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A GENETIC INTERPRETATION OF NEO-PYTHAGOREAN ARITHMETIC*

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The (Neo-)Pythagorean arithmetic tradition represents an important intellectual current, characterised by an approach to the construction of arithmetic, alternative to that of the Euclidean Elements. We take this tradition to embrace the following pieces of arithmetic theory:

- I- The theory of even and odd numbers;
- II- The theory of perfect numbers;
- III- The theory of numerical ratios;
- IV- The theory of figured numbers and
- V- The theory of means.

For the later three pieces, there are no analogues in Euclid's *Elements*. However, some pieces of the theory of polygonal numbers is found in

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Diophantus' treatise *On Polygonal Numbers*, which follows the style of the Euclidean tradition.

Our main sources for the Neo-Pythagorean arithmetic are:

- The *Introduction to Arithmetic* by Nicomachus of Gerasa (2nd century AD).
- The *Exposition of Useful Mathematical Knowledge for the Reading of Plato* by Theon of Smyrna (2nd century AD);
- Iamblichus' *Commentary on Nicomachus' 'Introduction to Arithmetic'* (4th century AD), which follows the order and the content of Nicomachus' treatise, but at times omits something¹ or adds new material;²
- The *Manual of Arithmetic Introduction* by Domninos of Larissa (5th century AD);
- The *Commentary on Nicomachus' 'Introduction to Arithmetic'* by Asclepius of Tralles;
- The *Commentary on Nicomachus' 'Introduction to Arithmetic'* by Ioannes Philoponus (5th-6th century AD).

The later two are in the form of scholia to the text of Nicomachus.

The style of arithmetic in the treatises the Neo-Pythagorean authors is strikingly different from that of the *Elements*. Namely, it is characterised by the absence of proof in the Euclidean sense and a specific genetic approach to the construction of arithmetic that we are going to describe in our paper. Lack of mathematical sophistication has led certain historians to consider this type of mathematics as a feature of decadence of mathematics in this period [Tannery 1887, 11-12; Heath 1921 I, 97-99].³ The alleged absence of originality in these works has also given grounds to believe that "the arithmetic presented in these works derives substantially from an ancient, primitive stage of Pythagorean arithmetic" and to use them as "an index of the character of arithmetic science in the 5th century" [Knorr 1975, 132].

In this paper, we take Nicomachus' *Introduction to Arithmetic* for point of departure, because it is the richest and most well organised treatise representing this tradition [Vitrac 1994, II, 474-475]. However, we also take into account the works of other Neo-Pythagorean authors. We are going to show

that the Neo-Pythagorean arithmetic might have been developed in a natural, self-contained manner as a simple theory of counting over a domain of concrete initial objects, designated by fixed signs. This approach can be comfortably realised without appealing to assumptions of axiomatic character, but relying upon some 'genetic' constructions intended to be carried out by means of the designated entities.

1- SEMIOTICS

Nicomachus' concept of number in the context of ancient semantic traditions

Nicomachus gives in his *Introduction to Arithmetic* a general semiotic scheme for the symbolic representation of numbers. He notes that the designation of numbers by means of letters is *conventional* (νόμος), not *natural* (οὐ φύσις σημαντικὸν ἐστὶ τοῦ ἀριθμοῦ). The *natural semeiosis* (φυσική σημείωσις) of numbers is by means of the representation of the units composing a number, one beside the other.

First, however, we must recognise that each letter by which we designate a number, such as iota, the sign for 10, kappa for 20, and omega for 800, signifies that number by man's convention and agreement, not by nature. On the other hand, the natural, unartificial, and therefore simplest designation of numbers would be the setting forth one beside the other of the units contained in each.⁴

The concepts of convention (νόμος) and nature (φύσις) go back to the pre-Socratics (Pythagoras, Democritus) and the Sophists. For the later, the distinction between convention and nature was a central element in their philosophy. Proclus ascribes to Democritus the view that the relation between names and things named is conventional, rather than natural. He supposed to have set forth four arguments, called *polysemy* (πολύσημον), *equipollence* (ισόρροπον),

⁴ Nicomachus, *Introductionis Arithmeticae*, II. vi, 2, Hoche ; D'Ooge, 1926, 832.

We follow D'Ooge's translation except the instances otherwise stated. Here our translation diverges from that of D'Ooge in two main points: we render the term "σημείωσις" by designation, whereas the term "σημαντικὸν" by the rather neutral term signifying. By the operation of σημείωσις a sign (σημεῖον) is assigned to a number; this sign signifies conventionally that number.

We also suggest another translation of the last paragraph of this passage, to avoid D'Ooge's rendering of μονὰς μὲν α by "the number 1", since the monad was not considered as number by the Greeks.

¹ E.g. the topics in *Nicomachus Introductionis Arithmeticae* I.vi and II.v, Hoche.

² E.g. the 'bloom' of Thymarithas.

³ The idea of the necessary decline of Greek mathematics in view of their inherent limitations that was reigning since the works of Zeuthen [1896] and Tannery was challenged by Bashmakova [1961] and Stein [1990, 182-197].

metonymy (μετωνυμία), and *anonymy* (ἀνώνυμος), respectively, in support of his viewpoint. Namely the occurrence of homonyms for different things, the existence of synonyms for one and the same thing, the possibility of renaming the same thing, and the occurrence of irregularities in the words-formation. Pythagoras reportedly holds that the assignment of names to things is not an arbitrary operation, but is imposed upon things by some kind of natural adequacy between the names and the things, so that "the name-giver can only be a man who contemplates the mind and the nature of being".⁵ Pythagoras' views are the earliest reported in favour of a particular semantic theory. Iamblichus ascribes also to Pythagoras "the symbolic and detached use of mathematical words" (τὴν συμβολικὴν καὶ ἀπεξενωμένην χρῆσιν τῶν μαθηματικῶν λέξεων)⁶ and considers that he was the first who tried to apply the naturalistic semantic viewpoint to mathematics.

The original Pythagorean 'naturalistic' viewpoint seems to be shared by Iamblichus, who accuses Philolaus for having adopted the 'conventionalistic' viewpoint that was not shared by Pythagoras himself.⁷ If his report is true, we have to allow certain divergence of the Neo-Pythagoreans from Pythagoras himself as it concerns matters of semantics.

The opposition of the 'naturalistic' and the 'conventionalistic' viewpoints is the point of departure in Plato's *Cratylus*, where the 'conventionalistic' semantic viewpoint is attributed to Hermogenes, while the 'naturalistic' one is supported by Cratylus.⁸ Therefore, it seems that Nicomachus' conventionalism has its roots in the semantic tradition of the 5th century, yet most probably cannot be reduced directly to Pythagoras himself. It should be noted that historians of mathematics up to-date have neglected this relation of arithmetic to semantics.⁹

Analysis of the Nicomachean semiotic scheme for number

One can distinguish the following aspects of the Nicomachean semiotic scheme:

⁵ Proclus, *In Platonis Cratylum commentaria*, 16, p. 5, 25, Pasquali.

⁶ Iamblichus, *De communi mathematica scientia*, 67, 16, Festa.

⁷ *Id.*, *In Nicomachi Arithmeticon introductionem*, 19, 24, Pistelli.

⁸ Cf. Aristotle *De Interpretatione*.

⁹ See, for instance, D'Ooge's treatment of number in Nicomachus, where this relation is completely overlooked [D'Ooge 1926, 111-123].

First, the description of a *generating process*, the setting forth one alpha beside the other (παράλληλος ἕκθεσις). Thus, the unit is the setting forth of one alpha; the number two is the setting forth of two alphas one beside the other; the number three is the setting forth of three alphas one beside the other, and so forth (see the next section). Asclepius of Thralles explains us that by παράλληλος ἕκθεσις is not meant the representation in a straight line, but the arrangement of units one beside the other.¹⁰

Second, the description of a process of *designating* the objects counted. This process of transformation of some initial collection of objects to a schematic pattern, in principle, cannot be free from acts of arbitrary choice. In this sense, Philolaus and Nicomachus are right to support the conventionalistic viewpoint, in contrast to Pythagoras. The essential point lies in the organization of the designata into readily identifiable wholes, so that when one is faced with a pattern of alphas to be able to recognize one of the defined patterns and associate it with the appropriate number. It should be noted, however, that there is no pattern available to express the idea of "any number".

Third, the patterns defined to represent numbers should have *internal associations*, namely to be clear how it is possible to pass from one pattern to the next one by performing a certain operation. This entails that each pattern should be contained in its successor.

The unit (μονὰς) in the Pythagorean arithmetical tradition is a "minimal entity"¹¹, "indivisible by nature"¹² and serves as the "natural beginning of numbers"¹³ that is, it precedes the concept of number. It is "potentially all the numbers".¹⁴

Number in Nicomachus possesses internal structure (σχῆμα - 'arrangement'). It is a (finite) 'suite' (or a schematic pattern) of signs, unbounded in the direction of increase and bounded below by the *monas* in the direction of decrease. In Theon's words, "a number is a collection of units, or a progression (προποδισμός) of multitude beginning from a unit and a retrogression

¹⁰ Asclepius of Thralles, *Commentary to Nicomachus' 'Introduction to Arithmetic'*, II, ζ 1-5, Tarán.

