

The Pythagorean Roots of Introductory Physics

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Abstract Much of the mathematical reasoning employed in the typical introductory physics course can be traced to Pythagorean roots planted over two thousand years ago. Besides obvious examples involving the Pythagorean theorem, I draw attention to standard physics problems and derivations which often unknowingly rely upon the Pythagoreans' work on proportion, music, geometry, harmony, the golden ratio, and cosmology. Examples are drawn from mechanics, electricity, sound, optics, energy conservation and relativity. An awareness of the primary sources of the mathematical techniques employed in the physics classroom could especially benefit students and educators at schools which encourage integration of their various courses in history, science, philosophy, and the arts.

1 Introduction

“All things accord in number.” Though uttered approximately 2,500 years ago by Pythagoras this maxim (Iamblichus 2003 [3rd century BCE], p. 97) resonates with many physicists even today. The most revolutionary idea spread by Pythagoras was not the theorem concerning triangles which bears his name. More sweeping was his teaching that nature is ruled not by *mythos* (μῦθος) but *logos* (λόγος), that “the nature of the universe be defined according to the reasons (λόγοι) and proportions of numbers” (Porphyry 1987 [3rd century CE], p. 133).

Whether explicitly stated in our lectures or textbooks, physicists still rightly adopt and pass on to our students this philosophy: the universe is *ordered*, and mathematics is among the most important and effective means by which to study this order. This outlook has led to the amazing predictive powers we enjoy today, powers not unknown to Pythagoras in the 6th century BCE who “instead of divining by the entrails of beasts, he revealed to [his students] the art of prognosticating by numbers” (Iamblichus 2003 [3rd century BCE], p. 68).

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Most of what we know about Pythagoras (approximately 572–497 BCE) is from secondary sources (Iamblichus 2003 [3rd century BCE]; Porphyry 1987 [3rd century CE]). Like Socrates and other ancient teachers he did not commit his doctrines to writing. What is clear from these sources is the man was born on the island of Samos in the Aegean Sea, and emigrated to Greek southern Italy (Croton, in Greater Greece) where he taught in a setting we today would probably describe as a mathematical cult. There he coined the terms “philosopher” and “mathematician” as monikers for himself and his students respectively. Much of his actual work is clouded in myth, which is also true of his contemporaries such as Bhudda and Confucius. Like many religious orders of the age the Pythagoreans were vegetarian and believed in an eternal soul which was subject to a re-incarnation.

Besides broadly sharing much of the Pythagorean natural philosophy that “all things accord in number” the usual exposition of introductory physics also relies heavily on explicit mathematical tools which Pythagoras and his school are generally credited with developing. Since it is difficult to know what results are actually from the man Pythagoras, as distinguished from the members and descendants of his brotherhood, I will use the term “Pythagorean” to signify both the historical man and his school.

There is considerable literature on the legacy of Pythagoras in various fields. In the context of physics, recent work has focused on its relation to the physics of sound (Caleon and Ramanathan 2008; Caleon and Subramaniam 2007) and quantum theory (Bunge 2003). This paper will focus on examples of Pythagorean mathematical reasoning typically passed along during the first year sequence in university physics. Although the influence can also be found in advanced courses, this is outside the scope of the present paper. Further, this work will focus on the classroom lecture component of the curriculum, as opposed to the other equally important aspects of teaching and learning physics (e.g., laboratory experiments, data analysis, homework, assessment). The result is a sort of “mathematical archeology” of common physics equations. The examples are restricted to mathematics generally accepted by historians of the subject as originating with the Pythagorean school.¹ As we will see this touches on many areas of the curriculum, e.g., from introductory topics such as kinematics and energy conservation to second semester topics in circuits and relativity. Many of the examples are well known, but taken as a whole may not be obvious to all.

I must note that the intent is not to erect a false idol, or suggest physics owes everything to ancient ideas. The purpose is simply to gather these examples in one place so their weight and connection to physics throughout the ages can be better appreciated.

2 The Pythagorean Theorem

Let us start with the most familiar. In right angled triangles the square of the side opposite the right angle equals the sum of the squares of the remaining two sides. That is, $c^2 = a^2 + b^2$. This result pre-dates Pythagoras and Greek mathematics.² Particular examples of three numbers which satisfy $c^2 = a^2 + b^2$ (so-called “Pythagorean triples”) have been discovered in tablets from Egypt and Old Babylonia circa 1800–1600 BCE (Neugebauer 1969, p. 36). However the general demonstration of this truth for *all* right

¹ See for example chapters III–V of Heath (1965).

² By the term “Greek” I signify the entire empire falling under ancient Hellenistic culture around the Mediterranean.

triangles—as a theorem—is attributed to Pythagoras’ school (Heath 1965, pp. 143–149). Indeed this idea of a general mathematical theorem, proven from logic and clear definitions, is the hallmark of the Pythagoreans. The first general proof of the proposition that the sum of interior angles in a triangle is 180° is also attributed to them. In fact early on in the 5th century BCE the Pythagoreans had completed the content in Books I, II, IV, VI (and probably III) of Euclid’s elements a good century before Euclid compiled and committed this wondrous mathematics to a general audience (Heath 1965, pp. 166–168).

The most obvious use of the Pythagorean theorem in a physics class is in calculating the magnitude of a vector in Cartesian coordinates, e.g., the total magnitude of a particle’s velocity or momentum in kinematics, or resultant forces or field intensities in dynamics. Then there are purely geometric settings such as finding the distance between two points, or path lengths in interference problems involving sound or light waves.

