio of A to B is conjoined of that which A has to Γ and Γ to B .

In fact, let the value Δ of ratio A, Γ and E of <ratio> Γ, B , be taken[; therefore Γ multiplying Δ makes A and B multiplying E makes Γ]. Then, let Δ multiplying E make Z . I say that Z is the value of the ratio of A to B , that is, that Z multiplying B makes A .

In fact, let B multiplying Z make H .

Now, since B multiplying Z turns out to make H and multiplying E <turns out to make> Γ , therefore, as Z is to E , H is to Γ . Again, since Δ multiplying E turns out to make Z and multiplying Γ turns out to make A , therefore, as E is to Γ , Z is to A ; alternately, as

⁶⁶ Translated from *in Sph. cyl. II.8 aliter*, edited in *AOO III*, 198.20–200.17.

⁶⁷ Translated from *in Sph. cyl. II.4*, edited in *AOO III*, 122.11–124.7. The clauses omitted in *in Con.*, *AGE II*, 218.27–220.16, are bracketed. For the underlined deduction also see the main text; slight variations in wording are not considered.

E is to Z, Γ is to A; and inversely, as Z is to E, so A is to Γ ; but as Z is to E, H was proved to Γ ; therefore, also, as H is to Γ , A is to Γ ; therefore A is equal to H; [but B multiplying Z turns out to make H;] therefore B multiplying Z also makes A; therefore Z is the value of the ratio of A to B; and Z is Δ multiplied by E, that is, the value of ratio A, Γ by the value of ratio Γ , B; therefore the ratio of A to B is compounded both of that which A has to Γ and Γ to B].

The proof at *in Con.* argues more effectively, replacing the underlined text with

Now, since Δ multiplying E turns out to make Z and multiplying Γ turns out to make A, therefore, as E is to Γ , Z is to A. Again, since B multiplying E turns out to make Γ and multiplying Z turns out to make H, therefore, as E is to Z, Γ is to H; alternately, as E is to Γ , Z is to H; and, as E is to Γ , Z was to A;

Document 8: Leo's Main Proof and Example of Removal⁶⁸

This being established as a preliminary, let there be a magnitude A having a ratio to B, let the value of this ratio be Γ , and between A, B let a chance magnitude Δ fall. I say that the ratio of A to B is conjoined both of that which A has to Δ and Δ to B.

In fact, it is clear that B, multiplying the value Γ of the ratio, made A. But again, since magnitude B, multiplying the value Z of the ratio of Δ , B, turns out to make Δ , but, also, magnitude Δ , multiplying the value E of the ratio of A, Δ , turns out to make A, therefore by the first lemma, since B, multiplying the <number resulting> from B, Z, exactly turns out to make A, therefore B, multiplying the <number resulting> from E, Z, also turns out to make A. But indeed, both the <number contained> by B, Γ is A, and again that <contained> by B, Z, E is A; therefore that <contained> by B, Γ is equal to that <contained> by B, Z, E; therefore the value Γ of the ratio of magnitudes A, B is equal to that resulting from the values E, Z; therefore the value Γ is compounded of <value> E multiplied by Z. We shall say the same also if a magnitude fall between A, Δ , and again if another fall between B, Δ , for it is the same procedure.

But now, thus, let it be supposed A having to B the ratio that number 17 has to 13, and <let it be supposed> to remove that which has the ratio of 19 to 11 from it. Now, as 19 is to 11, so I do 17 to 187 nineteenths; therefore the ratio of $\frac{187}{19}$ to 13 units stands as a remainder, that is, if we make 13 nineteen times, in smallest numbers, those of 187 to 247.

⁶⁸ Translated from *EOO* V, 716.24–717.18 and 718.9–22.

Document 9: George Pachymeres' Rules⁶⁹

When we want to find which ratio is compounded of any given different ratios, we first take their values and multiply them by one another, and we write down the number resulting from their multiplication (whether it is integer or it also has a part or parts). Then we search which ratio is denominated by such a number, and, whatever it is, that is the <ratio> compounded from the composition of the given ratios.