¹¹ Iamblichus, *In Nicomachi Arithmeticon introductionem*, 11, 1, Pistelli.

¹² Nicomachus, *Introductionis Arithmeticae*, I, viii, 4, Hoche.

¹³ *Ibid.*, I, viii, 2, Hoche.

¹⁴ *Ibid.*, I, xvi, 3, Hoche.

(ἀναποδισμὸς) ceasing at a unit".¹⁵ Nicomachus describes number as "a flow of quantity made up of units" (ποσότητος χύμα ἐκ μονάδων συγκείμενον). All these descriptions suggest an internal 'directional' order structure for the notion of number.

Iamblichus writes that the notion of number as 'collection of units' (μονάδων σύστημα) goes back to Thales who follows the Egyptian pattern¹⁶ on this point, whereas the notion of number as 'definite multitude' (πλήθος ὠρισμένον) was introduced by Eudoxus. Nevertheless, the characterisation of number by means of such terms like 'limit' (πέρας) or 'limiting quantity' (περαίνουσα ποσότης) – the later is ascribed by Iamblichus to Thymaridas¹⁷ – seems to tally with the Pythagorean philosophical tradition of the 5th century, particularly, Philolaus' doctrine of the Limit and the Unlimited (περαίνοντα).

Further, the finite sequence of such simple suites can be constructed, i.e. the sequence of the so called "properly ordered" (εὐτάκτους) numbers $\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta, \iota, \iota\alpha, \iota\beta, \iota\gamma, \iota\delta, \iota\epsilon$

that is the sequence (which we denote by natural numbers in *italic*)

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15,

which is called *the natural suite* (ὀφυστικὸς στίχος) by Nicomachus¹⁸ and serves as pattern exemplifying the mode of construction of the kind of number considered in each case. Although this concept is introduced in Book II, Nicomachus makes use of it already in Book I. The natural suite should not be confused with the natural series. In Nicomachus, it is a finite constructional element. The concept of the natural suite is intrinsically connected with the notion of "properly order". In Nicomachus, the natural suite is always a suite of "properly ordered" numbers, that is it embodies the specific regularity or rule according to which it is constructed.

¹⁵ Theon, *Expositio rerum mathematicarum ad legendum Platonem utilium*, I, iii, 1-5, Hiller; Dupuis 1966, 28-29.

¹⁶ See also [Herodotus, *Historiae*, II, 36, 1.18-20, Hude].

¹⁷ Iamblichus, *In Nicomachi Arithmeticum introductionem*, II, xi, Pistelli.

¹⁸ Nicomachus, *Introductionis Arithmeticae*, II, viii, 3, Hoche.

Number and iteration

The iterative operation of generation of successive numbers is described as "advance by a unit" (κατὰ μονάδα προβάλλειν, ἐπὶ ἓν προχωρεῖν),¹⁹ etc. This operation is illustrated:

- by the attachment of the sign α in a suite, or

- by the application of a specific schematic pattern (i.e. the *gnomon*).

The operation of attachment (or removal) of a sign is illustrated in the well-known fragment of Epicharmus, cited by many historians [Becker 1936, Knorr 1975] as an early evidence for the pebbles-representation technique.

If to an odd number or to an even,

you would like to add a pebble or to remove it,

do you think that it will remain the same?

In no case!²⁰

The "gnomon" on the other hand, has a similar function in the theory of figured numbers. The basic property of the gnomon is that its application is closed with respect to the kind of figured number (the kind of polygonal number). This property is clearly stated, for instance, by Iamblichus: "It is called *gnomon the augmentative* [because] when added it preserves the same kind of each of the polygonal [numbers]".²¹ Most of the available general definitions of gnomon are based on the property of similarity of the wholes after and before its placement.²²

¹⁹ Nicomachus, *Introductionis Arithmeticae*, II, vii, 3, Hoche; D'Ooge, 1926, 833.

²⁰ Diels, *Fragm. Epicharmus*, A.2.

²¹ Iamblichus, *In Nicomachi Arithmeticum introductionem*, m 58, 19-21, Pistelli.

²² For instance, in Aristotle *Categories* C, 14, 15 a 29-33, or in Heron *Def.* 58, t. IV, 225, Heiberg. An exception is the gnomon passage in Aristotle's *Physics*, where we find a use of gnomon not based on this property. Namely, the case of the construction of oblong numbers when the gnomons which generate them are always different [Heath 1970, 101]. An account of the different interpretations that have been suggested for this passage can be found, for instance, in Ross' *Aristotle* [1936, 542-545].

Theon also calls "gnomons" and "gnomonic numbers" (γνωμονικούς ἀριθμούς) all the successive numbers (differences) that generate triangular, square or polygonal numbers. [Theon, *Expositio rerum mathematicarum ad legendum Platonem utilium*, I, xxiii, Hiller; Dupuis 1966, 62-65].

Iterative process and the finite

The idea of iteration entails the possibility of unlimited continuation of the process of addition by a unit. In Theon, we find the iterative operation of adding by a unit in a context vividly expressing the endless character of this process: "The unit can increase by addition up to the infinite".²³ In the same fashion, Nicomachus states that "unity is the beginning of all number that advances unit by unit in one direction".²⁴ Further Nicomachus states that "the multitude never ceases to advance"²⁵ and Iamblichus that "the multitude can increase to the infinite".²⁶ Proclus, in the second chapter of his first Prologue, where he considers the Limit and the Unlimited as common ontological principles of mathematics states that although number can increase *ad infinitum* (ἐπ' ἄπειρον, μέχρις ἀπέριου), any given number is finite: "number, beginning from the unit, can increase without limit, yet any number taken [individually] is finite".²⁷

Thus, the property of unlimited continuability of the process of increase by a unit is applied to number or to *multitude* (πλῆθος). However, any given number is taken always to be 'finite' or 'definite' whereas multitude is 'infinite' or 'indefinite' in its nature and is associated by the Pythagorean authors with the medium of the discrete (τὸ διηρημένον).

The doctrine of the Limit and the Unlimited, to which Proclus and other Pythagoreans often mention, is rather obscure, insofar as it admits also a cosmological interpretation and, thereby, involves a cosmological interpretation of number and the generation of numbers. Use of these principles in a sense relevant, in some respect, to mathematics should be sought for, in our view, in the conception of the Limit as delimiting principle against the background of the potentially infinite medium of the Unlimited, which we find in Iamblichus and Proclus. Iamblichus, for instance, imitates the Limit with a defining, confining principle against the background of the advancing infinite (πρόσσει μὲν ἐπὶ τὸ ἄπειρον ἀεὶ, ὁρίζεται δὲ ὑπὸ τοῦ περαίνοντος).²⁸ Moreover,

²³ *Ibid.*, II, xxxi, Hiller; Dupuis 1966, 136-137.

²⁴ Nicomachus, *Introductionis Arithmeticae*, II, vii, 3, Hoche; D'Ooge, 1926 833.

²⁵ *Ibid.*, I, ii, 5, Hoche.

²⁶ Iamblichus *In Nicomachi Arithmeticum introductionem* 7, 20, Pistelli.

²⁷ Proclus, *In primum Euclidis Elementorum librum commentarii*, 6.5, Friedlein; Morrow, 1992, 5.

²⁸ Iamblichus, *De communi mathematica scientia*, 13, 20, Festa.

Proclus associates in a decisive way the Unlimited with the generative power of the monad when he says, "If there were no infinity... numbers could not exhibit the generative power of the monad".²⁹ Nevertheless, one cannot expect that the principles of the Limited and the Unlimited were applied, in some formal sense, to various mathematical situations. They rather reflect an abstract philosophical understanding of the idea of the finitude, against the background of a (potentially) infinite medium.

The scope of arithmetic and the finite

Nicomachus, Iamblichus and Proclus seem also to agree that only finite objects can fall under the scope of science:

The understanding from which our ideas and demonstrations proceed does not use the infinite for the purpose of knowing it, for the infinite is altogether incomprehensible to knowledge; rather it takes it hypothetically and uses only the finite for demonstration; that is, it assumes the infinite not for the sake of the infinite, but for the sake of the finite.³⁰

Proclus explains that what he calls "infinite" in this context is "whatever [imagination] dismisses as something that cannot be gone through". The thesis of the recognition of the infinite as incomprehensible in its completion is usually justified by an epistemological argument, namely, that it is necessary for the object of science to be *intuitable* (ἐνορᾶσθαι) and *comprehensible* (περιληπτὸν) by the intellect (ἀναγκαίου δὲ ὄντος ἐπιστήμης φύσιν ἐνορᾶσθαι τοῖς οὖσιν).³¹ This viewpoint is outspokenly ascribed to the Pythagoreans by Proclus:

The Pythagoreans consider quantity and magnitude not in their generality,... but only as finite in each case. ... For they say that the sciences study the finite in abstraction from infinite quantities and magnitudes, since it is impossible to comprehend infinity in either of them.³²

²⁹ Proclus, *In primum Euclidis Elementorum librum commentarii*, 6, Friedlein; Morrow, 1992, 5.