More subtle applications arise in conservation laws. Consider applying momentum and energy conservation to the elastic collision of two identical mass bodies. The Pythagorean theorem can be used to show that the final directions in an off-center collision are perpendicular (Tipler 1999, p. 238). Taking one body to be initially at rest (i.e., solving in one body’s rest frame) conservation of momentum gives $m_1 \mathbf{v}_{1i} = m_1 \mathbf{v}_{1f} + m_2 \mathbf{v}_{2f}$, using standard subscripts for particles and states. Dividing through by the identical mass $m = m_1 = m_2$ gives

$$\mathbf{v}_{1i} = \mathbf{v}_{1f} + \mathbf{v}_{2f}, \quad (1)$$

that is, these three vectors evidently form a triangle by the geometric interpretation of vector addition. Since conservation of energy requires $\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$, or

$$v_{1i}^2 = v_{1f}^2 + v_{2f}^2 \quad (2)$$

we conclude from the conservation conditions (Eqs. 1, 2) that these three velocities form a Pythagorean triple. Geometrically speaking the triangle of vectors is right angled. Therefore the two bodies must emerge at 90° to one other after colliding.

In certain special but fundamental cases the Pythagorean theorem can even be used to prove conservation laws, or at least arrive at them on geometric grounds. Consider the simple harmonic oscillator with angular frequency $\omega = \sqrt{k/m}$. The total energy, kinetic and potential, is $E = \frac{1}{2} m v^2 + \frac{1}{2} k x^2$. Since its position can be expressed $x(t) = A \cos(\omega t + \phi)$ and $v(t) = \dot{x}(t)$ we have

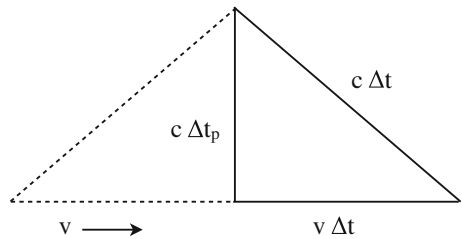
$$E = \frac{1}{2} m \omega^2 A^2 \sin^2(\omega t + \phi) + \frac{1}{2} k A^2 \cos^2(\omega t + \phi) \quad (3)$$

$$= \frac{1}{2} k A^2 [\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)] \quad (4)$$

$$= \frac{1}{2} k A^2. \quad (5)$$

That this energy is constant in time follows from the so-called “trig identity” $\sin^2 \theta + \cos^2 \theta = 1$ found in the back of the any student’s text, or back further in their memorizations of high school trigonometry identities. But of course the *source* of this relation is the Pythagorean theorem, applied to a right triangle inscribed inside a unit circle. Thus the Pythagorean theorem expresses the underlying orderliness of energy conservation in harmonic systems, from springs to LC electrical circuits.

Fig. 1 Right triangle geometry underlying relativistic time dilation



Other examples of the theorem's application include impedance in LRC circuits (Young et al. 2006, p. 1061), the generation of Pythagorean triples for manual and online test bank problems (Lavatelli 1964), and the propagation of experimental errors in quadrature (Allie et al. 2003).

Consider finally a few instances of the Pythagorean theorem drawn from special relativity. An elementary derivation of time dilation analyzes the round trip path taken by a light beam bouncing off a mirror fixed to the ceiling of a speeding train car (see Fig. 1). By symmetry we can focus on one leg of the journey. In the rest frame of the train this path is *vertical* and takes a proper time Δt_p . To an observer on the ground the path is along a *diagonal*, which being longer implies more time Δt elapses for this same process since the speed of light is postulated equal in both frames. Applying the Pythagorean theorem to the right triangle defined by these two paths gives

$$c^2 \Delta t^2 = c^2 \Delta t_p^2 + v^2 \Delta t^2. \quad (6)$$

Isolating Δt gives the famous relation between the two reference frames' time intervals,

$$\Delta t = \frac{\Delta t_p}{\sqrt{1 - v^2/c^2}}. \quad (7)$$

After one has worked this derivation a few times it becomes clear that the initially odd and pervasive Lorentz factor ($1/\sqrt{1 - v^2/c^2}$) in relativity is simply the residual radical from an underlying Pythagorean triangle between events in space and time. In some sense there is no formula to memorize.

A few pages later in the typical textbook telling of relativity the Pythagorean theorem rises again and underlies one of most famous equations in physics, the expression for the rest mass energy of a massive body: $E = mc^2$. Extending the Pythagorean distance formula to four-dimensional spacetime one is led to the invariant interval $(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$ between events.³ Dividing through by proper time and re-arranging gives the energy-momentum relation $(mc^2)^2 = E^2 - (p_x c)^2 - (p_y c)^2 - (p_z c)^2$. The famous mass-energy relation follows as the special case for a particle at rest. Ironically, or perhaps fittingly, the most recognizable equation in physics is an application of one of the most recognizable equations in elementary mathematics.

