

PYTHAGORAS BELIEVED THAT ‘THE GODS’ USED THE SMALL WHOLE NUMBERS TO DESIGN THE UNIVERSE. WHAT WAS HIS EVIDENCE?

Geoff Faux reconstructs his opening session at Conference 2013

As you read this article you may want to pause and work on some of the mathematics.

Pythagoras

Pythagoras is early amongst the Greek philosophers. So early that nowadays there is debate about whether he actually existed. His dates are ?569? – ?475? BCE. This places him as one of the first of the Greek philosopher mathematicians. The table below lists a selection of Greek Mathematicians with their dates. It is worth noticing that Pythagoras lived three hundred years before Euclid, a contemporary of Aristarchus, the first person to consider that the sun, not the earth, might be at the centre of the universe.

The table indicates the time over which Greek mathematics developed and flourished. Some 800 years after Pythagoras there was a rich vein of mathematical ideas. In 250 AD Diophantus set about collecting and publishing a set of whole number problems. These problems were then worked on and shared amongst the Renaissance mathematicians leading to further work and ideas.

Pythagoras about 569 – about 475 BCE
Euclid of Alexandria about 325 BCE - about 265 BCE
Aristarchus of Samos about 310 BCE - about 230 BCE
Archimedes of Syracuse 287 BCE - 212 BCE
Diophantus of Alexandria about 200 AD - about 284 AD

Pythagoras was born in the Isle of Samos. Samos, one of many independent rival Greek city states was ruled, at the time, by the tyrant Polycrates. Polycrates had the largest fleet in the whole Mediterranean, agreements with the Pharaoh in Egypt about mutual support in the event of an attack by Athens and so on.

The fortifications and organization in Samos were extraordinary and included a military harbour, walls around, and over, the adjoining hills and a tunnel right through a hill to bring fresh water to the town from a distant reservoir. The harbour and the tunnel still exist. See, <http://goo.gl/zVMwLM>

Myth and fact in terms of Pythagoras early life is hard to disentangle. The young Pythagoras is said to have travelled to Egypt where he became friendly with the priests. He possibly next moved over to Babylon before re-appearing in Samos about 12 years later. Pythagoras only stayed there for a short time. One version states that Polycrates attempted to involve Pythagoras in the political shenanigans. Pythagoras retired to a cave, preferring to work on philosophy and geometry before moving out to Croton, in southern Italy, where he founded the Pythagoreans and remained for the rest of his life.

So, to return to the title: *‘the Pythagoreans believed that ‘the gods’ had used the small whole numbers to design the universe’.*

Patterns in small numbers

Imagine you are with a group of teachers at conference

I write

$$1 + 2 = 3$$

... and after a pause write

$$4 + 5 + 6 = 7 + 8 \text{ underneath.}$$

In your head construct the next line.

You might now have

$$1 + 2 = 3$$

$$4 + 5 + 6 = 7 + 8$$

$$9 + 10 + 11 + 12 = 13 + 14 + 15$$

‘What do you notice?’ Have you any conjectures at this stage?

Before reading on it is worth spending time to consider your own ‘noticings’.

At Conference 2013, there was time for delegates to work both in groups and individually. When it came to ‘sharing’, ... some of the following were aired:

- The third row adds to 42. 42 is the answer to everything.
- An initial conjecture was that, because of the 42, this might be the last line that works. This conjecture was quickly abandoned!
- Many participants noticed a square number at the start of each line.
- One person noticed that the line starting with three squared has three times four before the equals sign and conjectured that the next line would have four times five before the equal sign.
- The third line had three numbers after three squared before the equal sign, and so on.

I asked participants to write down the line that included the number 40 somewhere within the run, as a way of steering them to work with a slightly more general special case, and as an antidote to those who wanted to immediately introduce 'n' into their talk.

We are talking about the gods. The gods live in a higher dimension, perhaps. So, starting with $1 + 2 = 3$ but trying to move it up a couple of dimensions I wrote $1^3 + 2^3$. I notice that this adds not to 3^3 , but 3^2 .