One must know that, whenever we set out to remove a ratio from a ratio—in any case the lesser from the greater—we first take the two terms of the greater ratio (the antecedent and the consequent), from which we set out to do the removal, and write them down. Then we take the lowest [πυθμενικούς] terms of the lesser ratio, which we also set out to remove, and similarly also write them down on a straight line with the said terms of the greater ratio. Then we multiply the consequent term of the greater by the antecedent of the lesser, and write down separately the number resulting from the multiplication of such terms. Then we take it and we divide it by the consequent term of the lesser ratio, and, taking the number resulting from its division, we write it down between the two terms of the greater ratio: for this will also necessarily have, to the consequent term of it, the ratio itself that we also set out to remove from it. Then we remove such a term [*scil.* the said consequent]—which, clearly, also serves as a consequent of the ratio taken by proportion, that we also set out, as we have said above, to remove from it [*scil.* the greater ratio]—so that such a ratio [*scil.* the lesser ratio] would also have been removed at the same time with it: and in fact, once whatever term is removed of those that are considered to one another in some ratio, necessarily the ratio itself, of which this <term> either is the antecedent or the consequent, is also removed at the same time.

Document 10: Maximus Planudes' Scholium⁷⁰

Of the most learned Maximus Planudes, to the definition 'ratio from ratios' of the sixth, namely, that every ratio can be compounded by two, three, or several ratios. For instance, the double <ratio> 12 to 6 is compounded of two ratios: sesquitercian and sesquialter (both 8 to 6 and 12 to 8), but is also compounded of three: sesquitercian 8 to 6, sesquiquartum 10 to 8, and sesquiquintum 12 to 10. Similarly also from several <ratios>. Now, once the <numbers> paronymous to the compounded ratios are taken and multiplied by one another, it results a number paronymous to the compounded ratio. For instance, since, as said, the double is compounded of sesquitercian and sesquialter, and the sesquitercian has once and a third of the whole under it, I take

⁶⁹ Translated from Tannery 1940, 352.25–31 and 353.31–354.14.

⁷⁰ Translated from *EOO* V, 327.12–329.17.

one unit instead of once and $\frac{1}{3}$ instead of a third. Again, since the sesquialter has once and a half of the whole under it, I similarly take one unit instead of once and $\frac{1}{2}$ instead of a half. Now, I multiply these numbers—of course, one unit and a third by the other one unit and a half, and it results two units, that are paronymous to the double. The multiplication can be performed as follows: once one, one: here it is, one unit; once a half, a half; and again a-third times the unit, that is, a third of the unit, a third; and a-third times the half, namely, a third of the half, a sixth. A half and a third and a sixth <give> one unit, that, compounded with that before it, gives two.

In this way, also, from a double and a triple it results a sextuple, for I take two units instead of double, three instead of triple, and I multiply these by one another, and it results six.

If instead the double be compounded of three <ratios>, as shown above: sesquitercian, sesquiquartum, and sesquiquintum, I take again one unit and a third instead of sesquitercian, one unit and a fourth instead of sesquiquartum, one unit and a fifth instead of sesquiquintum, and I multiply these by one another, and it results two units. The multiplication can be performed as follows: in the first place, the unit and $\frac{1}{3}$ by the unit and $\frac{1}{4}$: once one, one; once $\frac{1}{4}$, $\frac{1}{4}$; a-third times one, namely, a third of one, a third; a-third times $\frac{1}{4}$, namely, $\frac{1}{3}$ of $\frac{1}{4}$, $\frac{1}{12}$; and here it is, a unit, $\frac{1}{4}$, $\frac{1}{3}$, and $\frac{1}{12}$. Afterward I multiply the unit and $\frac{1}{5}$ by the unit and $\frac{1}{4}$ $\frac{1}{3}$ $\frac{1}{12}$, and I say: once one, one; once a fourth, a fourth; once a third, a third; once a twelfth, a twelfth; again, a-fifth times one, namely, a fifth of the unit, a fifth; a fifth of a fourth, a twentieth; a fifth of a third, $\frac{1}{15}$; a fifth of a twelfth, a sixtieth. All these parts give one unit as a result, that, conjoined with that before it, gives two as a result. You shall know in the following way that these parts give a unit as a result. One must find the number (let it be taken as one unit) having these parts in the first place starting from a unit, and it is sixty. Indeed, a fourth of this fifteen, a third twenty, a twelfth five, a fifth twelve, a twentieth three, a fiftieth four, a sixtieth one—and fifteen, twenty, five, twelve, three, four, and one <give> sixty.

And in this way, too, from a double, a triple, and a quadruple it results a twenty-fourfold, for instance 2 4 12 48. I take two instead of double, three instead of triple, four instead of quadruple, and I multiply two by three, and it results six. Next, four by six, and it results twenty-four, which is paronymous to twenty-fourfold.