There is evidently a profound connection between the Pythagorean theorem and many physical laws and relations. Although the purpose of the present paper is not to answer Wigner's question on "the unreasonable effectiveness of mathematics in the natural sciences" (Wigner 1960) let us reflect a bit on the effectiveness in the above examples. In the

³ The signatures on the space and time dimensions take a bit of hand-waving at the introductory level, but can be interpreted with Minkowski's original convention of using an imaginary unit for the time coordinate in spacetime.

simplest cases there is no real mystery. For example when we agree to describe the position of a body by a vector, then the entire apparatus of Euclidean space comes along automatically. In the other examples addressed above the connection is less obvious. It is perhaps too strong to say we have derived, or proven, a given physical result since one might invert the logic and conclude, for example, that the Pythagorean theorem is a consequence of energy conservation in harmonic systems. Such logical inversions in the case of mechanics are explored in (Levi 2012, pp. 9–26); for example the physical fact that a fish-tank full of water remains in equilibrium regardless of its shape (including a tank with a right-triangle for base) is used to *derive* the Pythagorean theorem. At first this seems either amusing or puzzling. In most of these instances the deep connections between triangle geometry and physics actually follow from simple ideas of conservation or invariance. Euclid's proofs of the Pythagorean theorem (Book I, Proposition 47, and especially his alternative proof in Book X, Lemma to Proposition 33) rely upon similar triangles and the intuitive idea that areas are conserved despite orientations or placements in the plane. Considering the fish tank example, since one of the conditions for equilibrium of a rigid body is the torques, or moments, sum to zero (or angular momentum is conserved) we see that a conservation law (angular momentum) underlies the connection; this connection between areas and angular momentum is well known from Kepler's second law, which can either be expressed as a constancy of areas as Kepler did, or equivalently as a constancy or conservation of the physical quantity angular momentum. In the above cited example of time-dilation it is evident that its derivation also rests upon the assumption that some physical quantity, the speed of light, is a constant in all reference frames. In fact Einstein privately preferred the name *invariantentheorie* (theory of invariants) to relativity theory (Holton and Elkana 1997, p. xv) since it is based upon constants, such as the speed of light and the invariant inertial mass (Hecht 2009).

Lest one extrapolate that the Pythagorean theorem underlies all physics, there are clear cases where it does not apply. One is cautioned against applying the theorem to constructed spaces where the different axes do not have the same measure for units. An example of such a non-metric, or affine, space is the PV-diagram in classical thermodynamics (Papastavridis 1998, pp. 90–91). Although not often stressed to our students, all the examples in this section at some stage of their formulation are careful to force the same measure on all axes (e.g., by taking ct as the appropriate fourth coordinate in spacetime geometry). Besides non-metric spaces one must also take care in non-Euclidean spaces; for example the classical sum of squares distance formulae must be modified in the curved spaces of general relativity, even though the generalization is clearly motivated from its root Pythagorean form.

3 Music, Harmony, and Sound

The theorem on right angled triangles is but one specific piece of Pythagorean mathematics. More generally Pythagoras and his school were pre-occupied with investigations and theories of music and harmony. See Caleon and Ramanathan (2008) and Caleon and Subramaniam (2007) for a comprehensive discussion. What Pythagoras found was that the most harmonious sounds to our human ear are those whose tones (be it quantified by string length or tension) are in the simplest whole number ratios. Experimenting with a monochord and ruler Pythagoras discovered that the three perfect consonances—octave, fifth and fourth—are in ratio 1:2 (octave), 2:3 (fifth) and 3:4 (fourth) relative to the string's root length. To this day the foundations of Western musical scales are founded on this triad of

notes (e.g., C, F, G in the key of C). That something as beautiful and emotive as music could exhibit an accordance with number was the prototype for the Pythagorean maxim “all things accord in number.” This principle was extended beyond the music from strings and air instruments into a more general theory of harmony applying to the entire universe.

In modern times little focus is given in introductory physics to music or harmony per se, besides the typical textbook section or lecture on the more generic topic of sound. Nevertheless a legacy to the past conventions, and even mathematics, of harmony remains in the broader guise of the fundamental importance physicists continue to place on vibrations, waves, and harmonic analysis; this includes the idea that discreteness naturally arises, or at least accompanies, the dynamics of continuous systems.

Consider the standard contemporary derivation of the allowed vibrations for a string with length L fixed at both ends. Standing waves form when two waves propagate on a string with identical amplitude, wavelength and frequency, but in opposite directions and phase relations due to reflection from the fixed ends. Their superposition gives,

$$y(x, t) = y_1(x, t) + y_2(x, t) \quad (8)$$

$$= A \cos(kx - \omega t) - A \cos(kx + \omega t). \quad (9)$$

As universally noted in any physics text a “trig identity” reduces this to the form

$$y(x, t) = 2A \sin(kx) \sin(\omega t). \quad (10)$$

Since $y(x, t)$ must vanish at the ends for all time, then $\sin(kL) = 0$, which only happens if $kL = n\pi$ for integer n . Hence only discrete wavelengths $\lambda_n = 2\pi/k = 2L/n$ are possible. Unlike Pythagoras’ musical “discovery” of integer relationships in the plucked string, in this trigonometric analysis there is nothing mysterious or even surprising about integers arising; due to the simple boundary conditions imposed on the wave equation the possible modes are quantized in whole number harmonics.