So, what about

$$1^3 + 2^3 = 3^2$$

... followed by

$$1^3 + 2^3 + 3^3 = 6^2$$

... and the next line? Is it?

$$1^3 + 2^3 + 3^3 + 4^3 = 10^2$$



If you are happy with a generalization that you have made from these three ordered special cases then watch the film at atm.org.uk/mt237

Watching the film suggests that the formula for the sum of the cubes is accessible to many people well before it appears in the current curriculum at early A-level. In fact both of the primary classes I worked with in Cumbria, before the conference, seemed really at home with the idea of slicing their *Cuisenaire* cubes up, and filling up a growing square in an organized way.

It would be wonderful to receive feedback on any work, by learners, inspired by this type of work.

Staying with small numbers, start with

$$3^2 + 4^2 = 5^2$$

... but write underneath,

$$3^3 + 4^3 + 5^3 = 6^3$$

... and then ask yourself what could be next?

In preparing for the session I had looked at this pair of equations asking myself *if there were other triples of cubes?* ... and then moving to conjecturing about fourth powers.

I had used some of this number pattern work in a morning with the Birmingham branch of ATM. At the end of that session Tom Francome 'shared' that he had noticed a connection between the odd numbers and the cubes.

$$1 = 1^3$$

$$3 + 5 = 2^3$$

$$7 + 9 + 11 = 3^3$$

There was not time to work on this at conference. I allowed it to briefly appear and thanked Tom. This is another pattern, I now see, like the first one that is really accessible via *Cuisenaire* rods, at least that is how I gained an insight.

Prime Numbers

The Pythagoreans worked on Prime numbers. I want to look briefly at prime numbers from a geometric point of view. Start by imagining a collection of cubes. The Pythagoreans were interested in the fact that some collections of cubes could only be arranged in a line, whilst others like 6 cubes, or 15 cubes could be arranged into rectangular arrays.

Not many of the small numbers can be arranged in arrays, four is the first, so we need a systematic way of finding out which can, and which cannot, be so arranged.

Use the grid shown in Figure 1. Start by ringing 2, and then cross out all the numbers that can be arranged in groups of two, 4, 6, 8 ... Next ring 3, and cross out all the multiples of 3. Now on to 5, ring 5 and cross out all the multiples of 5; 10, 15, 20..., **even the ones that are already crossed out**, watching yourself and noticing where the numbers lie. And next for 7, where do the multiples of 7 lie? Watch yourself crossing out 14, 21, 28...

2	3	4	5	6	7
8	9	10	11	12	13
14	15	16	17	18	19
20	21	22	23	24	25
26	27	28	29	30	31
32	33	34	35	36	37
38	39	40	41	42	43
44	45	46	47	48	49
50	51	52	53	54	55
56	57	58	59	60	61
62	63	64	65	66	67
68	69	70	71	72	73
74	75	76	77	78	79
80	81	82	83	84	85
86	87	88	89	90	91
92	93	94	95	96	97
98	99	100	101	102	103

Figure 1

Do you need to work with multiples of 11?

Using this layout the prime numbers, larger than 3, appear in the two columns either side of the six times table.

The Pythagoreans were interested in twin primes, 17 and 19, 29 and 31 etc. Move to one of the spirals in Figure 2. Use the results from your first sheet to circle all the primes on one spiral sheet. A startling result is that on both sheets the bottom left to top right diagonal consists of primes.

Why?

185	184	183	182	181	180	179	178	177	176	175	174	173
186	141	140	139	138	137	136	135	134	133	132	131	172
187	142	105	104	103	102	101	100	99	98	97	130	171
188	143	106	77	76	75	74	73	72	71	96	129	170
189	144	107	78	57	56	55	54	53	70	95	128	169
190	145	108	79	58	45	44	43	52	69	94	127	168
191	146	109	80	59	46	41	42	51	68	93	126	167
192	147	110	81	60	47	48	49	50	67	92	125	166
193	148	111	82	61	62	63	64	65	66	91	124	165
194	149	112	83	84	85	86	87	88	89	90	123	164
195	150	113	114	115	116	117	118	119	120	121	122	163
196	151	152	153	154	155	156	157	158	159	160	161	162
197	198	199	200	201	202	203	204	205	206	207		

Figure 2

... or,

Does this work with other pairs of twin primes?

