

Mathematics Before and After Pythagoras

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Exploring the Foundations and Evolution
of Mathematical Thought



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Truth shall prevail despite all attempts at suppression.

*No subject loses more than mathematics
by any attempt to dissociate it from its history.*

James Whitbread Lee Glaisher (1848–1928, England)

*Dedicated to my teachers who taught me how to stand in
the mathematical world:*

Swami Dayal Nigam (1924–2009, India)

Vangipuram Lakshmikantham (1924–2012, India-USA)

Günther Hämmerlin (1928–1997, Germany)

Roberto Conti (1923–2006, Italy)

Foreword

This is an impressive book that provides a comprehensive treatment of a plethora of mathematical ideas and results from different civilizations and cultures before the time of Pythagoras to the present. Since numbers have held the attention of humans from the dawn of civilization, the history of mathematics is intertwined with the history of civilization. And among the numerous mathematical luminaries in history, Pythagoras occupies a lofty position as one of the most influential thinkers—both in terms of his mathematics and his philosophy. Thus Professor Agarwal has done a great service by writing this book focusing on the intellectual contributions of Pythagoras, and providing a global and historical perspective by discussing the mathematical developments before his time, and touching upon a variety of significant problems that have engaged, and continue to engage, many of the most gifted scientific minds of later generations up to the present.

Pythagoras was both a mathematician and a philosopher. He had a number of students and followers (the Pythagoreans), and his teachings influenced the development of mathematics and philosophy throughout the Mediterranean region for several centuries. Prof. Agarwal provides in Chap. 1, a detailed account of the fascinating life, work, and teachings of Pythagoras, describing also what the Greek philosopher-mathematician learned from other cultures during his travels.

The name of Pythagoras is most famously associated with *The Pythagoras Theorem* which states that for a right-angled triangle, the square on the hypotenuse is the sum of the squares on the other two sides. Pythagoras neither discovered nor proved this theorem, but he and his pupils were interested in right-angled triangles with positive integer sides. The Pythagoras theorem is central to all developments in geometry including the study of the lengths of curves, since distance between two points in two and higher dimensional spaces is determined to using this fact for right-angled triangles. In Chap. 5, Prof. Agarwal provides a detailed and thorough account of the history of the Pythagoras theorem, and gives five different proofs of it. And with the Pythagorean equation with integer solutions as a starting point, Prof. Agarwal analyzes in Chap. 6 more general Diophantine equations including Fermat's assertion whose resolution after 300 years is one of the crowning achievements of twentieth-century mathematics.

The Pythagoreans also studied sequences of integers associated with geometrical figures, such as triangular numbers, squares, pentagonal numbers, and so on, and more generally, figurate numbers. Chapter 7 is a detailed treatment of such figurate numbers and certain number theoretic questions involving figurate numbers. The proof of the irrationality of $\sqrt{2}$ is attributed to Pythagoras in classic number theory textbooks such as that by Hardy and Wright. Regardless of who first proved the irrationality of $\sqrt{2}$, it was this realization that began the theory of irrational numbers, a subject that remains an active area of research to this day. In general, it is very difficult to confirm the irrationality of a given number. The irrationality and the transcendence of π were established only in the nineteenth century, and that finally settled in the negative one of the three problems of Greek antiquity, namely to construct using only a ruler and compass, a square equal in area to a given circle. The final Chap. 8 of the book is an account of some major developments in the study of irrational and transcendental numbers.

In summary, this book has a lot to offer—mathematically, historically, and even philosophically. It is written in a style that would appeal to lay persons, yet has a substantial amount on the history of various mathematical developments that will be useful even for researchers. We should be thankful that Prof. Agarwal, a very prolific and reputed researcher in the field of differential equations, has spent so much time in writing this book in the area of mathematical history, which experts and non-experts will definitely enjoy.