Nonetheless it is curious to note that underling the above contemporary derivation is a crucial trigonometric relation, the sum and difference formula $\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$. If pressed to justify this identity most of us would appeal to relatively advanced methods such as complex numbers and Euler’s formula. But the identity’s historical roots and proof are much earlier than Euler and the eighteenth century. This and indeed most trigonometric relations we rely upon follow from the more general Ptolemy’s Theorem⁴ developed in 2nd century Hellenistic Egypt by Ptolemy and proved in his thirteen volume *Mathematical Syntaxis*, known to most by its Arabic name *Almagest* (Ptolemy 1998 [2nd century CE]). Therefore even a modern treatment of waves contains a largely invisible legacy from ancient Greek, and primarily Pythagorean mathematics. Firstly, Ptolemy’s Theorem shines in the history of geometry as a generalization of the Pythagorean theorem, providing an alternate proof of the right triangle theorem as a special case. Secondly, Ptolemy’s proof employs similar triangles, part of the powerful geometric technique of “similarity of figures” developed by the Pythagorean school. Lastly, and more

⁴ For a quadrilateral $ABCD$ inscribed inside a circle the product of the diagonals equals the sum of the products of the opposite sides, or $AC \cdot BD = AB \cdot CD + AD \cdot BC$. Ptolemy stated his theorem in terms of chords on a circle (the analog to our sines), not in the modern language of trigonometric or circular functions, which are after all sophisticated analytic abbreviations encoding the definite ratios attaining between lengths (or chords) defined on circles. Ptolemy motivates the importance of this theorem for his, and indeed all, work on geometry and astronomy by first stating, “But we see that it is first necessary to explain the method of determining chords: we shall demonstrate the whole topic geometrically once and for all” (Ptolemy 1998 [2nd century CE], (I 9), p. 47).

generally, Ptolemy's mathematical work was highly motivated by problems in Pythagorean harmony, as embodied in his 2nd century CE book *Harmonica* where Ptolemy assigned musical notes to every celestial body in such a way that the solar system rang with a consonance of fourths and octaves (Ptolemy 1980 [2nd century CE], Book III, Chapter 13. pp. 107–109).

Joseph Fourier generalized these simple harmonic relations of a string to a staggering array of systems which we tend to take for granted but which Fourier's contemporaries objected to at the time. If not only strings but the flow of heat, and even pathological shapes like square waves and delta functions, can be reduced to waves then in some sense Fourier provides a further justification for why "all things accord in number" namely the discrete wave numbers labeling the spectral series of modes or harmonics comprising a system. Terminology inherited from harmony and music still lingers: harmonic, mode, series. With Bessel and Laplace's higher dimensional generalizations to circular and spherical harmonics these principles attain even more force. For example with quantum theory's wave mechanics the colorful spectra of gases radiate in accord with simple whole number ratios, the quantum numbers associated with atomic spherical harmonics. For further exposition of this Pythagorean legacy in harmonic analysis see Bunge (2003), who goes so far as to claim that, "In fact, the first to discover quanta was not Planck in 1900, but Pythagoras in the 6th century B.C. . . . This must be emphasized to debunk the myth that only exotic microphysical objects have quantal properties." Pluck a string, or perturb an atom, and number attends.

4 Pythagorean Proportions and Means

Greek mathematics contains a rich theory of proportions and means which still pervades any introductory physics course. This is far from obvious today. In general the terms "proportion" and "ratio" are now used interchangeably,⁵ equated with our division symbolism $a \div b$. Often "mean" is understood as a synonym for "average." There is a practical need for physicists, and indeed all scientists, to "think in averages" when studying complex systems. Statistical mechanics is perhaps the most obvious instance, where macroscopic thermodynamic properties are calculated from averages over the microscopic constituents. Yet even when studying something as simple as a ball in free-fall there is a step (often implicit at the introductory level) where the myriad atoms comprising earth and ball are replaced by average quantities, i.e., the centers of mass which bridge the microscopic and macroscopic physics. Most of the ways we combine elements in physical systems fall into one of the mathematical procedures set forth millennia ago for mediating a generic set of quantities, procedures which defined a given *mean* based upon the corresponding *proportion* for mediating the quantities. By far the so-called "arithmetic mean" is the most familiar today, which corresponds to the operation we often associate with an "average." However physicists often combine physical elements in other ways, using means which we often do not explicitly name but which are in fact examples of the other means formulated in Greek mathematical science.

⁵ In his *Harmonices Mundi* (Kepler 1997 [1619], p. 264) Kepler writes a digression on the translation of Greek mathematical terms which concludes "We must therefore keep the custom introduced by the barbarian commentators on the Greek Elements." This is essentially the terminology carried to this day. At least in mathematical texts the word "proportion" derives from *proportio*, the common Latin translation for the Greek *ἀναλογία*. The word "ratio" translates the Greek *λόγος*, a notoriously potent word which, depending upon ones discipline and context outside science and mathematics, unpacks variously as ratio, reason, proportion, number, meaning, means, cause, wisdom, or word.

Though proportions were known and used earlier in antiquity, the Pythagoreans are credited with systematizing and generalizing the possible proportions to ten, each of which defines a corresponding mean (Boyer 1968, p. 61). Of these ten the three most commonly applied, then and now, are the arithmetic, geometric, and harmonic. In modern notation, for two numbers $a < c$ the mean b can mediate these numbers according to the following proportions:⁶

$$\frac{b-a}{c-b} = \frac{a}{a} \quad (\text{arithmetic mean}) \quad (11)$$

$$\frac{b-a}{c-b} = \frac{a}{c} \quad (\text{harmonic mean}) \quad (12)$$

$$\frac{b-a}{c-b} = \frac{a}{b} \quad (\text{geometric mean}) \quad (13)$$

These three means were intimately bound with Pythagorean music theory as an alternate way of defining the musical scale.⁷ Note the symmetry in their definitions. Re-arranging these definitions to a more modern and recognizable form,

$$b = \frac{1}{2}(a + c) \quad (\text{arithmetic mean}) \quad (14)$$

$$\frac{1}{b} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{c}\right) \quad (\text{harmonic mean}) \quad (15)$$

$$b = (a \cdot c)^{1/2} \quad (\text{geometric mean}) \quad (16)$$

blurs this original symmetry but does allow us to see how these various means arise in elementary physical systems. Keep in mind the natural generalization to mediation between not two but N quantities is achieved by replacing $1/2$ by $1/N$ in the above expressions.