These are two questions that may be worth working on at this point.

The spiral in an interactive spreadsheet, can be found on-line at www.atm.org.uk/mt237.

Circles

In addition to lines and areas the Pythagoreans were also interested in circles.

Inside a circle it is possible to draw a number of regular polygons as in Figure 3.

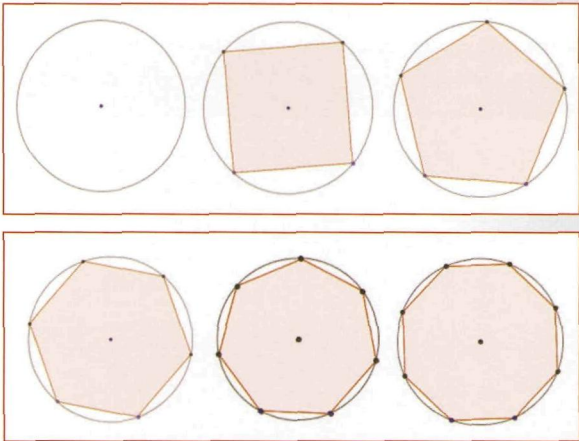


Figure 3

Of the regular polygons, squares are the most readily available so I suggest starting with a square.

- Find a square of paper.
- ... Mark the centre of your square (by folding perhaps).
- Fold each vertex, in turn, to the centre forming a new quadrilateral.
- Is this new quadrilateral also a square?

As it is twice the thickness of the original square is this a proof that the new square is half the area of the first square?

Now, we move on to a hexagon. There are two routes; either cut and reason with a cut out hexagon, or watch the film at atm.org.uk/mt237



It is worth cutting out the hexagon, but looking at the two images in Figure 4, is it possible to agree that the triangle in the first picture is half the area of the hexagon and, by drawing in some extra triangles in the second figure that the smaller hexagon is $\frac{2}{3}$ of the triangle, so the smaller hexagon is $\frac{2}{3}$ of $\frac{1}{2}$ of the larger one. For the hexagon the fraction is $\frac{1}{3}$.

Cutting out and folding gives a very different feel to this 'proof'

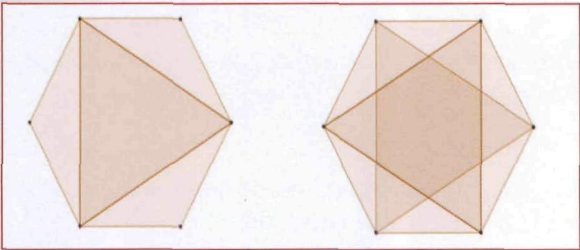


Figure 4a

After working on these two regular polygons consider the table in Figure 4b;

Sides of regular polygon	Ratio of areas
3	A half
4	
5	
6	A third
7	
8	
9	
10	

Figure 4b

There seem to be some questions here.

In particular, ‘what could be a fraction for the pentagon?’

One reason for asking about the pentagon is that the Pythagoreans used the pentagram as a symbol to recognize each other. I have no idea why, but one conjecture might be that they saw this as an unfinished task that real mathematicians were working on – the ratio of the smaller red pentagon to the larger one.