³⁰ *Ibid.*, 285 24, Friedlein; Morrow, 1992, 222.

³¹ Iamblichus, *In Nicomachi Arithmeticum introductionem*, 7, 26, Pistelli; *De communi mathematica scientia*, 7, 31, Festa.

³² Proclus, *In primum Euclidis Elementorum librum commentarii*, 36, Friedlein; Morrow, 1992, 30.

In another passage, Proclus stresses that the subject matter of the Pythagorean science is focused on the study of the Limited: "[The Pythagorean sciences] exclude infinite from plurality and magnitude and concern themselves straightway with the Limited".³³ The same view is also shared by Nicomachus:

Since, however, all multitude and magnitude are by their own nature of necessity infinite ... and since sciences are always sciences of limited things, and never of infinities, it is accordingly evident that science dealing either with magnitude, *per se*, or with multitude, *per se*, could never be formulated, for each of them is limitless in itself, multitude in the direction of the more, and magnitude in the direction of the less.³⁴

Proceeding from such assumptions, the subject of the (Neo-)Pythagorean arithmetic should be understood as the study of finite "fragments" of the infinite: *quantity* (ποσόν), for instance, should be viewed as finite segment of the (potentially) infinite multitude. Accordingly, arithmetic as science of the *quantity*,³⁵ studies only finite (definite) numbers, yet multitude is potentially infinite (indefinite). In this sense, it can be stated that arithmetic studies the Limited in an Unlimited medium.

II- PRINCIPLES OF NEO-PYTHAGOREAN ARITHMETIC

The doctrine of the absolute quantity

The first basic classification (τομή) of numbers is into even and odd. Nicomachus gives three different descriptions of the even and the odd.³⁶ These definitions are not accompanied by any graphical illustration or procedure for their construction. However, their construction is obvious. The suite of even numbers, for instance, can be obtained from the natural suite by multiplication

³³ *Ibid.*, 37, Friedlein ; Morrow, 1992, 31.

³⁴ Nicomachus, *Introductionis Arithmeticae*, I, ii, 5, Hoche; D'Ooge, 1926, 812.

³⁵ Iamblichus, *In Nicomachi Arithmetikum introductionem*, 7, 26, Pistelli ; *De communis mathematica scientia*, 7, 31, Festa.

³⁶ Nicomachus, *Introductionis Arithmeticae*, I, vii, 2, Hoche; D'Ooge 1926, 814. Cf. Theon, *Expositio rerum mathematicarum ad legendum Platonem utilium*, I, v, Hiller; Dupuis 1966, 34-35. Nicomachus, *Introductionis Arithmeticae*, I, vii, 3, Hoche; D'Ooge 1926, 814. Cf. Iamblichus, *In Nicomachi Arithmetikum introductionem*, 12, 2-9, Pistelli. Nicomachus, *Introductionis Arithmeticae*, I, vii, 4, Hoche ; D'Ooge 1926, 814.

of each term by 2. The construction of odd numbers is made by Nicomachus when he proceeds to the construction of the even-times odd numbers.

The theory of the odd and even is exposed in considerable detail in the first Book of Nicomachus' *Introduction to Arithmetic*. Apart from the even and the odd number, Nicomachus introduces various kinds of numbers, which are illustrated by means of the corresponding pattern of the natural suite and accompanied by the description of an effective procedure for their construction. The kinds of numbers are defined by means of two different basic concepts: 'dividing' (διαμεθεῖναι) or 'partitioning' (μερισθῆναι), and 'measuring' (μετρεῖν). Nicomachus uses the operation of dividing for the classification of the genus of the even and the concept of measuring for the classification of the odd numbers.

Classification of the genus of the even

The even number is further subdivided into three disjoint classes of numbers: the "even-times even", the "even times-odd", and the "odd times-even numbers". The basic concept used for classification of the even number is the operation of 'dividing' (διχασθῆναι, δίχα διαμετεῖν).

Even-times even numbers. According to Nicomachus, an even-times even number is a number that admits successive divisions into two equal parts that terminate in the unit.³⁷ Theon gives an analogous definition, but it is based on the inverse concept of multiplication. Thus even-times even numbers are those generated by the multiplication of two even numbers, have all their parts even, and have no part homonymous to odd number.³⁸

It is noteworthy that the definition of even-times even number in the Neo-Pythagorean tradition differs significantly from that of Euclid. The point comes clear in Proposition IX. 34 of the *Elements*, where it is proved that a

³⁷ Nicomachus, *Introductionis Arithmeticae*, I, viii, 4, Hoche ; D'Ooge 1926, 814.

³⁸ Multiplication itself is described by Theon as 'circuit' formed by numbers (περιοχὴν δὲ καλοῦσιν ἀριθμῶν τὸν δι' ἀλλήλων αὐτῶν πολυπλασιασμόν) [Theon, *Expositio rerum mathematicarum ad legendum Platonem utilium*, I, vii, Hiller ; Dupuis 1966, 40-41]. It should be noted that this definition of 'multiplication' is not, strictly speaking, a closed operation, since when it is performed on one-dimensional objects, gives a two-dimensional object.

certain kind of number is both even-times even and even-times odd. Euclid's definition is criticised by Asclepius³⁹ and Philoponus.⁴⁰

Nicomachus gives an effective procedure for the construction of all even-times even numbers: beginning from the unit we proceed by the double ratio to infinity. The pattern of this construction is exemplified by the following natural suite:

1, 2, 4, 8, 16, 32, 64, 128, 256, 512, ...

Instead of the sign of ellipsis, Nicomachus writes in words: καὶ ἐπ' ὅσονοῦν ("and so forth").

Moreover, Nicomachus declares that the numbers constructed in this way are the only even-times even numbers that can be constructed, "so that none will escape, but all successively fall under it".⁴¹

We can distinguish two aspects in the above definition:

- A statement that defines the mode of construction of the even-times even numbers;
- A statement asserting that by this method we can potentially construct all the successive even-times even numbers.

The pattern of the natural suite cannot be considered by itself as an explicit definition of the concept of even-times even numbers for the following reason. Since the set of even-times even numbers is infinite, such a definition through a finite segment of it would be incomplete, if not supplemented by a rule for calculating the successive even-times even numbers. This rule is always explicitly stated in words.

Even-times odd numbers. The second class of even numbers is the even-times odd numbers that is defined as "a number of which, though it admits of the division into two equal halves, after the fashion of the genus common to it and the even-times even, the halves are not immediately divisible into two equal parts".⁴² Nicomachus gives here also an effective procedure for the construction of all even-times odd numbers. This procedure is realised in two stages:

³⁹ Asclepius of Thralles, *Commentary to Nicomachus' 'Introduction to Arithmetic'*, I, ξα, Tarán.

⁴⁰ Ioannes Philoponus, *Commentary on Nicomachus' 'Introduction to Arithmetic'*, I, ξη, Hoche.

⁴¹ Nicomachus, *Introductionis Arithmeticae*, I, viii, 8-9, Hoche; D'Ooge 1926, 815.

⁴² *Ibid.*, I, ix, 1, Hoche; D'Ooge 1926, 815-16.

- The natural suite of odd numbers is constructed, i.e. beginning from the unit and proceeding with a difference of 2 to infinity;
- The suite obtained is transformed into a new suite by multiplication of each term of the suite of odd numbers by 2.

The resulting construction is exemplified by the following natural suite:

6, 10, 14, 18, 22, 26, 30, ...

The sign of ellipsis above is expressed by Nicomachus in words in the following way: "proceed as far as you wish" (καὶ μέχρι ὅν προχωρεῖν θέλῃς).

Odd-times even numbers. The last class of even numbers is the odd-times even number, which is "An even number which can be divided into two equal parts, whose parts also can be divided, and sometimes even the parts of its parts, but it cannot carry the division of its parts as far as unity".⁴³ The procedure for their construction given by Nicomachus is more sophisticated than the previous constructions. It is realised in three stages:

First, the natural suite the odd numbers from 3 onwards is constructed:

3, 5, 7, 9, 11, 13, 15, ...

Next, the natural suite of the even-times even numbers, beginning from 4 is constructed:

4, 8, 16, 32, 64, 128, 256, ...

The result is obtained by a kind of composition of the two natural suites by the following rule: each number of the first suite is multiplied by all the numbers of the second suite. The outcome is presented by Nicomachus in the form of table.

Odd numbers	3	5	7	9	11	13	15
Even-times even	4	8	16	32	64	128	256
Odd-times even numbers	12	24	48	96	192	384	768
	20	40	80	160	320	640	1280
	25	56	112	224	448	896	1792
	36	72	144	288	576	1152	2304
	44	88	176	352	704	1408	2816

Table 1. The construction of the odd-times even numbers

⁴³ *Ibid.*, I, x, 2, Hoche; D'Ooge 1926, 816. Theon again gives a definition in terms of multiplication, that is the odd-times even number is defined as the number generated by multiplication of odd and even numbers.