4.1 Arithmetic Mean

The first mean, the arithmetic, is the most commonly recognized. Basic examples arise in the first week of physics classes: kinematics lectures define average position and velocity, while labs use mean values in data analysis. At the conclusion of classes the notion arises again as grades are computed. Trivial as they are, these examples point out the close

⁶ This form matches their original definitions which employed words. For example Kepler (Kepler 1997 [1619], p. 166) recalls the ancients' definition of "harmonic proportion as that in which, three numbers being placed in their natural order, the amounts by which one of the pair of neighbors exceeds the other are in the same proportion as the outer numbers." The harmonic mean is the middle term of the three numbers (that is, the symbol b in Eq. 12).

⁷ For example, the Pythagoreans demonstrated that the musical scale can be constructed theoretically (as opposed to empirically with the human ear) using the three classical means: arithmetic, geometric, and harmonic. Consider their application to a string with length 1 and another string of length $1/2$ (octave). It turns out the *arithmetic* mean between these two extremes is $3/4$ (fourth) and the *harmonic* mean is $2/3$ (fifth). Thus the aesthetic foundation of octave, fourth and fifth discussed in previous section "falls out" mathematically. The remainder of the major scale follows by considering the third classical mean, the *geometric*, which fills in the remaining whole notes on scale (e.g., D is geometric mean between C and E). See (Fideler 1987, pp. 327–328).

connection between mechanics and statistics which underlies arithmetic proportion. Two additional examples further illuminate this connection. In rigid body dynamics the center of mass and moment of inertia rely upon the arithmetic mean, extended to infinite systems. Mathematicians inherit this mechanical terminology in speaking of statistical “weights” and “moments.” Boltzmann, Maxwell and Gibbs borrowed back the statisticians tools (as well as the Pythagorean theorem) to derive the fundamental relationship between microscopic molecular kinetic energy and macroscopic thermodynamic temperature, $\frac{1}{2}m\langle v^2 \rangle = \frac{3}{2}k_B T$.

One way to conceptualize any mean defined from N values is that one could replace each individual value by the mean and obtain in some sense the same cumulative result. Mathematically speaking,

$$\sum_{i=1}^N x_i = N \cdot \frac{1}{N} \sum_{i=1}^N x_i \quad (17)$$

$$= N \cdot \bar{x}_{arith} \quad (18)$$

$$= \sum_{i=1}^N \bar{x}_{arith}. \quad (19)$$

An example of such a conceptual replacement of each member by the mean is afforded by the center of mass, an idea which dates back to Archimedes’ ancient demonstration that the moment (or torque) exerted on a lever by a collection of weights resting at different spots on the lever is the same as if all weights are placed at a single point on the lever: the center of mass or gravity.

4.2 Harmonic Mean

This property of replacement is in fact true for all the Pythagorean means, including the harmonic. Consider a collection of resistors in parallel. The equivalent resistance is

$$\frac{1}{R_{eq}} = \sum_{i=1}^N \frac{1}{R_i} \quad (20)$$

$$= N \cdot \frac{1}{N} \sum_{i=1}^N \frac{1}{R_i}. \quad (21)$$

From the definition of the harmonic mean (Eq. 15, generalized to mediation of N quantities) this equivalent resistance for parallel circuits can be re-written

$$\frac{1}{R_{eq}} = N \cdot \frac{1}{\bar{R}_{harm}} \quad (22)$$

$$= \sum_{i=1}^N \frac{1}{\bar{R}_{harm}} \quad (23)$$

demonstrating that for a parallel topology the equivalent circuit is where each element is replaced by the harmonic mean, $\bar{R}_{harm}$. That is, resistors combine arithmetically in series and harmonically in parallel. Similar reasoning applies for capacitors and inductors.

Other instances of the harmonic mean appear in optics. For example the distance between the two foci of a thin lens corresponds to the harmonic mean between the image and object distances. For a concave mirror, which has only a single focal length, the mirror's radius of curvature corresponds to the harmonic mean of the image and object distances.

4.3 Geometric Mean

The geometric mean commonly arises when the physical quantities underlie geometric shapes (e.g., area) as opposed to a simple scalars (e.g., mass). For instance if one asks for the side of a square with the same area as the rectangle with sides a and c the answer is the geometric mean $(a \cdot c)^{1/2}$. Or consider capacitors, whose physics is strongly determined by their geometry. A simple capacitor comprised of parallel plates with area A separated by d has the form $C = \epsilon A/d$ where ϵ is the permittivity. But for a capacitor whose shape is more complex this simple formula does not immediately hold. Nonetheless if one is aware of the geometric mean then for many geometries the simple form $C = \epsilon A/d$ can be recovered. Take the case of a spherical capacitor consisting of two concentric spheres holding opposite charges of magnitude Q , the inner sphere with radius r_a and outer with radius r_b . Since the potential between the plates is

$$\Delta V = \frac{Q}{4\pi\epsilon} \left(\frac{1}{r_a} - \frac{1}{r_b} \right) \quad (24)$$

the capacitance $C = Q/\Delta V$ can be written, after some algebra,

$$C = \epsilon \frac{4\pi r_a r_b}{r_b - r_a}. \quad (25)$$

Despite the apparent complexity of this form compared to the case of simple parallel plates, the factor in the numerator can be expressed in terms of the geometric mean $\bar{r}_{geom} = \sqrt{r_a r_b}$ as $4\pi r_a r_b = 4\pi \bar{r}_{geom}^2$. Thus the simple form $C = \epsilon A/d$ still holds provided we identify A with the appropriate (geometric) mean surface area of the two concentric plates with separation d .