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Preface

Eric Temple Bell (1883–1960, USA) in his treatise [61] *The Magic of Numbers* remarked, “If one man more than another is to be credited with starting the mathematical and physical sciences on their course from antiquity to the present it is Pythagoras. And if western civilization means the technology and commerce of recurrent industrial revolutions detonated by the application of experiment and mathematics to the physical world, Pythagoras was its prime mover. All this is on the strictly scientific side. On the side of purely intellectual activity, the numerology (number mysticism) of Pythagoras and his Brotherhood is the source of essential germinal ideas in metaphysics of the sciences of Plato of Athens (around 427–347 BC, Greece).” According to Aristotle (around 384–322 BC, Greece) “The so-called Pythagoreans, who were the first to take up mathematics, not only advanced this subject, but saturated with it, they fancied that the principles of mathematics were the principles of all things.” Bertrand Arthur William Russell (1872–1970, England-USA) in [440] *A History of Western Philosophy* contends that the influence of Pythagoras on Plato and others was so great that he should be considered the most influential philosopher of all time. He concludes that “I do not know of any other man who has been as influential as he was in the school of thought.”

Besides philosophy, the following two attributes are due to Pythagoras: The explicit recognition that proof by deductive reasoning offers a foundation for the structures of number and form (in the sense we still know and follow it), and the daring conjecture that nature can be understood by human beings through mathematics, and that mathematics is the language most adequate for idealizing the complexity of nature into appropriable simplicity. Once deductive mathematics was accepted as real mathematics all the saints and sages (ancient philosophers, who by study, experiments, concentration of minds, and perhaps intuition [unreliable source of knowledge], arrived at the fixation of certain laws governing life) previous work was condemned or called trivial applied to practical problems such as land surveying, commerce, and counting.

To glorify Pythagoras achievements, and as a whole of Greeks, on all prior mathematical works several damaging remarks have been written, for example, according to Henry James Sumner Maine (1822–1888, Scotland-France), “Except

the blind forces of nature, nothing moves in this world which is not Greek in its origin”; Walter William Rouse Ball (1850–1925, England) [53] wrote “Oriental mathematics may be an interesting curiosity, but Greek mathematics is the real thing”; “The Hindoos, like the Chinese, have pretended that they are the most ancient people on the face of the earth, and that to them all sciences owe their creation. But it is probably that these pretensions have no foundation; and in fact no science or useful art (except a rather fantastic architecture and sculpture) can be definitely traced back to the inhabitants of the Indian peninsula prior to the Aryan invasion”; Godfrey Harold Hardy (1877–1947, England) recorded “The Greeks are not clever schoolboys or scholarship candidates, but fellows at another college”; John Edensor Littlewood (1885–1977, England) noted “Compared with what the Greeks achieved, the mathematics of Egypt and Babylonia is the scrawling of children just learning to write, as opposed to great literature. These civilizations barely recognized mathematics as a distinct discipline, so that for over a period of 4000 years hardly any progress was made in the subject”; Anthropologist Ralph Linton (1893–1953, USA) stated hypothetically that “... if Albert Einstein (1879–1955, Germany-USA) had been born into a primitive tribe which was unable to count beyond three, life-long application to mathematics probably would not have carried him beyond the development of a decimal system based on fingers and toes”; in 1967, Árpád Szabó (1913–2001, Hungary) writes “Before the development of Greek culture the concept of deductive science was unknown to the Eastern people of antiquity. In the mathematical documents which have come down to us from these peoples, there are no theorems or demonstrations, and the fundamental concepts of deduction, definition, and axiom have not yet been formed. These fundamental concepts made their first appearance only with the Greek mathematics.” Most importantly, he proclaimed that if one means by a proof any explanatory note that serves to convince and to enlighten, then one finds an abundance of proofs in ancient mathematical texts other than those of the Greeks (it also means that mathematics was existing before Greeks). Further, “We must not forget that what constitutes ‘proof’ varies from culture to culture, as well as from age to age.” In addition, Bell in [61] writes “A proof that convinces the greatest mathematicians of one generation may be glaringly fallacious or incomplete to a schoolboy of a later generation.”

We also note that the history of most mathematical subjects often traces a long way back in the timeline, where a process of slow evolution and introduction of new ideas led to some major discovery, which affects the foundations of modern mathematics. In fact, most of the times, ideas existed in the past, and were even applied in problem solving; however, it took a long time period to generalize the theory and systematically prove these ideas. Thus, mathematics has never been a single person process. For example, we will see that Isaac Newton (1642–1727, England) used simple interpolation in 1665 to generalize millennia-old binomial expansion, which was proved by Niels Henrik Abel (1802–1829, Norway) only in 1826; several results of Leonhard Euler (1707–1783, Switzerland) are based on simple (sometimes tedious) calculations, which were proved several years later; Srinivasa Ramanujan (1887–1920, India) compiled nearly 3900 results (mostly identities and equations) during his short lifetime, a small number of these results