Nicomachus stresses again here that this procedure uniquely determines the construction of the odd-times even numbers: "however far you go you will get nothing but the odd-times even numbers";⁴⁴

Classification of the genus of the odd

Now the genus of the odd number is subdivided into "prime and incomposite"? "secondary and composite" and "that which is secondary and composite in itself, but prime and incomposite in relation to another"; This classification is no longer made by means of the concept of 'dividing', but that of 'measuring' which is also undefined in Nicomachus.⁴⁵ Moreover, he uses two auxiliary notions: the "paronymous part" (παρώμιον μέρους) and the "heteronymous part" (ἑτερώμιον μέρους) of a number. A p -th part of unity is called the *paronymous part* of the number p . For instance, the number three has the third (of unity) as its paronymous part. A q -th part of unity is called *heteronymous part* of number p if q is different from p and q measures p . For instance, the number 9 has only one heteronymous part, the third part of unity; the number 15 has two heteronymous parts, the third and the fifth parts of unity.

Prime and composite numbers. The *prime and incomposite number* is defined by Nicomachus as an odd number having no other parts, except the paronymous to itself. According to this definition, these are odd numbers, which can be measured only by the unit.⁴⁶ The *secondary and composite number* is an odd number having in addition one or more heteronymous parts, apart from the paronymous one. Finally, the number which is *secondary and composite in itself, but prime and incomposite in relation to another number* is that which is measured, besides unity, by some other number as well. There-

⁴⁴ *Ibid.*, I, x, 8, Hoche ; D'Ooge 1926, 817.

⁴⁵ Another classification is suggested by Theon, in which prime and composite numbers appear as subclasses of number, and not of the odd number, as in Nicomachus. The principle of classification remains the notion of measure. However, this classification enables Theon to derive in a nice way the definitions of linear, plane and solid numbers: linear numbers are taken to be the prime numbers; plane numbers and solid numbers are taken as subclasses of composite numbers. Plane numbers are those contained by two numbers and solid those contained by three numbers [Theon, *Expositio rerum mathematicarum ad legendum Platonem utilium*, I, vi, Hiller; Dupuis 1966, 36-39].

⁴⁶ Note that, according to this definition 2 is not prime number, in contrast to the Euclidean concept of prime numbers [*Elements* VII, Def. 11]. Iamblichus follows Nicomachus in his criticism of Euclid.

fore, this number can have heteronymous part or parts as well as paronymous part besides unity.

This classification does not establish disjoint classes. The second concept includes the third [Nesselmann 1842, 194]. However, the example given by Nicomachus below suggest that the second concept is conceived as characterising a number, whereas the third seems to apply rather to a pair of numbers.

Nicomachus gives a unified method of construction of all the aforementioned kinds of odd numbers by means of the *sieve of Eratosthenes* (Table 2). The natural suite of odd numbers is set forth, beginning with 3, *i.e.*

3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, ...

Then, starting with the first we observe that the terms it can measure are by two places apart as far as we proceed, according to the following rule: it measures the first such number occurred, *i.e.* 9, by the quantity that stands first in the suite, *i.e.* 3 times; it measures the next such number occurred, *i.e.* 15, by the quantity of the second in order, *i.e.* 5 times; and again the next such number, *i.e.* 21 by the quantity of the third in order, *i.e.* 7 times, "and so on *ad infinitum* in the same way".

Now, we come to the second number 5 and observe that it can measure all the terms by four places apart, the first, *i.e.* 15 by a quantity of the first in order, *i.e.* by 3; the second, *i.e.* 25 by the quantity of the second in order, *i.e.* by 5; the third, *i.e.* 35 by that of the third, *i.e.* by 7; and so on in the same way.

Again, the third term 7 measures all terms six places apart, and the first by the first quantity in order, the second, by the second quantity, the third, by the third quantity. This process can be continued without interruption, so that

The numbers will succeed to the measuring function in accordance with their fixed position in the suite; the interval separating terms measured is determined by the orderly progress of the even numbers from 2 to infinity, or by the doubling of the position in the suite occupied by the measuring term, and the number of times a term is measured is fixed by the orderly advance of the odd numbers in the suite from 3.⁴⁷

⁴⁷ Nicomachus, *Introductionis Arithmeticae*, I, xiii, 6, Hoche ; D'Ooge 1926, 819.

3	5	7	9	1	1	1	1	1	2	2	2	2	2	3	3	3	3	3	3
				1	3	5	7	9	1	3	5	7	9	1	3	5	7	9	
3			3			5			7			9			1			1	
	5					3					5				7			3	
		7							3							5			
			9								3								

4	4	4	4	4	5	5	5	5	5	6	6	6	6	6	7	7	7	7	7
1	3	5	7	9	1	3	5	7	9	1	3	5	7	9	1	3	5	7	
		1			1					2				2			2		
		5			7					1				3			5		
		9				1					1						1		
					1						3						5		
				7						9								1	
		5								7								1	

Table 2. The sieve of Eratosthenes

Now the numbers that are measured by no number are prime and incomposite, e.g. 11, 13, 17, etc. Those measured by only one number in accordance with their own quantity will have one heteronymous part in addition to the paronymous one, e.g. 9 has one third as heteronymous part in addition to the paronymous part one ninth. Those measured by only one number, but in accordance with the quantity of some other number, different from their own, or measured by two numbers, will have several heteronymous parts in addition to the paronymous one, e.g. 15 that has two heteronymous parts, one third and one fifth, in addition to its paronymous one fifteenth. These will be secondary and composite. Finally, numbers which are in itself secondary and composite but primary and incomposite in relation to another are the numbers that are measured by some prime and incomposite number in accordance with its own quantity, if one thus constructed be compared to another of similar construction. For example, the numbers 9 and 25 that are constructed from 3 and 5, respectively, measuring their own quantity, when compared between each other have no common measure except the unit.

The theory of perfect numbers

Numbers are classified by Nicomachus into "over-perfect", "perfect" and "deficient". *Over-perfect number* (ὑπερτελής) is one, which is greater than the sum of its parts; *deficient number* (ἐλλιπής) is that which is less than the sum of its parts; *perfect number* (τέλειος) is that which is equal to the sum of its parts.

Nicomachus gives also an effective "neat and secure" procedure for the construction of perfect numbers, which "neither passes by any of the perfect numbers, nor fails to differentiate any of those that are not such".⁴⁸ First, he constructs the natural suite of the even-times even numbers

1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, ...

Then he adds them together one at a time and each time he considers their sum. If it is prime and incomposite number, then he multiplies it by the last number added, and the result will be perfect number. If the sum is secondary and composite, instead of multiplying, he adds the next number and considers the sum; if it is secondary and composite he adds the next; but if it is prime and incomposite, he multiplies it by the last term added and the result will be perfect number; and so on *ad infinitum*.

The doctrine of the relative quantity

Equality and Inequality. *Equality* (ισότης) and *inequality* (ἀνισότης) is introduced by Nicomachus as the highest generic divisions of the relative quantity. "Any thing when compared with another thing can be either equal or unequal, *tertium non datur*".⁴⁹ These concepts are considered to be of the same level of abstraction as unity and the dyad for the absolute quantity.⁵⁰

Inequality is further divided into two classes, the greater and the less. The class of the greater includes multiples, superparticulars, superpartients, multiple superparticulars, and multiple superpartients. The class of the less embraces the reciprocal ratios of the class of the greater. They are denoted as submultiples, subsuperparticulars, subsuperpartients, submultiple-subsuperpar-

⁴⁸ *Ibid.*, I, xvi, 4, Hoche; D'Ooge 1926, 821.

⁴⁹ *Ibid.*, I, xvii, 2-3, Hoche. About this reading see Sec. 0. D'Ooge translates: "everything viewed in comparison with another thing is either equal or unequal, and there is no third thing besides these" [D'Ooge 1926, 822].

⁵⁰ Nicomachus *Introductionis Arithmeticae* II, i, 1-2, Hoche; D'Ooge 1926, 829.

ticulars, and submultiple-subsuperpartients. In the end of Book I, Nicomachus states a general procedure for the construction of all forms of inequality of ratio from a suite of three given equal terms, and conversely that equality may be obtained from any given suite exemplifying a form of inequality. These procedures are also described by Theon.

Equality is considered as fundamental, very first (ἀρχικωτάτη) and indefinable concept, inasmuch as there is no specific difference for it to be defined (διαφορὰν γὰρ οὐδεμίαν ἐπιδέχεται).⁵¹

It is considered that equality holds when one of the compared things neither exceeds (ὑπερέχει), nor falls short (ἐλλείπει) in 'relation' (παραβολή) to the other, ... either in bulk, length, weight or any kind of quantity.⁵²

Iamblichus in analogous manner defines equality as "quantity, which in comparison with the corresponding [quantity] has nothing greater or smaller".⁵³ Nicomachus gives the following examples: one hundred is equal to one hundred, ten is equal to ten, two is equal to two, a mina to a mina, etc. Thus by equality Nicomachus means primarily numerical identity.