This example from electrostatics has pedagogical and practical value. What are at first sight more complex capacitance problems involving spherical charge separations (e.g., biological cell walls, planetary atmospheres) can be simplified if a student identifies the geometric mean for the two charged areas, then leverages the simpler capacitance formula for parallel plates which has already been mastered in simpler problems. This example also provides a concrete physical illustration of why all means are not arithmetic. Despite most students initial guess, the appropriate "mean area" to employ in $C = \epsilon A/d$ is not computed as one-half the sum.

Less geometric examples abound and are best seen with the following equivalent expression⁸ for the geometric mean: if a is to b as b is to c then the three quantities are in geometric proportion, with mean b . For example a rigid body's center of mass, radius of gyration, and center of percussion are in such geometric proportion, as is evident from the formula for the center of percussion $r_P = k_G^2/r_{cm}$ where k_G is the radius gyration and r_{cm} the center of mass.⁹

⁸ From Eq. 15 we have $ac = b^2$ which is equivalent to stating $a/b = b/c$.

⁹ This form holds for the center of percussion r_P relative to the body's center of mass. See (Hibbeler 2009, p. 508).

As a final example consider the kinematics of circular motion. Through our physics, astronomy, or even philosophy courses most of us are familiar with the ancient Greek's historical fixation on uniform circular motion. Aristotle devotes several chapters to circular motion in his *Physics*, concluding "this is the only perfect motion" (Aristotle 1941c [4th century BCE], Bk. VIII, Chapters 8,9). Two centuries earlier Pythagoras taught this ideal in his cosmology (Heath 1991, pp. 11–12). We soberly shy from such aesthetic terms as perfect, or eternal, in contemporary instruction. Nonetheless the equation we learn and subsequently teach for centripetal acceleration,

$$a = \frac{v^2}{r}, \quad (26)$$

still hides a certain Pythagorean legacy. If one were to inquire what sort of motion has the magnitudes of the three kinematic variables—position, velocity, acceleration—in precise geometric proportion the answer would be uniform circular motion. For $r/v = v/a$ implies Eq. 26. In this sense there is something uniquely proportioned about circular motions from a strictly numerical standpoint. I am not suggesting we should "derive" the kinematics of circular motion in this fashion today but merely point out the affinity to Pythagorean mathematical ideas. Today's more rigorous treatment employs Newton's infinitesimal analysis to the kinematic vectors on a circle, yet ironically this derivation relies upon applying "similar triangles," a proof technique inherited from ancient Greek geometers.

One may wonder whether any mathematical expression can be rearranged into this Pythagorean language of means, and thus cast doubt on the relevance of these examples. It is true that any sum can be written as the sum of a mean (e.g., Eq. 19) but it does not follow that the physics examples given here of the three means are simply trivial mathematical ways of reformulating sums or products. For instance, though mathematically possible it is not necessarily physical meaningful to rewrite any product as a geometric mean. In the physics of two body gravitation the product of masses appears in the inverse square law. Mathematically this product could be expressed in terms of a geometric mean mass $\bar{m}_{geom} = \sqrt{m_1 m_2}$. But this reformulated quantity $\bar{m}_{geom}$ plays no meaningful physical role in our approach and solution to the dynamics. In fact it is instead the so-called "reduced mass" μ which is the important physical quantity, which allows the two-body problem to be "reduced" to an equivalent problem involving a single body with the combined mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}. \quad (27)$$

Rearranging this expression and comparing with Eq. 15 shows that this combination μ is actually related to the harmonic mean of the two masses, which is the most meaningful mediation between the two masses from the standpoint of physics.

Perhaps in addition to passing on to students various mnemonics (e.g., RASCAP. Resistors Add in Series and Capacitors Add in Parallel) we could also serve university level students by teaching them the various classical means, which provide a natural language for talking about mathematical formulas. Knowledge of these various means can also serve as a guard against the tendency to believe *combination* is a synonym for *addition*. For example individual magnifications, $M_1, M_2, \dots, M_N$, in a compound lens system combine geometrically (the product) and not arithmetically (the sum) as most students intuit.

4.4 Golden Ratio

Before closing this discussion of proportions and means it is worth mentioning the so-called “golden ratio” or “divine proportion” traditionally attributed to Pythagoras (Livio 2002, pp. 35–36). According to Euclid “a straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less” (Elements, Book VI, Definition 3, Heath 1956, Vol. 2, p. 188). Numerically this golden ratio corresponds to the constant $\phi = (1 + \sqrt{5})/2 \approx 1.618 \dots$, a common element in ancient mathematical and architectural constructions. Kepler considered this ratio and the Pythagorean theorem to be the “two treasure houses of geometry” (Kepler 1981 [1596], p. 133). Though claims for its ubiquity in the natural world are often abused there are in fact physical settings where ϕ is observed and has a clear dynamical explanation. Such examples are found in optics (Hoggatt Jr and Bicknell-Johnson 1979), crystallography (Levine and Steinhardt 1984; Caspar and Klug, 1962), phyllotaxis (Douady and Couder 1992), atomic physics (Heyrovska 2005), and recently in several exotic quantum mechanical settings involving quasiparticles (Coldea et al. 2010) and Bell’s theorem (Styer 2000, p. 55).