were actually false and some were already known, in recent years most of his claims have now been proven correct. George Gheverghese Joseph (born 1928, India) in his book [289] of 1991 focused mainly on the achievements of Kerala (India) in astronomy and mathematics and the transmission of mathematics from India to Europe. In his book “Beyond Numeracy” of 1992, John Allen Paulos (born 1945, USA) tells this story: “A German merchant of the fifteenth century asked an eminent professor where he should send his son for a good business education. The professor responded that German universities would be sufficient to teach the boy addition and subtraction, but he would have to go to Italy to learn multiplication and division. Before you smile indulgently, try multiplying or even just adding the Roman numerals CCLXIV, MDCCCIX, DCL, and MLXXXI without first translating them.” While Paulos provides no source for this story, there seems to be some truth as the whole of Europe was waking up from the dark ages between the fifth and fourteenth centuries.

In 2005, Vangipuram Lakshmikantham (1924–2012, India-USA), Srinivasa Leela (born 1936, India-USA), and Jonnalagadda Vasundhara Devi (born 1964, India) in [329] focused on the origin of the mathematics and corrected the chronology which was distorted by Western historians of mathematics. They have specially reported several pre-Pythagoras accomplishments of Indians for which historians have credited to Pythagoras and other Europeans. Their work was further extended and explained by Agarwal and Sen in 2014, see [14]. This book also lights on the very humanity of almost 400 mathematicians, their mode of thought, and struggle in their achievement. David Gray (USA) in his article *Indic Mathematics: India and the Scientific Revolution* of 2011 writes “The study of mathematics in the West has long been characterized by a certain ethnocentric bias, a bias which most often manifests not in explicit racism, but in a tendency toward undermining or eliding the real contributions made by non-Western civilizations. The debt owed by the West to other civilizations, and to India in particular, go back to the earliest epoch of the ‘Western’ scientific tradition, the age of the classical Greeks, and continued up until the dawn of the modern era, the renaissance, when Europe was awakening from its dark ages.” He concludes by asserting that “the role played by India in the development (of the scientific revolution in Europe) is no mere footnote, easily and inconsequentially swept under the rug of Eurocentric bias. To do so is to distort history, and to deny India one of its greatest contributions to world civilization.”

In Chap. 1, we present a comprehensive study of Pythagoras, Pythagoreanism, and the early Pythagoreans through an analysis of the many representations of the individual and his followers, allowing them to complement and critique each other. This includes major events and struggles in his life since birth till death, details of his philosophy (The Golden Verses and Symbols), and dramatic mathematical and astronomical achievements which made him immortal. We shall also report the origin of most of his accomplishments with supporting statements of distinguished scholars.

In mathematics one of the major contributions of Pythagoras is to give *divine significance* to most of the natural numbers, and an attempt to find mathematical

explanations for everything in the Universe in terms of numbers (natural and rational) including in geometry. In Chap. 2, we begin with natural numbers whose very origin is a mystery; however, it is generally perceived that they have in some philosophical sense a natural/divine existence independent of man. This is followed by the number sense which is intuitive understanding of the natural numbers, their magnitude, their patterns and relationships, and how they are affected by the basic operations (addition, subtraction, multiplication, and division). We shall exhibit that number sense is not only the natural ability of primitive man and children, but also there are recorded incidences of birds, animals, insects, and aquatic creatures who show through their behavior a rudimentary number sense, namely, comparing/sorting. Next, we shall provide the origin of negative numbers, and Brahmagupta's (born 30 BC, India) treatment of positive and negative numbers in terms of "fortunes" (dhana) and "debts" (rina), also his rules for dealing with negative numbers (very similar to those we still use today). We shall convince the reader that only through continuous effort and struggle from the middle of the nineteenth century negative numbers received their relevance logically across the world.

This is followed by the origin of zero to whom the status of a number was given by Hindus. Its discovery took place within an environment that was at once mystical, philosophical, religious, cosmological, mythological, and metaphysical. Brahmagupta defined zero as the result of the subtraction of a number by itself, and laid down the basic rules; however, he struggled when it came to division by zero. In fact, it took several centuries to realize that mathematically $0/0$ is neither meaningful nor meaningless, it is indeterminate, and it may have any value but only in the limiting sense. Most importantly the number zero led to the decimal system. From the thirteenth century, when calculations could be performed "in writing," slowly the importance of zero and the place-value system was recognized all over the world, and prominent mathematicians and philosophers started to understand their importance and making constructive comments.