Document 11: Bryennios' Fifth Scholium⁷¹

Since the ratio of KA to $\Delta\Gamma$, as Ptolemy says, is conjoined of the ratio of KA to ΛM and of that of ME to $\text{E}\Gamma$ and the ratio of KA to ΛM is given whereas that of ME to $\text{E}\Gamma$ is not given, yet it is given by means of the removal of the said given ratio of KA to ΛM (as again he himself says), it is necessary to briefly expound how we shall exactly remove it.

⁷¹ Translated from Acerbi & Pérez Martín 2015, 136.

So, one must know that, when we are performing the removal of such a ratio so as to find the remaining ratio (that is to say, ME), we first draw through two straight lines equal to one another and meeting one another in the middle, and we make the sign of the letter chi, in order for the procedure of removal of such a ratio to result perspicuous and invariable. Next, we put, at its upper extremes,⁷² both the antecedent and the consequent of the conjoined ratio, from which we remove that which has to be removed (I mean the antecedent on the left and the consequent on the right, as here⁷³ 97;4,56 e 70;32,3), and again, at its lower left extreme, the consequent of the removed ratio (as here 24;15,57). Next, we multiply the consequent of the removed ratio (namely, 24;15,57) by the consequent of the conjoined ratio (namely, 70;32,3), and we divide (that is to say we apply⁷⁴) the number resulting from the multiplication by one another of both such consequents by the antecedent of the conjoined ratio (namely, 97;4,56), and we put, at its lower right extreme, the fourth proportional resulting from such a division (as here 17;37,45, for, as 70;32,3 is to 97;4,56, so 17;37,45 is also to 24;15,57). Next, we take the antecedent of the removed ratio (as here 117;31,15) and we put it between its⁷⁵ two lower extremes.

And since the ratio of 17;37,45 to 24;15,57 is identical to the ratio of KA to ΔΓ and the ratio of 17;37,45 to 117;31,15 is again⁷⁶ identical to the ratio of ME to EΓ, and the ratio of 117;31,15 to 24;15,57 is again identical to the ratio of KA to ΛΜ, we remove the ratio of 17;37,45 to 24;15,57 (namely, its consequent 24;15,57⁷⁷) and the ratio of 117;31,15 to 17;37,45 (namely, of ME to EΓ) is left out (that is to say, it is given).

Now if straight line KA were equal to EΓ, immediately we would also exactly have ME, which would be 17;37,45 parts. Since instead it is unequal (for this one is 117;31,15, that one is 120), we necessarily transform the ratio of 117;31,15 to 17;37,45 into that <having> 120 <as antecedent> and we also have in this way of how many parts exactly is ME. We transform it handily as follows: we again form first, as above, the analogous sign of chi and we put at its two upper extremes both the antecedent and the consequent of the ratio left out (I mean the antecedent on the left and the consequent on the right, namely, 117;31,15 and 17;37,45) and 120 at its lower left extreme. Next, we

⁷² The extremes of ‘the sign of the letter chi’ are meant.

⁷³ Here, as elsewhere, Bryennios inverts the terms of the ratio. The mistake will not be corrected until the end of this first taking of a fourth proportional. The numerical operations are correct, their correlates in terms of designators are not.

⁷⁴ The verb ‘to apply’ (παραβάλλειν) denotes the geometric construction corresponding to a numerical division: a plane region is ‘applied’ to a straight line when it is transformed into a rectangle having the straight line as a side. The ‘result’ of the application is the other side of the rectangle.

⁷⁵ Again, the extremes of ‘the sign of the letter chi’ are meant.

⁷⁶ Note that the second and third identification of the numerical values with the compounding ratios are false, as it must be since KA ≠ EΓ.

⁷⁷ If we consider the compounded ratio ‘17;37,45 to 24;15,57’, the procedure of removal simply amounts to eliminating its consequent.

multiply 120 itself by the said consequent of the ratio left out (namely, 17;37,45) and we divide the number resulting from the multiplication by one another of both such <numbers> by the antecedent of the ratio left out (namely, 117;31,15), and taking the number resulting from such a division (as here 18;0,15), which is the fourth proportional (and in fact, as 17;37,45 is to 117;31,15, so 18;0,15 is to 120) we also exactly have ME, of 18;0,15 parts. Such a transmogrification and manipulation is called transformation of similar ratio starting from a proportion.

Document 12: Some propositions of Barlaam's *Treatise of Logistic*⁷⁸

1. To find the value of a given ratio.

Let there be a given ratio A to B. Then one has to find the value of the ratio which A has to B.

Let A be divided by B and let Γ result.

Now then, Γ multiplying B will make A; but indeed, we call value of ratio a number that, multiplied by the consequent, makes the antecedent; and the antecedent of the ratio which A has to B is A, the consequent is B; therefore Γ is the value of the ratio which A has to B.