One instance germane to introductory physics is the case of an infinite parallel-series resistor network. Although a classic textbook exercise (Alonso and Finn 1970, p. 498), only relatively recently was it noticed that such a network can give rise to the golden ratio, and in the simplest case where every resistor has unit value. Such a circuit’s equivalent resistance is

$$R = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}} , \quad (28)$$

the continued fraction representation for the golden ratio (Srinivasan 1992).

5 Cosmology and Music of the Spheres

The example of circular motion has shown up more than once in this paper. In fact most common methods for arriving at the centripetal acceleration $a_c = v^2/r$ for a body in circular motion rely upon millennia-old Pythagorean mathematics. Two were discussed above (Sect. 4.3), one relying on the geometric mean, and the other upon similar triangles. Two other methods are the following: (1) express the body’s position as a Cartesian vector $\mathbf{r}(t) = r \cos(\omega t) \hat{\mathbf{i}} + r \sin(\omega t) \hat{\mathbf{j}}$ then differentiate twice to determine acceleration; or (2) follow what was essentially Newton’s procedure (Tipler 1999, p. 125) of analyzing how much a satellite “falls” as it executes circular orbit. It turns out these additional two approaches both rely upon $c^2 = a^2 + b^2$ at some stage of the demonstration. Thus the mathematization of circular kinematics seems almost inescapably Pythagorean.

In Pythagorean cosmology uniform circular motion is the most proportioned and harmonious, epitomized by bodies moving on the celestial spheres. “For everything moves in proportion; and this proportion of equality is the only one which, when it occurs, produces circles and spheres, because it returns on itself” (Archytas 1987 [4th century BCE], p. 183). This harmonic motion was understood to give rise to the so-called “music of the spheres,” a mystical yet remarkably enduring Pythagorean notion.¹⁰ “There is geometry in

¹⁰ Shakespeare alludes to this music: *Look how the floor of heaven / Is thick inlaid with patines of bright gold: / There’s not the smallest orb which thou behold’st / But in his motion like an angel sings,* (Shakespeare, The Merchant of Venice, act 5, scene 1.)

the humming of the strings, there is music in the orbits of the spheres.” The ancient historian Porphyry (Porphyry 1987 [3rd century CE], p. 129) wrote of Pythagoras that “He himself could hear the Harmony of the Universe, and understood the universal music of the spheres, and of the stars which move in concert with them.” Today of course we see no necessary connection between circular planetary motions and music, but it is worth pointing out that through much of the modern era the technology behind music has remained steadfastly tied to the dynamics of circular motion: be it Edison’s original phonograph, vinyl LPs, laser CDs, computer hard disks, and even the iPod scroll-wheel.

Besides the specific mathematical examples described in this paper, there is also a more general sense in which the historical development of physics and astronomy reveals the echo left by Pythagoras’ cosmology and mathematical philosophy. His school was the first to apply the Greek word *cosmos* (order) to the entire universe, standing in contrast to the opposing mythical Greek term *chaos*. Much of this cosmology is reported by the later Aristotle, who for the most part dismisses it where it contradicts his own physics. For instance as opposed to the four Aristotelian elements of earth, air, fire and water “They [Pythagoreans] supposed the elements of numbers to be the elements of all things, and the whole heaven to be a musical scale and a number” (Aristotle 1941a [4th century BCE], Book 1, Ch. 5. p. 698). Unlike most ancient models, both earlier and later, Pythagorean cosmology was not geocentric but rather placed a spherical earth in a circular motion which caused day and night. Again Aristotle, after laying out his geocentric view, comments “But the Italian philosophers known as Pythagoreans take the contrary view. At the centre, they say, is fire, and the earth is one of the stars, creating night and day by its circular motion about the centre” (Aristotle 1941b [4th century BCE], Book II, Ch. 13. 293a. p. 428).¹¹

Many of these Pythagorean cosmological ideas propagate in one form or another for the next two millennia. Ptolemy reverts to geocentrism yet still adheres to uniform circular motion and associated music of the spheres. As noted above (Sect. 3), in his 2nd century book *Harmonica* (Ptolemy 1980 [2nd century CE]) Ptolemy assigned musical notes to every celestial body in such a way that the solar system rings with fourths and octaves. Centuries later, Copernicus, in the preface to *On The Revolutions of Heavenly Spheres* (Copernicus 2004 [1543], p. 4) specifically mentions researching “Philolaus the Pythagorean” and concluding “Therefore I also, having found occasion, began to meditate upon the mobility of the Earth.” This newfound heliocentrism continues with Kepler. Despite abandoning the ancient ideal of circular planetary motions for his famous ellipses,¹² Kepler is otherwise unabashedly Pythagorean. In *The Secret of the Universe* he acknowledges that his famous model for the planetary spacings in terms of the five regular polyhedra relies upon Pythagoras “Because the doctrine of the five geometric figures being distributed among the bodies of the universe is traced back to Pythagoras, from whom Plato borrowed

¹¹ There is debate and historical confusion on whether this “central fire” was the Sun or a separate “central hearth.” Regardless of which, the Pythagorean model is striking in considering earth as a spherical, mobile body.