However, number in Nicomachus has structure, it is a 'suite' or a 'configuration'. It is noticeable that his terminology – 'exceeds' (ὑπερέχει), 'falls short' (ἐλλείπει), 'relation' (παραβολή) – is not completely free from geometric connotations. A comparison of numerical values can be expressed more appropriately by the terms 'greater' (μείζων) and 'less' (ἐλλάσσων), instead of 'exceeds' (ὑπερέχει) and 'falls short' (ἐλλείπει). Thus, Nicomachus seems to allow an interpretation of equality as *graphical equality*, that is the possibility of coincidence of the patterns serving as their designata. According to an explanation by Iamblichus,⁵⁴ the process of comparison should be considered as "one by one" (τὸ ἓκαστον ἐν πρὸς ἐν ἑστί).⁵⁵ This description suggests that in the case when the corresponding configurations do not coincide, then one of them will coincide with an initial segment of the other.

⁵¹ Cf. Iamblichus *In Nicomachi Arithmeticum introductionem* 43.23, Pistelli.

⁵² Nicomachus *Introductionis Arithmeticae* I, xvii, 2-3, Hoche.

⁵³ Iamblichus *In Nicomachi Arithmeticum introductionem* 36, 16-18, Pistelli.

⁵⁴ Iamblichus in analogous manner defines equality as "quantity which in comparison with the corresponding [quantity] has nothing greater or smaller" [Iamblichus *In Nicomachi Arithmeticum introductionem* 36, 16-18, Pistelli].

⁵⁵ Iamblichus *In Nicomachi Arithmeticum introductionem* 43.24, Pistelli.

In case equality is understood as involving a geometric pattern, the number ('suite') 3, for instance, cannot be equal to the triangular number 3, for these numbers are numerically identical, but differ in configuration, that is they are not graphically equal. Hence, the fundamental trichotomy of numbers into linear, plane, and solid (see below) demarcate three types of objects in Neo-Pythagorean arithmetic, according to the dimension of their graphical representation.

Accordingly, equality should be possibly assumed as applying only to objects of the same type. Nevertheless, this is not enough either. The plane number 12, for instance, is an even number, but also an oblong number (see p. 133), as well as a pentagonal number (see p. 133). Consequently, equality, as graphical equality, can actually apply only to numbers of the same kind, that is to numbers of similar combinatorial complexity. Nicomachus and the other Neo-Pythagorean authors are not explicit on that point. Truly, they never state the contrary proposition: that, for instance, the oblong number 12 is equal to the pentagonal number 12. We find, however, propositions, where Nicomachus talks about polygonal numbers as 'system of numbers of the same rank' (σύστημα ὁμοταγῶν)⁵⁶ numbers.

This vagueness in the definition of equality can be eliminated by the adoption of the following assumption: any process of counting of the numbers of a collection is a process of transformation of the initial pattern into a pattern of a different type (presumably linear) with the same numbers. However, one does not find in the texts of the Neo-Pythagoreans any stipulation of similar kind.

The theory of figured numbers

Turning back to the inquiry of properties of absolute numbers, Nicomachus introduces the concepts of "plane", "surface", and "solid" thing, based on the notion of dimension (διάστημα). Accordingly, by *linear*, *plane*, and *solid numbers* are meant those having one-, two-, or three-dimensional geometrical configuration, respectively.

The generation of triangular numbers. Among plane numbers, the triangular are considered by Nicomachus the most primary and elementary. A *triangular number* is defined as "one which, when analysed into units, shapes into

⁵⁶ Nicomachus *Introductionis Arithmeticae* II, xii, 5-8, Hoche; D'Ooge 1926, 836.

triangular form the equilateral placement of its parts in a plane".⁵⁷ These numbers are constructed according to the following pattern:

3, 6, 10, 15, 21, 28, ...

Nicomachus describes the following procedure for the construction of triangular numbers. Beginning from the natural suite:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15

We take the first term and have the potentially first triangular number, which is 1) Then adding the next term, *i.e.* 2) we get the actually first triangular number 3) its configuration is constructed by setting two units, side by side, beneath one unit. Then by addition of the next term of the natural suite, *i.e.* 3, arranged into units, and joined to the former configuration, it gives the second triangular number 6. In the same way, the next term, *i.e.* 4, arranged into units, and joined to the former configuration, it gives the third triangular number 10. We proceed in the same way by adding all the successive terms of the natural suite, arranged into units, and joining them to the preceding configuration.⁵⁸



$$1, 3=1+2, 6=1+2+3, 10=1+2+3+4, \dots, \frac{n(n+1)}{2} = 1+2+\dots+n.$$

Theon explains that this procedure can go on *ad infinitum* (καὶ μέχρις ἀπείρου ὁ αὐτὸς λόγος).

The generation of square and oblong numbers. Theon describes two procedures for the construction of *square numbers*: by *episyntesis* (κατ' ἐπισύνθεσιν) and by *multiplication* (κατὰ πολλαπλασιασμόν).

The method by *episyntesis* runs as follows: Beginning from the suite of odd numbers we form the sum of the first two terms; then this sum is added to the third term; the procedure continues by adding each new sum to the next term. Thus beginning from the suite

1, 3, 5, 7, 9, 11, ...

We can form the successive terms of square numbers:

⁵⁷ Nicomachus *Introductionis Arithmeticae* II, viii, 1, Hoche; D'Ooge 1926, 833.

⁵⁸ Nicomachus *Introductionis Arithmeticae* II, viii, 3, Hoche; D'Ooge 1926, 833-34.

$$1+3=4, 4+5=9, 9+7=16, 16+9=25, \dots$$

Theon adds again that the same reasoning can be continued *ad infinitum* (καὶ μέχρις ἀπείρου ὁ αὐτὸς λόγος).

By multiplication, the square numbers are obtained as products of the same successive numbers, that is

$$2 \cdot 2=4, 3 \cdot 3=9, 4 \cdot 4=16, \dots$$

The successive configurations of square numbers are constructed below:

$$\begin{array}{c} \alpha \quad \alpha \quad \alpha \quad \alpha \\ \alpha \quad \alpha \quad \alpha \quad \alpha \\ \alpha \quad \alpha \quad \alpha \quad \alpha \\ \alpha \quad \alpha \quad \alpha \quad \alpha \end{array} \quad \begin{array}{c} \alpha \quad \alpha \quad \alpha \\ \alpha \quad \alpha \quad \alpha \\ \alpha \quad \alpha \quad \alpha \\ \alpha \quad \alpha \quad \alpha \end{array} \quad \begin{array}{c} \alpha \quad \alpha \quad \alpha \quad \alpha \\ \alpha \quad \alpha \quad \alpha \quad \alpha \\ \alpha \quad \alpha \quad \alpha \quad \alpha \\ \alpha \quad \alpha \quad \alpha \quad \alpha \end{array}$$

$$1, 2^2=1+3, 3^2=1+3+5, 4^2=1+3+5+7, \dots, n^2 = 1+3+5+\dots+(2n-1)$$

It should be noted that the methods of *episyntesis* and multiplication are conceived by Theon as schemes of reasoning that may apply to different things. The later scheme applied to the suite of odd numbers as described above generates square numbers, whereas applied to the suite of even numbers generates the suite of *oblong* (ἐτερομήκεις) numbers.⁵⁹ Indeed, let

$$2, 4, 6, 8, 10, 12, 14, 16, 18, \dots$$

The suite of even numbers; then the successive oblong numbers are constructed in the following way

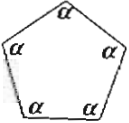
$$2+4=6, 6+6=12, 12+8=20, 20+10=30, \dots$$

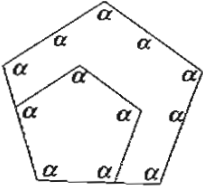
The generation of polygonal numbers. In analogous way, *pentagonal*, *hexagonal*, *heptagonal*, *octagonal*, *enneagonal* and *decagonal numbers* are constructed. As it is noted by Nicomachus "the doctrine of these numbers is to the highest degree in accord with their geometrical representation, and not out of harmony with it",⁶⁰ that is the arithmetical theory develops in accordance to its combinatorial model.

Like the triangular and the square numbers, which are constructed from the terms of the natural suite that differ by 1 and 2 respectively, pentagonal numbers are constructed from terms that differ by 3.

⁵⁹ Theon *Expositio rerum mathematicarum ad legendum Platonem utilium*, I, xix, Hiller; Dupuis 1966, 52-57.