In Sect. 2.7, we shall mainly preset several examples from physics, mathematics, games, and puzzles where large numbers appear in a very natural process. Large numbers will appear in later chapters routinely. One of the major struggles in mathematics has been to accept that infinity is a legitimate concept. In Sect. 2.8, we shall begin with Hindu mythology according to which zero is also a term Ananta, which means infinite (infinite void or void infinite), and in Hindu philosophy God is infinite and within us. The infinite remains the same, even though the infinite Universe which has no beginning or end has come out of it, for details see Lakshmikantham [330]. For general reading see the exceptional book [439] of Rudy Rucker (born 1946, USA). We shall discuss Jainas classification of numbers into three groups enumerable, innumerable, and infinite (nearly infinite, truly infinite, and infinitely infinite). We shall carefully define and illustrate potential and actual infinity. We shall show that from the beginning Greek philosophers and mathematicians refused to accept or confused with the concept of infinity and this continued till eighteenth century. In fact, during this period several prominent mathematicians perpetrated all sorts of blunders, made false proofs and drew incorrect conclusions. Finally, Georg

Ferdinand Ludwig Philipp Cantor (1845–1918, Russia-Germany) during 1871–84 systematically mathematized the concept of infinity. His classification of countable and uncountable sets became a turning point in whole of mathematics. In this section we shall also introduce infinitely small numbers or infinitesimals, which eventually led to the discovery of calculus.

Section 2.9 deals with number mysticism, which is based on the idealistic belief that numbers are not only symbols of reality, but the final substance of real things, and possess spiritual and magical powers. While the origin of number mysticism is unknown, but it is believed that it started along with the birth of natural numbers. For Pythagoras only first ten numbers were of spiritual significance (some claim first 50) and some human attribute. We shall discuss a special geometric arrangement of the numbers ten, which Pythagoreans called Tetraktys and recognized it as fate, the Universe, the heaven, and even God, and honored it by never gathering in groups larger than ten. We shall also discuss about numerology, which is an offshoot of number mysticism and to this day persists in otherwise unaccountable omens and superstitions in most of the religions. In Sect. 2.10, we have collected several numbers which have some special properties. This includes palindromic numbers, and magic squares which have been considered strong talismans against evil, and possession of a magic square was thought to insure health and wealth. Finally, in Sect. 2.11, we have introduced complex numbers. This includes their origin, basic rules, representations, Euler's most curious formula, and roots of unity.

To make this book accessible to wider audiences, in Chap. 3, some basic questions which are vaguely discussed in existing books have been clearly explained and embellished through interesting examples from several diverse fields. These questions will also pave the way to appreciate the later chapters. To summarize, we shall show that despite of numerous attempts from primordial to modern leading philosophers and mathematicians, the word mathematics is too subtle to define exactly; however, a mathematics teacher and a mathematician can be differentiated and defined assuredly. We shall reveal that history of mathematics deepens our respect for human cultures and collaboration across time regardless of their location, and presents us with role models. We shall also exhibit the human nature of mathematicians who are very often believed to be bizarre individuals. We shall detail basic prerequisites for the deductive mathematics such as a mathematical statement and a mathematical definition. We shall rigorously define axioms and list them for geometry, natural numbers, fields, and sets. We shall establish that occasionally eliminating or changing an axiom from the earlier assumed axioms has led to altogether new mathematics, which is as consistent as earlier, and often more useful.

Then we shall define only that segment of logic that is necessary in mathematics. This prepares us to define the terms theorem/result/proposition, lemma, and corollary, which are the heart of whole mathematics. Even an obvious proposition in mathematics without its proof is meaningless so we shall carefully study the term mathematical proof. Then we shall discuss several widely used methods to prove theorems and illustrate each with elementary, but of paramount interest, examples. In mathematics there are many innocent looking problems for which

classical mathematical proofs are not within the reach of humans. For one of such problems, namely, four color theorem, a major breakthrough came in 1976 with the assistance of electronic computer. Since then such proofs have been added in the vocabulary of mathematics as computer-based proofs, and have been successfully applied to several unsolved problems. This has meticulously filled the gap between mathematicians and computer scientists. However, among mathematicians there is a disagreement whether to accept computer-based proofs 100%. Certainly, such proofs provide guidance in understanding the problem better, but loses the flavor of classical mathematics.