¹² The names for the conic sections originate (Heath 1965, p. 150–151) from the Pythagorean technique in *geometrical algebra* known as “application of areas.” It is essentially a geometric way to solve quadratic equations by finding an equivalent area to a given figure, one easier to reckon than the given figure. In more general use this *application* (παράβολη) could be *exceeding* (υπερβολή) or *falling short* (ελλειψις) of the given figure. I include the Greek to show the origin of our terms *parabola*, *hyperbola* and *ellipse*. Indeed these terms were adopted by Apollonius (3rd century BCE) in his famous treatise on conics since his constructions employed the Pythagorean application of areas, where the three possible applications (equal, exceeding or falling short) lead to the most general construction of the three conics.

this part of his philosophy” (Kepler 1981 [1596], p. 61). In Kepler’s later *The Harmony of the World* (Kepler 1997 [1619]), where he consolidates his eponymous three laws of planetary motion, the author in fact derives a detailed musical score for the harmony of the spheres based upon his third law, or harmonic law, relating a planet’s period and semi-major axis.¹³ When Galileo enters the historical scene he, like Copernicus, stirs trouble for his “spreading and acceptance . . . of the false Pythagorean doctrine, altogether contrary to the Holy Scripture, that the earth moves and the sun is motionless” (Decree of General Congregation of the Index, 5 March 1616, in Finocchiaro 2008, p. 177). More generally Galileo spreads the ancient “All is number” natural philosophy with his famous imagery of the universe being a book. “It is written in mathematical language, and its characters are triangles, circles and other geometric figures” (*The Assayer*, 1623, in Finocchiaro 2008, p. 183). Newton completes this mathematization of the natural world and philosophy quite literally with his *Mathematical Principles of Natural Philosophy* (Newton 1995 [1687]). The majestic calculus in some way displaces the Pythagorean musical ideas; which is perhaps fitting since the fruit of so many centuries of mental labors had finally matured, and could detach from the old tree. Nonetheless, Newton fully relies upon the authority of what he calls the primary “phenomena” of Kepler’s laws, in particular the harmonic law (or third law) concerning the periods and mean radii of planetary orbits.

Thus the long historical narrative leading to what is traditionally taught in introductory, classical physics carries forth not only specific mathematical techniques, but a mathematical inspiration and response to the ancient ideas of Pythagoras and his school. Finally, though it is far from our treatment of introductory physics, one cannot help but wonder what Pythagoras would make of a certain contemporary theory of the universe whose elements are highly mathematical vibrating strings.

6 Concluding Remarks

Physics and mathematics are wedded together. As Uhden et al. (2012) conclude in their recent study of mathematical reasoning in physics education, there is an “entanglement of physics with its mathematical structure” and “for both prospective teaching and further research, a focus on deeply exploring such interdependency can significantly improve the understanding of physics.” This entanglement is practical (e.g., the need for physics students to take mathematics pre-requisites) but as evidenced in this paper it is also historical.

Most students are aware of a vague connection between physics and some ancient civilization, due to the Greek symbols used by their textbooks and professors. As shown above through numerous examples this connection goes much deeper, so prevalent it is

¹³ Although Kepler’s first two laws can already be seen in his *New Astronomy* (see Kepler (1992 [1609])) he introduces his third law ($p^2 \propto a^3$) in *The Harmony of the World*. The entire contents of the final Book V (On the Most Perfect Harmony of the Heavenly Motions) lays out the consequences of this third “harmonic law.” There he derives several musical scores for the planetary motions (e.g., see Kepler 1997 [1619], p. 457) accompanied by demonstrations such as “Proposition XI: The proportion of the motion of Saturn at aphelion to that at perihelion ought to have been 4:5, a major third, but that of Jupiter’s motions 5:6, a minor third.” At the risk of digression we will not unravel the question of what extent this music of the spheres could or should actually be heard. The most creative answer to this question is related by Aristotle who writes that “the sound is in our ears from the very moment of birth and is thus indistinguishable from its contrary silence” (Aristotle 1941b [4th century BCE], Book II, Ch. 9). Porphyry holds that these are sounds “which we cannot hear because of the limitations of our weak nature” (Porphyry 1987 [3rd century CE], p. 129). For a modern treatment of this question using astrophysical examples see Caleon and Ramanathan (2008).

easily taken for granted. Indeed, many of the mathematical methods used in reasoning through a typical introductory physics course have their root in Greek thought from the 6th century BCE, particularly Pythagoras and his school. Applications to physics of the elementary Pythagorean theorem abound, and point to a useful lesson: fancy mathematics does not necessarily underly deep physics. Even revolutionary results such as Einstein's special theory of relativity require surprisingly little mathematical sophistication. What is novel is the *physics*: careful reasoning about natural principles and experiments. Beyond plane geometry, Pythagoras' experiments and theories concerning music, harmony and sound shed light on the dual relation between continuous and discrete quantity, important in reasoning about vibrations and waves, and more generally harmonic analysis and quantum theory. The rich theory of Pythagorean proportions and means exemplifies the important principle that not all averages, or means, are arithmetic; sometimes physical quantities combine harmonically (e.g., reduced mass) or geometrically (e.g., areas of capacitive plates).

Beyond these specific mathematical methods, we also see the more general echo left by Pythagoras' cosmology and mathematical philosophy (e.g., the music of the spheres) on the historical development of physics and astronomy. Understanding this legacy can potentially afford a deeper appreciation of more modern developments in physics. For example Einstein's remark that Bohr's mathematical model for the atom "is the highest form of musicality in the sphere of thought" (Pais 2005, p. 416) is seen not simply as poetic praise but as an allusion to a long history of ideas on the nature of the physical world. More so than many sciences, physics has a long narrative stretching back thousands of years. The discipline of physics inherited and carries forth specific ways of reasoning based upon an ancient idea: that the entire universe is ordered in a mathematically intelligible way, a cosmos, where "all things accord in number."

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