⁶⁰ Nicomachus *Introductionis Arithmeticae* II, xii, 1; D'Ooge, 1926, 835.



$$1, \quad 5=1+4=\frac{2 \cdot (3 \cdot 2 - 1)}{2},$$


$$12=1+4+7=\frac{3 \cdot (3 \cdot 3 - 1)}{2}, \dots, \quad \frac{n(3n-1)}{2} = 1 + 4 + 7 + \dots + (3n-2).$$

Likewise, hexagonal numbers are constructed from the terms of the natural suite, which differ by 4, heptagonal from terms which differ by 5, octagonal by 6, enneagonal by 7, decagonal by 8. In general, the k -gonal number (ὁσάγωνος)⁶¹ is constructed according to the following rule:

The root-number [gnomons or gnomonic numbers] of any polygonal differ by 2 less than the number of the angles shown by the name of the polygonal – that is, by 1 in the triangle, 2 in the square, 3 in the pentagon, 4 in the hexagon, 5 in the heptagon, and so on, with similar increase.⁶²

Consequently, a k -gonal number of n sides is $n + \frac{(k-2)n(n-1)}{2}$ and the successive gnomons are:

$$1, 1+(k-2), 1+2(k-2), \dots, 1+(r-1)(k-2), \dots$$

III- THE NEO-PYTHAGOREAN ARITHMETICAL REASONING

The genetic approach

On the grounds of the above analysis, we can conclude that the Neo-Pythagorean approach to arithmetic is predominantly non-axiomatic. This is clear

⁶¹ Theon *Expositio rerum mathematicarum ad legendum Platonem utilium*, I, xxvii, Hiller; Dupuis 1966, 68-69.

⁶² Nicomachus *Introductionis Arithmeticae* II, xi, 4; D'Ooge, 1926, 835. A similar rule is also described by Theon [*Expositio rerum mathematicarum ad legendum Platonem utilium*, I, xxvii, Hiller; Dupuis 1966, 68-69] and Iamblichus [*In Nicomachi Arithmeticon introductionem* 62.10-18, Pistelli]. Diophantus ascribes this rule to Hypsicles (about 170 B.C.) [Diophantus I, 470-71, Tannery; Heath 1964, 252].

- From the way that the objects of the theory, that is numbers of various kinds are introduced;

- From the logical technique used for the development of arithmetic.

According to the axiomatic approach, as described, for instance, by Aristotle, one should proceed from some initial concepts, which are evident and a set of propositions (axioms), describing relations between these objects, which are true, immediate, more familiar, prior and causative of the theorems.⁶³

However, in the Neo-Pythagorean arithmetic we meet another approach. The point of departure of the theory is not some initial concepts and statements, but a given object – the monad, designated by a letter – and a generative operation (more precisely a set of generative operations) that serve as rules for the construction of new objects (suites or configurations) out of given ones.⁶⁴

The « alphabet » of Neo-Pythagorean arithmetic

From semiotic point of view, the 'alphabet' of Neo-Pythagorean arithmetic consists of solely one object, i.e. the monad, denoted by an alpha, and taken to be a designated object, that is

$$L = \{\alpha\}.$$

Over this set an iterative procedure of attaching an alpha is admitted. Numbers can be defined then as suites of the form

$$k \approx \langle \alpha, \alpha, \alpha, \dots, \alpha \rangle$$

where $\approx$ means that k is an abbreviation for the suite

$$\alpha, \alpha, \alpha, \dots, \alpha$$

consisting of k signs.

It is important to stress the difference between this concept of number and the modern conceptualisation. In modern number theory the arithmetic concepts are relative to some set theory and thereby depend on set-theoretic axioms. Moreover, the reduction of arithmetic to set theory, as is implemented, for instance, by Frege and Dedekind, involves quantification over infinite sets, since the set of natural numbers is defined as the intersection of all infinite sets, each of which contains the empty set and the successor of each member of

⁶³ Aristotle *Posterior Analytics* I, ii, 71b.

⁶⁴ A general characterisation of the genetic approach to the construction of a scientific theory can be found in [Smirnov 1962].

it. In the Neo-Pythagorean approach, on the contrary, the arithmetical concepts do not rely on any concept of set, but on the notion of the "suite", and arithmetic depends on constructional genetic possibilities rather than on axioms of set-theoretic character. Moreover, the "suite" is only *potentially infinite* object, whereas infinite sets, in the modern sense, are taken to be *actually infinite*.

Further, the natural suite can be introduced as a sequence of the form

$$\langle 1, 2, 3, \dots, k \rangle.$$

and the various kinds of numbers can then be specified as suites constructed according to certain rules.

From this point of view, Neo-Pythagorean arithmetic is an informal theory of counting having a distinctive combinatorial character, insofar as it concerns constructions of various finite schematic patterns, that is suites and configurations. In this context, arithmetical reasoning is conducted as theoretical informal reasoning concerning the possibility to carry out certain genetic constructions over a domain of concrete objects. Such type of arithmetical reasoning admits the representation by (configurations of) letters as intended interpretation for the ultimate confirmation of the arithmetical statements and develops in accordance with its combinatorial model. However, it should be noted that very little attention is paid by the Neo-Pythagorean authors to the practice of representation by letters itself, that is to the 'experimental' part of the theory. In their works, arithmetical reasoning appears to be predominantly contemplative activity [Vitrac 1994, II, 483].

As we have already noted, the objects of the Neo-Pythagorean arithmetic are divided into three levels depending on the dimension of their representation. Linear numbers are generated from the monad; plane numbers are generated from the linear ones; solid numbers are generated from the appropriate combination of plane and linear numbers. Such a stratification of objects into linear, plane and solid numbers makes necessary the use of different generative operations at each level. At the level of linear numbers, for instance, the basic generative operation is that of the addition by a unit, whereas at the level of plane numbers the use of gnomon plays the role of basic generative operation. Moreover, equality can apply only to objects of the same level and similar combinatorial complexity.

Genetic definition of arithmetic concepts

The definitions of all kinds of numbers considered in Neo-Pythagorean arithmetic are not given by our common set-theoretic *abstraction*, that is by a property, which defines a class of numbers possessing that property. Instead, all the arithmetic predicates are accompanied by an effective procedure for the genetic construction of the considered set of numbers. Whence, the focus is not on the *existence* of certain numbers, possessing certain property, but on the *mode of the genetic construction* of the numbers. In this sense, the specifications of numbers in Neo-Pythagorean arithmetic can be considered as *genetic definitions*. The genetic rule is illustrated in a number of initial cases and the ancient author closes his reasoning by the conclusion that the construction according to this rule can go on *ad infinitum*. In virtue of the constructive nature of the objects to which this kind of reasoning is applied, these definitions bear an inductive character of the type discussed in the next section.

Yet, in Neo-Pythagorean arithmetic the process of construction does not take place in *time*. In the modern constructive case, that is in intuitionistic arithmetic, the succession of stages of construction is succession in time, and incompleteness arises from the fact that the theory is a theory of an idealised *finite mind*, which is located at some point in time and has available only what it has constructed in the past and its intentional attitudes towards the future. Thus, there is no stage at which all constructions are complete. Such a semantic picture, however, is not found in the Neo-Pythagorean arithmetic, although it can be characterised as arithmetic of finite mind (see section on the scope of arithmetic and the finite, p. 121).

General features of genetic constructions

All genetic constructions of the Neo-Pythagorean arithmetic have the following features:

- (1) They begin from the same initial object, i.e. the monad;⁶⁵
- (2) Provided that the result of the application of certain iterative operations (addition by a unit, application of the gnomon or other derivative operations) generate numbers of the same kind, new numbers are constructed;

⁶⁵ There are constructions that begin not directly from the unit, but from a natural suite. However, in these cases the construction of that suite, beginning from the unit, is obvious or has been realised previously and is reasonably omitted.

(3) It is stated that the method of generation of a particular kind of number can generate all the numbers of the kind required.

These features can be taken as essential characteristics of Neo-Pythagorean arithmetical reasoning.

The first two clauses enable one to construct new numbers out of given ones. The third clause (sometimes omitted by the Neo-Pythagorean authors) says that the method of construction described by the first two clauses "exhausts" all the numbers that have to be constructed. Special attention should be paid on the formulation of the third clause. In modern genetic definitions, the analogous clause is usually phrased as follows:

(3*) There are no other objects except those generated by the application of the first two clauses.

Thus the clauses (1), (2), and (3*) are taken to define a set of objects that are specified by means of the clauses (1) and (2). The possibility for an object to possess the defining property is based on clause (3*). This clause implicitly assumes the universe of all objects as given beforehand. In virtue of the law of excluded middle, every object of the universe either possesses the defining property or does not. Then clause (3*) is taken to mean that there are no objects in the set determined by the defining property except those that have been constructed in the way described by (1) and (2).

However, this does not seem to be the way of thinking of the ancient authors, for whom the totality of all numbers was not considered as given beforehand. It could not be for one more reason: the statement "a number such that..." (in the sense of its effective construction) was for the Neo-Pythagorean arithmeticians an "experimental" fact, whereas the statement "there are no numbers such that..." is not an "experimental" fact. Accordingly, clause (3) is never phrased by the Neo-Pythagoreans in the form of negative existential sentence.

The objects introduced in Neo-Pythagorean arithmetic by genetic definitions are infinite sequences, usually incompletely exemplified by a natural suite or certain combinatorial configuration. Insofar as the objects discussed (suites and combinatorial configurations) are always finite instances and never considered in their completion, what is implicitly involved here is the *abstraction of potential infinity* that allows reasoning about however long genetic processes. The realizability of the genetic constructions is taken to be potential. The natural suites, for instance, are taken to be extendible "as far as

you wish", and the process of construction of successive combinatorial configurations can continue *ad infinitum*. Moreover, the objects introduced in Neo-Pythagorean arithmetic are not only (potentially) infinite, but also intrinsically associated with the rule for their genetic construction from the monad. In this sense, we can say that the Neo-Pythagorean approach to arithmetic is not purely extensional.