8. If a ratio be compounded of two ratios and their values be taken, the same ratio is also compounded of the two ratios which its [*scil.* of the compounded ratio] value has to each of the remaining values.

In fact, the ratio of A, B be compounded both of the ratio of Γ , Δ and of that of E, Z, and let the value of the ratio of A to B be H and that of the <ratio> of Γ to Δ be Θ and that of the <ratio> of E to Z be K. I say that the ratio of A to B is also compounded of two ratios, both that which H has to Θ and that which H has to K.

In fact, since the ratio of A, B is compounded both of that of Γ , Δ and of that of E, Z, and the value of the ratio of A, B is H and that of the <ratio> of Γ , Δ is Θ and that of the <ratio> of E, Z is K, therefore K multiplying Θ makes H. Similarly, also, Θ multiplying K makes H; therefore K is the value of the ratio which H has to Θ and Θ the value of the ratio which H has to K; therefore the value of the ratio which H has to Θ , multiplying the value of the ratio which H has to K, makes the value of the ratio which A has to B; therefore the ratio of A to B is compounded both of that of H to Θ and of that of H to K.

9. The values of identical ratios are equal, and the ratios whose values are equal are identical.

In fact, let the ratio of A to B be identical to that of Γ to Δ , and let the value of the ratio of A, B be E, that of the <ratio> of Γ , Δ be Z. I say that E is equal to Z.

⁷⁸ Translated from Carelos 1996, 69.15–24, 75.4–76.15, 80.10–18 and 81.7–20, 82.22–83.7.

In fact, since E multiplying B makes A, therefore, as E is to the unit, so A is to B. Exactly for the same reasons, as Z is to the unit, so Γ is also to Δ ; but indeed, the ratio of A, B has also been supposed identical to that of Γ , Δ ; therefore, as E is to the unit, so Z is to the unit; and those having identical ratios to the same are equal to one another; therefore E is equal to Z.

Again, let E be supposed equal to Z. I say that, as A is to B, so Γ is to Δ .

In fact, since, as A is to B, so E is to the unit, and, as E is to the unit, so Z is to the unit—because E has been supposed equal to Z—and, as Z is to the unit, so Γ is to Δ , therefore, as A is to B, so Γ is to Δ .

14. If the values of two ratios be taken, any of them is compounded both of the other and of the ratio which its value has to the value of the former.

In fact, let the values of two ratios—both that which A has to B and that which Γ has to Δ —be taken, and let them be E, Z. I say that the ratio of A to B is compounded both of that which Γ has to Δ and of that which E has to Z.

[I translate the alternative proof]

In fact, let the value of the ratio of E, Z be H.

Therefore H multiplying Z turns out to make E; and H is the value of the ratio which E has to Z and Z the value of the ratio of Γ , Δ and E that of A, B; therefore the value of the ratio of E, Z, multiplying the value of the ratio of Γ , Δ , made the value of the ratio of A, B, so that the ratio of A to B is compounded both of the ratio of Γ , Δ and of that of E, Z. Again, let the value of the ratio of Z, E be Θ . Therefore Θ multiplying E turns out to make Z, so that the value of the ratio of Z, E, multiplying the value of the ratio of A, B, turns out to make the value of the ratio of Γ , Δ ; therefore the ratio of Γ to Δ is compounded both of the ratio of A, B and of that of E, Z.

17. A given <ratio> being removed from a given ratio, to find which is the ratio left out.

Let the given ratio, from which one has to remove, be that of A, B, the <ratio> that one has to remove be that of Γ , Δ . Then, once the ratio of Γ , Δ is removed from that of A, B, one has to investigate which ratio is left out.

Let the value of the ratio of A, B, be E and that of the <ratio> of Γ , Δ be Z.

And since the ratio of A to B is compounded both of that of Γ to Δ and of that of E to Z, therefore, if from the ratio of A to B we subtract the ratio of Γ to Δ , the ratio of E to Z will be left out for us.

Document 13: Demetrios Cydones' General Rules⁷⁹

Since Ptolemy—even if, in the mathematical theory of what moves spherically, he sets out lemmas for the removals of ratios—does not quite seem to refer there to all removals he appears to employ in the subsequent books, it is necessary to preliminarily set

⁷⁹ Translated from my own critical text of Cydones' scholia.

out them, too, so that nothing can stand against those who approach such a theory with research aims. Accordingly, let the diagram of the first rectilinear lemma be set out.