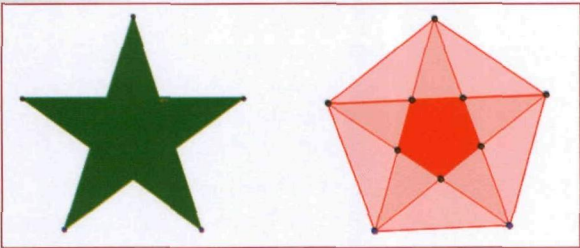


Figure 5

More work with a square

Take a new square of paper. Mark the middle of the top, and then fold one of the bottom corners up to that point, see Figure 6

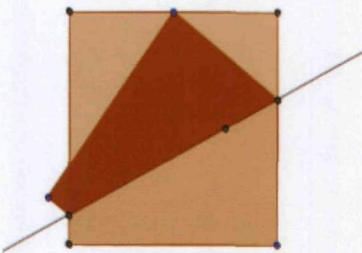


Figure 6

The task is then to find the ratio of the three sides in the triangle in the top right hand corner, as shown in Figure 6. I am conscious that this is a problem posed by NRICH – go to Napkin fold, <http://nrich.maths.org/700> – so, if you are familiar with that task fold the bottom right hand corner up to the point one third of the way from the left hand end along the top, and search for the ratio of the three sides on that new triangle.

Before reading on it would be sensible to work on both these tasks.

Some results are shown in Figure 7.

Distance from left hand end	Triangle	
$\frac{1}{2}$	3, 4, 5	4 along the top
$\frac{1}{3}$	5, 12, 13	12 along the top
$\frac{1}{4}$	7, 24, 25	24 along the top
$\frac{1}{5}$	9, 40, 41	24 along the top
And so on		
But also for $\frac{2}{3}$	3, 4, 5	3 along the top

Figure 7

The final entry raises the question, what fraction will give 5, 12, 13 the other way around?, ... that is 5 along the top. An astonishing fact, for me, is that there is a Pythagorean triple associated with every rational fraction along the top and vice-versa. There is also a wonderful connecting idea between the 3,4,5; the 5,12,13; the 7, 24,25 ... and Pythagoras’ work on classical notes and fractions along a string.

Pythagorean triples

Because Pythagoras was in the title of this piece, I think it is time to look at;

$$3^2 + 4^2 = 5^2$$

What is the next Pythagorean triple that springs immediately to mind?

At Conference 2013 there were various responses, but no-one suggested

$20^2 + 21^2 = 29^2$, which is the one that I would like to place immediately under 3,4,5.

My reason for going to this triple next is that (20,21,29) is the next 'most square' rectangle with an integer diagonal. This raises the question; *what is the next most square integer sided rectangle with an integer diagonal length? And the next?* Development on this idea can be found, in this edition of MT, as a separate article entitled *Folding a Square*, Page 29.

In-circles

Returning to the idea of working with circles 'Imagine a horizontal line and cut it with a vertical line to make two corners, one above and one below.' Now take a circle with radius one and push it gently into the top corner. Take a second circle, radius two, and push it into the corner below the horizontal line. Lean a line against the two circles to make a triangle. See Figure 8.

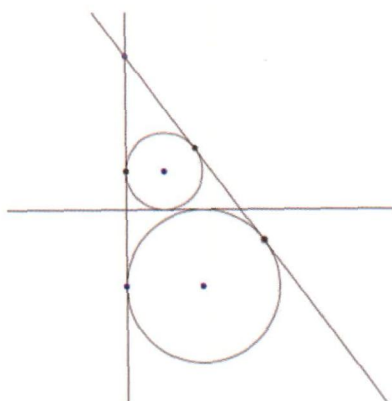


Figure 8

Find the lengths of the sides of the triangle.

It would be helpful to work on this task before reading on.

As delegates found a solution they felt at ease with, I asked them to look at a similar diagram with circles of radius 2 and 3, then 3 and 4, ..., and so on.

After a little work I suggested that although there are an infinity of triangles around a unit circle I now know that there is only one integer triangle around the unit circle and this is the right angle 3,4,5 triangle. Algebra feels to be a *heavy* approach to proving this uniqueness. Are there any geometric proofs?

Around a circle of radius 2 there is clearly 6,8,10, twice 3,4,5 and, from the work that has just been done, 5,12,13. But, it is also possible to put two Pythagorean triangles together, either overlapping, or side-by-side, to form a new triangle with integer sides and an in-circle whose radius we can find.