Generative operations

Insofar as natural numbers are constructed beginning from the monad by addition of a unit, the iterative operation of transition from a number to its successive plays a fundamental role in Neo-Pythagorean arithmetic. Definitions of arithmetical concepts are reduced to the demonstration how a definite combinatorial rule works, when one passes from a number to its successor, in the process of construction of the considered kind of number. Arithmetic theorems are ultimately reduced to the demonstration that the transition from a number to its successor follows a definite combinatorial rule.

Starting from the fundamental iterative operation of "addition by a unit" the elementary operations of arithmetic – addition, subtraction, and multiplication – admit an evident interpretation by means of the genetic technique. The addition of a number by another number can be carried out by means of the successive attachment of the units contained in the second number to those of the first. The subtraction of a number b from another number a , where b is less than a , can be carried out by means of the successive removal of the units in b until we obtain a . In a similar fashion, the multiplication of a number a by another number b can be performed by means of the attachment of a to itself as many times as there are units in b . Evidence about the later definition of multiplication provides the Definition 15 in Book VII of Euclid's *Elements*.

In addition to the above basic arithmetical operations, one can find in the works of Neo-Pythagorean authors some derivative generative operations, such as:

- Addition of parts of a number (see p.128);
 - Multiplication of the terms of a suite by a number (see p. 125);
 - Term by term multiplication of two natural suites (see p. 125);
 - The construction by episyntesis or by multiplication (see p. 133);
- as well as manipulations of constructive objects of the following kind
- Transformation of natural suites according to a rule (see p.125).

Genetic construction vs. proof

It is obvious that the method outlined above is not grounded upon the idea of proof (in the strict sense of the word). In the context of Neo-Pythagorean arithmetic, numbers are conceived as given and any statement about them asserts something, which is confirmed in each instance by simple combinatorial means. If, for example, two numbers are given, it is sufficient to *confirm* by of the construction of the corresponding configuration (or by observations over the exemplary suite), whether what has been stated about these numbers is correct or not. The demonstration is performed by inspection over a finite fragment of a usually (potential) infinite object. In this sense, we are going to say that the statements of Neo-Pythagorean arithmetic have *finitary* meaning.

Therefore, the foundation of Neo-Pythagorean arithmetic is not proof (ἀπόδειξις) in the style of Euclidean *Elements* and the works of other mathematicians of the classical antiquity, but the idea of effective genetic construction, by means of which the correctness of arithmetical statements is confirmed. This type of arithmetical reasoning about given numbers can be realized without assumptions of axiomatic character. The confirmation of arithmetical statements is realizable by a specific 'experiment'.

The rule of mathematical induction

The question whether mathematical induction is used in Greek mathematics has puzzled historians of mathematics. B.L. van der Waerden [1954] admits the use of some kind of mathematical induction in Greek mathematics, yet only "essentially". H. Freudenthal rather cautiously ascribes to Euclid the knowledge of only a "*quasi-allgemeiner* [our emphasis] Induktionsschluss" [Freudenthal 1953, 28]. On the other hand, J. Itard [1961] seems to ascribe unreservedly to Euclid the use of complete induction in Propositions VIII 2, 4, 13, 27, 36, IX 8, 9. A discussion of his claims is found in [Vitrac 1994, 467-72]. A survey of various forms of mathematical induction and their interpretations by the historians of mathematics is given by Rashed [1994, 62-84].

D.D. Mordukhaja-Boltovskoj [1948-50], the editor of the Russian translation of Euclid's *Elements*, excludes the use of mathematical induction in Greek mathematics, because the concept of the infinite and, particularly, the concept

of the (infinite) set of "all natural numbers" was alien to the Greek mathematical thinking.⁶⁶ The same view is supported by Unguru [1991, 278; 1994], who, in addition, stresses the difference between the ancient and the modern conceptualisation of number. Unguru proceeds from the modern conception of the principle of mathematical induction (see the induction scheme suggested by him in [Unguru 1991, 274]), which, however, is valid in the context of axiomatic theory of Peano arithmetic and presupposes the modern concept of set. Obviously, use of mathematical induction in this sense cannot be found in Greek arithmetic. As Rashed has put it:

If we confine ourselves to a rigorous formulation – which is essential – attempts that do not state the argument of induction – $P(n) \Rightarrow P(n+1)$ – for any n in an explicit way will be rejected as outside mathematical induction. But as this rigour is related to a complete system of axioms – known as Peano's system – which includes precisely the exact formulation of the principle of mathematical induction, all earlier formulations are necessarily naive [Rashed 1994, 77-78].

However, Unguru seems to maintain a stronger thesis, namely that use of mathematical induction necessarily needs the general abstract concept of natural number (understood as independent variable), which is not true.

On the other hand, Mueller [1981] admits a pre-structural form of mathematical induction. Fowler [1994] has suggested an actually finitary form of mathematical induction (called by him "relaxed form of mathematical induction"), resting on a liberate passage from one case to its next in succession, and has argued that such a scheme could have been used by the Greeks. Finally, Acerbi [2000] has suggested regarding the Platonic passage *Parmenides* 149 a 7-c 3, as a full-fledged example of proof by complete induction.

A property of polygonal numbers

An interesting property of polygonal numbers is found in Theon:

From the multiples of the unit, that is, the doubles, the triples and the subsequent, all numbers which successively leave out one are square, all those which successively leave out two are cubes, and those which successively leave out

⁶⁶ Concerning Mordukhaja-Boltovskoj's viewpoint, see [Medvedev 1990].

five are cubes and squares at once; and they have sides square numbers, being cubes, and being square have cubic sides.⁶⁷

This statement is especially interesting, because it is the Neo-Pythagorean analogue of the Proposition IX, 8 of Euclid's *Elements*. Both propositions, in Theon and in Euclid, are proved by a mode of reasoning that we have called "finitary form of mathematical induction".⁶⁸ The proof of Theon runs as follows:

That from the multiple numbers those which beginning from the unit leave out one are square, those leave out two cubes, and those leave out five cubes and squares at once is clear by the following: set forth the suite of double numbers, that is

1, 2, 4, 8, 16, 32, 64, 128, 256.

In this suite the first double is 2; it follows 4 which is square; then follows 8 which is cube; then 16 which is square; then 32; after it 64 which is square and cube; then 128; then 256 which is square and the same reasoning *ad infinitum*.

Set forth the [suite of] triple numbers, that is

1, 3, 9, 27, 81, 243, 729, ...

In this suite the [numbers] leave out one are square, and in the [suite of] five-fold numbers and the subsequent [suites of] multiples. In like manner, in the [suites of] multiples it will be found that all those leave out two are cubes, those leave out five are cubes and square at once.

It should be noted that Theon does not give a unified method of proof for all the multiples of the unit. He does not refer to the Euclidean concepts of "ratio" and "continuous proportion". Instead, he constructs the corresponding suites: the suite of doubles, the suite of triples, and so forth, and confirms the required property for each suite separately. On these grounds, he concludes that the required property holds for any suite of multiples. Thus, induction is used:

- On the sequence of the suites of multiples;
- To establish that the required property holds in each suite.

⁶⁷ Theon *Expositio rerum mathematicarum ad legendum Platonem utilium*, I, xx, Hiller, Dupuis 1966, 56-59.

⁶⁸ For a discussion of the relevant Euclidean proposition, see [Vandoulakis 1998, 172-177].

The rule of mathematical induction

Let us elucidate how mathematical induction could be understood in the context of Neo-Pythagorean arithmetic. Let statement P states something about numbers. As we have seen the Neo-Pythagorean arithmeticians, beginning from the unit confirmed that the statement P holds for the first case. Assuming that it holds for the first case, the next number is constructed and this statement is confirmed for the second case. Again, provided that it holds for the second case, they confirmed it for the next case by the construction of the next number. Finally, it is concluded in one or another way that this process can be continued *ad infinitum*.

What we actually encounter here is not the application of an abstract logical law, but a type of reasoning that establishes a conclusion as drawn from the fact that the corresponding constructions in the successive cases follow certain regularity. Therefore, the establishment of a general property of numbers is reduced (after its confirmation that the monad possesses the considered property) to the demonstration that the considered property is "inheritable" when passing from a number to its successor. If every number transmits a certain property to the next number, irrespective of any particular quality of the numbers themselves but in virtue of the rule of their construction, then all numbers will have the property. This mode of reasoning we are going to call *finitary form of mathematical induction* or *rule of mathematical induction*.

Informally speaking, such a rule says that if some initial objects have some property and this property is "hereditary", because of a certain rule of construction, then any object has this property. The property of numbers is established by a procedure of genetic construction which constructs $P(1)$, $P(2)$, ... in a uniform manner that does not depend on the actual stage of the procedure.