Then it is manifest from this that, if a ratio be compounded of 2 ratios, the inverse is also compounded of the inverse ratios; [Ptolemy] turns out to employ this removal in the second theorem of the second book, and elsewhere.

Then it is manifest from this that, if a ratio be compounded of 2 ratios, one of the compounding <ratios> is also compounded both of the compounded <ratio> and of the remaining among the compounding <ratios>, taken in the inverse.

{*in the margin*: this is the inverse of the first porism, for the inverse is compounded of the inverses} Then it is manifest from this that, if a ratio be compounded of 2 ratios, one of the compounding <ratios> taken in the inverse is also compounded both of the remaining among the compounding <ratios> and of the compounded <ratio>, taken in the inverse. [Ptolemy] employs this removal in the theorem in which he seeks the setting angle <contained> by the beginning of Taurus and the horizon.

Document 14: The Definitions of George Gemistus Plethon⁸⁰

A composition of ratios occurs in three terms, once a middle term is taken (either less than one of the extremes and greater than the other, or even greater than both, or even less than both) and this is removed away [ὑπεξαίρουμένου] in the composition of ratios.

A removal of a ratio from a ratio (once three terms have been set out, one of which is shared by the removed ratio and by that <ratio> from which this removed ratio has to be removed, and next a fourth proportional has been found in addition) leaves out the remaining term between the shared one among those previously set out and this fourth <proportional> found in addition, taken as a middle of the terms that contain the ratio from which the removed <ratio> has to be removed (and next either of the extremes has been removed away [ὑπεξηρημένου]).

The fourth proportional term is found in addition once two terms have been multiplied by one another and what turns out to result from the multiplication has been divided by the remaining <term>; the <number> that turns out to result from such a division is in fact the fourth proportional term, which—whenever the extremes among the original terms (that is, the greatest and the least) multiply one another and the division occur by the middle—will be taken as a middle of either of the extremes and of the middle among the original terms—but whenever the middle among the original terms multiply either of the extremes and the division occur by the remaining <extreme>—the <terms> multiplying one another will be the middle <terms>, the one

⁸⁰ Translated from Acerbi *et al.* 2016, 443.

by which the division occurs and this one that turns out to result from the division, will be the extremes.

Document 15: George of Trebizond's 'Chiasmus Rule'⁸¹

I say chiasmus when we get the ratio left out by multiplying the first <term> by the fourth and the second by the third. For instance, in the case of the continuous <compounded ratio>⁸² 2 4 8, writing down the compound <ratio> first, I write underneath one of the compounding <ratios>, thus [*diagram*], and multiplying by chiasmus I get 8 16; therefore the <ratio> left out is double, and once either of the extremes is removed, say 8, a double <ratio> is left out. We must always use the chiasmus in this way, whenever the proportion is from three terms, namely, by doubling one of the extremes and by writing underneath one of the compounding <ratios> after writing down the compound <ratio>.

The procedure is as follows. Four numbers being given, if we divide by the first the result of the multiplication of the second by the third, we put it after the fourth and the ratio of the fourth to the result of this division is the <ratio> left out; if, instead, we divide by the fourth, we put it after the first and the sought <ratio> is that of the first to it. And if the multiplication occur to us the first being made by the fourth, if we divide by the second the result of such a multiplication, the result of the division is put before the third; if, instead, the division occur by the third, the result of the division is put before the second.

If one likes to operate by chiasmus, one must know that the compound <ratio>, once the terms are written down in a row, is the first to the third. Let four numbers be given in a row 4 6 8 20; by chiasmus [*diagram*]. And the left out <ratio> is two-thirds-more [*ἐπιμερίζων δύο τρίτα*], 80 to 48, and this itself is found in four ways according to Ptolemy's practice. In fact, multiplying the second (namely, 6) by 8 (namely, the third) I get 48, and if I divide it by the first (namely, 4), I get 12, and, as 80 is to 48, the fourth (namely, 20) is to 12. If, instead, I divide by the fourth (namely, 20), I get 2;24, and, once I have put it after the first, I get 4 to 2;24, identical to 20 to 12. Again, I multiply the first by the fourth (namely, 4 by 20), and it results 80; if I divide this by the second (namely, 6), it results 13;20, which is to 8 (for one must put it before the third) as 20 is to 12; if, instead, <I divide> by the third (namely, 8), I get 10; once I have put this before the second (namely, 6), the ratio of 10 to 6 is as that of 20 to 12.

⁸¹ Translated from my own critical text of the *Isagoge* to the *Almagest*.

⁸² Namely, a compounded ratio written in standard form.

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