As, for instance, in Figure 9

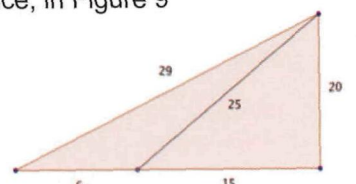


Figure 9

The 6, 25, 29 triangle can be seen as a 20, 21, 29 triangle with a 15, 20, 25 triangle removed.

There is a piece of bookwork in the A-level syllabus, from many years ago, and I have just found a use for it! The bookwork states that:

In any triangle the connection between radius of the in-circle and area of the triangle, with sides a , b , c is given by...

$$\Delta = rs$$

where Δ = area of the triangle,

r = radius of the in-circle

and

$$s = \text{semi perimeter} = \frac{1}{2}(a + b + c)$$

The image for this is shown in Figure 10

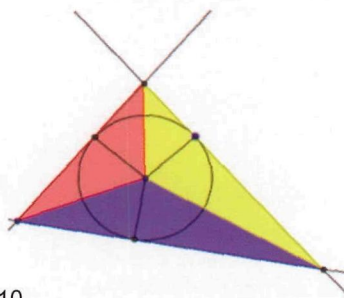


Figure 10

... from which it is reasonable to write that

$$\Delta = \frac{ra}{2} + \frac{rb}{2} + \frac{rc}{2}$$

where r is the radius of the in-circle. The formula follows.

A consequence of putting this formula together with the half-base-times-height formula for the area of a triangle is that the only integer sided triangles, with an integer length in-circle radius, must also have a rational height.

The reasoning for this is as follows.

$$\Delta = r \times s = a \times h$$

If, $r \times s = a \times h$ we require r , s , and a ... all to be integers, or at least rational, so h must also be an integer, or rational. If h is rational then it lies on a grid point and the Pythagorean triangle is assured. From this the in-circle in the 6, 25, 29 triangle has radius 2.

I know that there is only one integer triangle around the unit circle. This is presented as an assertion and the *proof* is left to the reader!

I would like to pass on a 'wondering' from Alan Beardon '*Is there anything known about integer triangles round a circle of radius 2?*'

Having had a number of e-mail exchanges with John Mason after conference we have five different integer triangles round a radius 2, but sorting out that this is all of them... is another matter entirely. This is left open as work in progress.

Proving the uniqueness of the 3,4,5 triangle around the unit circle is a more straightforward task.

Multiple in-circles

I cannot resist showing this image next.

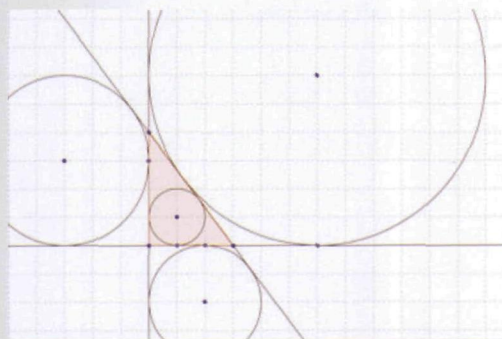


Figure 11

Here we have a 3,4,5 triangle with an incircle of radius one, and the other three circles, which touch all three sides of the triangle, having radii, two, three and six. So, a perfect result 1,2,3,6 and $1 + 2 + 3 = 6$. It leaves me

conjecturing about 28, (1, 2, 4, 7, 14) the next perfect number, and what the dimensions of an associated right angle triangle might be?

Tessellation

Figure 12 shows another way of looking at the 3,4,5 triangle.