In modern terms this type of reasoning can be presented by the following scheme of inference:

For the first case: $P(1)$;
 For the second case: $\frac{P(1) \& P(1) \rightarrow P(2)}{P(2)}$;
 For the third case: $\frac{P(2) \& P(2) \rightarrow P(3)}{P(3)}$;

and so on *ad infinitum*.

It is clear that the inferential scheme that is repeated in all these cases (δ αὐτὸς λόγος) is the following :

$$\frac{P(t) \& P(t) \rightarrow P(St)}{P(St)},$$

where t stands for an arbitrary (given) number (suite) and St its successor. This inference scheme states that on the assumption that the statement P is confirmed for any given number t it can be confirmed for the next number St as well. Consequently, the statement P can be understood as asserting something of *any* (πᾶντες) given number.

In comparison to the modern principle of mathematical induction in the axiomatic system of Peano arithmetic, we would like to stress the following differences:

First, the modern form of mathematical induction is understood in the context of the axiomatic system of Peano arithmetic, that presupposes the concept of set (in particular, the set of natural numbers). On the contrary, the Neo-Pythagorean form neither is stated in the context of any axiomatic system of arithmetic, nor the totality of natural numbers is presupposed as given beforehand. In the modern case,

in each system of set theory which contains number theory, the principle of induction becomes a set-theoretic principle derivable from the axioms of the system; and whether induction is applicable to a certain sentence of the system depends both on the strength of the axioms of the system and the definitions for the arithmetic notions [Wang 1970, 466].

Second, in contrast to the modern form of mathematical induction the Neo-Pythagorean form does not use variables of any kind (number is not a variable in Neo-Pythagorean arithmetic, but is always given). Accordingly, there is no use of any quantifiers ranging over the (actual) infinite set of "all" natural numbers. The quantificational words used, for example, by Theon, do not actually act as quantifiers, but simply express the fact that a statement P asserts something of *any* given number. Consequently the Neo-Pythagorean form of mathematical induction is, essentially, a quantifier-free form of induction.

Third, in the Neo-Pythagorean form of mathematical induction a statement P about numbers is never asserted of the complete totality of natural numbers.

Generality

In the expositions of Neo-Pythagorean arithmetic, one does not find enunciation of universal theorems, that is formal statements beginning with quantificational words of the type "all", "every", etc. The problem primarily concerns the case when a general property is asserted of an infinite (in the sense described above) domain of objects. In these cases the general property is established by induction, that is by means of certain constructions of such a character that the possibility to repeat a similar reasoning so that to implement the corresponding construction for *any* other given number of the same kind is evident. On that ground one can conclude that whatever number of the kind might have been given, it is possible to confirm (by analogous line of reasoning) that this number has the property in question.

A generality inferential scheme that might have been used in arithmetic is described in *Posterior Analytics* by Aristotle.

An attribute belongs to a subject universally when it can be shown to belong to an arbitrary and primary instance of that subject (τὸ καθόλου δὲ ὑπάρχει τότε, ὅταν ἐπὶ τοῦ τυχόντος καὶ πρώτου δεκνῇται).⁶⁹

The term τυχόν in this passage is usually rendered as "chance instance" or "random instance". This rendering is misguided by the etymological analysis of the word, which derives from the word τύχη. However, the use of τυχόν in this context should be understood as "arbitrary". Such a reading is also suggested by its use in Euclid's *Elements*.⁷⁰ The focus is on the individual subject chosen, rather than on the random character of the act of choice.

The use of such a rule of generality is necessary for the development of Neo-Pythagorean arithmetic, in absence of specific quantification. In this way, the statements of Neo-Pythagorean arithmetic can be understood as general declarations of our capability to implement certain constructions for any given number of certain kind.

The belief that we can implement the required construction for any given number (combinatorial pattern) might have been rooted in the experience gained from the realization of such constructions. As a result of this kind of "experimentation" it becomes clear how one has to proceed in each case, i.e. at

⁶⁹ Aristotle *Posterior Analytics* 73b 25-74a.

⁷⁰ See, for instance, the use of the term 'arbitrary point' (τυχόν σημείον) and 'arbitrary angle' (τυχοῦσα γωνία) in the *Elements* I, 5 and VI, 9, respectively.

each case it is clear what we have to do. In this sense, the statements established in Neo-Pythagorean arithmetic are *general*, that is they establish that for any number of certain kind the outcome of the construction, according to a definite rule, is a number possessing the required property.

Negation and *tertium non datur*

It is obvious from the evidence of arithmetical reasoning discussed so far that Neo-Pythagorean arithmetic is about affirmative sentences stating something 'positive' that can be confirmed by means of the construction of the corresponding configuration. We do not meet in Neo-Pythagorean arithmetic any kind of 'negative' sentences, that is statements asserting existence of a number specified by a negative property, or statements asserting impossibility of a construction. Such statements are not of such straightforward "experimental" character. The lack of negative statements in the canonical exposition of Neo-Pythagorean arithmetic lends to it certain peculiar semantic characteristics, namely all sentences in Neo-Pythagorean arithmetic are affirmative statements asserting the possibility of certain constructions. Therefore, the Neo-Pythagorean arithmetic contains only such "experimental" truths that describe the actual state of affairs.

Accordingly, the arithmetical Being of the Neo-Pythagoreans is a genetic true Being, generated from a minimal, indivisible entity – the *monas*. Moreover, it is a descriptive Being, since it is based on "indication" (δείξις), which comes to be by a timeless generation process that follows definite rules. The entities to which these rules are applied are finite, insofar as only limited things are comprehensible and knowable, yet they are potentially infinite. The 'universe' of these entities is not supposed to be given beforehand, but they come to be as a result of construction. One does not find in the Neo-Pythagorean writings any attempt to establish a dichotomy between true and false Being (or between true and false statements) and, thereby no conscious use of the principle *tertium non datur* can be attributed to them.

The use of what we called *tertium non datur* in the definition of equality (Part I) should be understood not as an application of the abstract law of excluded middle. We used the term in order to provoke the question "is *tertium non datur* used in Neo-Pythagorean arithmetic?" However, our answer to this question is negative. The aforementioned use should be understood as a fact

drawn from the comparison of the graphical representations of numbers of the same type.

Let two numbers of the same kind be given. Both of them begin from the unit. If we begin to construct them carrying out the operation of addition by a unit (or by placing the gnomon), then the two procedures will continue in the same way and two cases are possible:

- The two processes will stop at the same time;
- One process will continue while the other will stop.

In the first case the corresponding suites or configurations of numbers can be get into coincidence, whereas in the second case the configuration of one number can be get into coincidence only with an initial segment of the other. Accordingly, in the former case the numbers are equal, while in the later, are unequal.

CONCLUSIONS

In our understanding, the Neo-Pythagorean arithmetic is a positive finitary fragment of arithmetic that can be developed without assumptions of axiomatic character, but as informal theory of counting over a domain of concrete finite objects of combinatorial character. These objects are intrinsically associated with a certain rule for their construction and characterised by a specific mode of organisation of a designated entity, taken for unit, into wholes possessing internal structure.

The role of these objects for the elaboration of the arithmetical theory is double: Firstly, they are combinatorial patterns, that is they obey certain regularities of combinatorial character. Secondly, they are functional, namely, they are, so to say, a mode of existence of number; numbers have not been yet abstracted from their combinatorial model.

The definition of arithmetical concepts and the establishment of theorems in this type of arithmetic is reduced to the implementation of certain genetic constructions that are carried out, starting from the monad and following a definite combinatorial rule described in words. Insofar as the objects involved in arithmetical reasoning are infinite, the corresponding constructions are incomplete and implicitly assumed potentially realisable. To this effect, a finitary quantifier-free form of induction is applied when a general property is asserted of "all" numbers of a certain kind. This specific form of induction is

not only compatible with the type of arithmetic developed by the Pythagoreans and the structure of their universe, but also sufficient for the requirements of arithmetic elaborated.

It is important to stress that the Arabic mathematicians perceived the difference in approach between Nicomachus and Euclid. Ibn al-Haytham states explicitly the distinction between the two versions of arithmetic: while Nicomachus' arithmetic proceeds by inductive reasoning, Euclid's version is developed by deductive reasoning.

Properties of numbers are shown in two ways: the first is by induction, since if we take the numbers one by one and if we distinguish between them, we find by distinguishing and considering all their properties, and to find the number in this way is called *al-arithmāṭiqī*. This is shown in a book on *al-arithmāṭiqī* [Nicomachus of Gerasa]. The other way of showing properties of numbers proceeds by proofs and deductions. All properties of numbers grasped by proofs are contained in these three books [of Euclid] or in those which refer to them.⁷¹

The Nicomachean type of arithmetic was called *al-arithmāṭiqī*, which is the phonetic Arabic transcription of the Greek term ἀριθμητική, whereas the Euclidean variety of arithmetic was called *ilm al-'adād*, that is "the science of numbers" [Rashed 1994, 246].

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⁷¹ Ibn al-Haytham, *Sharḥ muṣādarāt Kitāb Uqlīdis*, (Commentary on the Premises of Euclid's "Elements"). Feyzullah, Istanbul, MS 1359, f. 213'. Quoted from [Rashed 1994, 246].

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