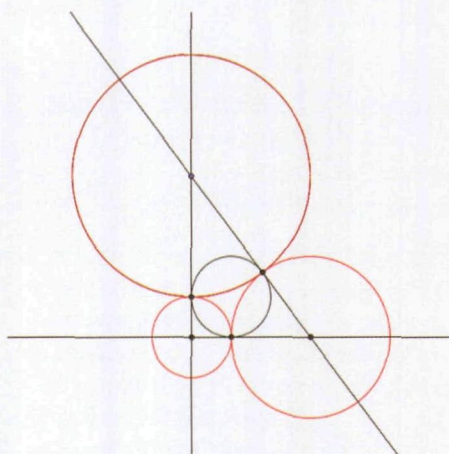


Figure 12

The three red circles show a way of accessing the connections between a 3, 4, 5 triangle and the associated in-circle. But, standing back, it is clear that if this triangle tessellates the plane, which it does, then it should be possible to pack the plane with circles of radius one, two and three. This is a dynamic geometry task that, at conference, I left open to be worked on in workshop time.

Ratio in three dimensions

When you elect to set hares running at the start of a conference there are always choices to be made.

I decided that there was time for one further activity. Unfortunately, I needed to set the activity up using some introductory ideas.

I wanted to work on the ancient Greek geometric challenge of constructing an altar that is just twice the volume of an existing altar. In traditional ruler-and-compass work this is impossible, but by using some of the ideas from origami, and looking in the ATM publication, '*Ideas for 6th Form Mathematics*', as a starting point, much of this is opened up.

I started by folding pieces of A4 paper into a cylinder in two different ways. In the first approach I joined the long lengths together to produce one cylinder, and as second approach by joining the short lengths to produce a different cylinder.

Holding up the two cylinders I asked; *which of these containers would hold the most Maltesers?* In a straw-poll most delegates believed the 'more squat' cylinder would hold the most, but there were a few votes for 'the same', and one or two for the taller cylinder as holding the most.



I creased both cylinders so that each was now a square based prism, and asked; *will this special case preserve the ratio of the 'volumes' between the two paper models?*



I left the task, knowing that the side lengths were in the ratio $1:\sqrt{2}$, for finding the volume ratio, but suggested that cutting a large square in half to get a pair of rectangles with lengths in the ratio $1:2$ might be a special case that could provide insight into the first, two cylinders task, maybe generalising along a chain of specialisations.

I next produced a *brick* folded from a large square. Then, I unfolded the *brick* to show the creases. I decided to finish the session with this challenge; *is it possible to find the side of a square that will fold to make a brick of half the volume of the original brick?*

For instructions on how to fold, and produce, the brick go to www.atm.org.uk/mt237.



Geoff Faux

And finally

A possible different second line under $1 + 2 = 3$

Instead of writing $4 + 5 + 6$ underneath would it be possible to start with 2 and the next line starting with 3

So we get $2 + 3 + 4 = 4 + 5$

$$3 + 4 + 5 + 6 = 5 + 6 + 7$$

Readers might like to work on the samenesses and differences between the infinity suggested by these three and the three at the start of this article.

References

Ideas for 6th Form Mathematics, ATM

Resources

Linked to this article there are a number of resources on-line. www.atm.org.uk/mt237

Instructions for making a brick from a square

A list of activities for delegates at conference following Geoff's plenary.

Other Resources

The following are web-based sources that I regret were not included in the opening conference plenary because of time restrictions.

Part of Bronowski from Ascent of Man that deals with Pythagoras theorem (BBC) <https://www.youtube.com/watch?v=mOvpV0CuEdc>

... or you can see the whole episode

<https://www.youtube.com/watch?v=IENM8u47zsl&playnext=1&list=PL9D6C6564285AFD14>

Donald in Mathmagicland – golden section piece

<http://www.youtube.com/watch?v=D5n6iT2Aqrl>

... or the whole film

<http://www.youtube.com/watch?feature=endscreen&NR=1&v=nav0kVa66xk>

Archimedes volume of a sphere

<http://www.cut-the-knot.org/pythagoras/Archimedes.shtml>

<http://www.youtube.com/watch?v=giUk9IeseBs>

There are two DVDs I often use with primary learners, although I did not do so on with the group I worked with prior to conference

The dot and the line Norton Juster'

<http://www.youtube.com/watch?v=hgqUya0kGPA>

The Phantom Tollboth

<http://www.youtube.com/watch?v=Llg5VODW6n4>

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