An example that disproves a mathematical statement (shows that it is false) is called a counterexample. It is beyond doubt that often the construction of a counterexample is challenging. We shall provide a few simple examples to clear up this important concept in mathematics. Next we shall take up one of the most demanding questions in mathematics “can proofs be exact.” We shall conclude that today’s proof of a theorem is never permanent, within a few years (sometimes several years) it is modified/simplified/generalized, and later (often) you as well as your proof is being criticized. Contemplating this in mind, we shall mention several proofs that are excessively long for which mathematicians are searching for shorter proofs. A mathematical statement that has not yet been rigorously proved is called a conjecture. We shall cite and explain several conjectures, some of which are challenging from the last several years. A statement for which different valid logical arguments lead to different conclusions (namely true and false) is called a paradox. We shall discuss several paradoxes, some of which are entertaining. We shall also discuss in detail four paradoxes of Zeno of Elea (around 495–435 BC, Greece) which require the acceptance of infinity. While deciding of bad, good, and beautiful mathematics is individualistic, several mathematicians/philosophers have tried to response conclusively. We have tried to recognize the difference between bad, good, and beautiful mathematics through simple examples. In the last Sect. 3.20, we shall take up mainly three classical problems of antiquity. We shall show that Euclidean tools are not enough to solve these problems. The most important aspect of these problems is that the failure of solving these problems has led to substantial amount of new and deeper mathematics.

In Chap. 4, we shall study subsets of natural numbers. We shall begin with the sets of prime and composite numbers whose union is the set of natural numbers. In Sect. 4.2, we shall discuss Eratosthenes of Rhodes’ (around 276–194 BC, Greece) method known as Sieve of Eratosthenes which is apparently the first methodical attempt to separate the primes from the set of natural numbers; Ramanujan highly composite numbers; Square spirals of Stanislaw Marcin Ulam (1909–1984, Poland-USA) and his co-workers; Two jewels in number theory proved by Euclid of Alexandria (around 325–265 BC, Egypt-Greece), namely, Fundamental Theorem of Arithmetic which ensures every integer $n \geq 2$ is either prime or can be expressed as a product of primes (thus prime numbers are the “atoms” of the natural numbers), and Infinity of Prime Numbers (which makes their study fascinating); Theorem of Peter Gustav Lejeune Dirichlet (1805–1859, France) which ensures every arithmetic sequence $a + nd$, $n = 1, 2, \dots$ in which a and d are relatively prime (no common

factors other than 1) contains an infinitude of primes; Present status of Joseph Louis François Bertrand's (1822–1900, France) assertion that between any number and its double there exists at least one prime; and palindromic primes.

In Sect. 4.3, we shall provide easily verifiable divisible tests by certain integers, especially for all primes up to 50, which help in confirming for a given number of reasonable size to be composite. In Sect. 4.4, we shall examine Père Marin Mersenne (1588–1648, France) numbers and primes denoted as $M_n = 2^n - 1$, $n \geq 1$. We shall affirm that $M_{82589933}$ is the largest known prime. It is not known whether there exist infinitely many Mersenne primes, if every Mersenne number is square free, and if there are infinitely many composite Mersenne numbers. An integer $n \geq 2$ is said to be perfect (the nomenclature is due to Pythagoras) if it is equal to the sum of its proper divisors (excluding itself and including 1). In Sect. 4.5, we shall prove Euclid's result which provides the construction of all even perfect numbers, and its stronger version due to Euler. The largest known even perfect number is $2^{82,589,932}(2^{82,589,933} - 1)$.

In 1640, the father of modern number theory, Pierre de Fermat (1601–1665, France), also known as *the prince of amateurs and mischievous genius* (see Michael Sean Mahoney, 1939–2008, USA [356,357]), conjectured that Fermat numbers $F_n = 2^{2^n} + 1$, $n \geq 0$ without exception are prime. In Sect. 4.6, we shall follow Euler to show that $F_5 = 641 \times 6700417$ and hence composite. In fact, no other Fermat primes F_n with $n > 4$ have been found. In Sect. 4.7, we shall provide the proof of Fermat's Little Theorem: If p is prime and a any positive integer, then p divides $a^p - a$. We shall also show that the converse of this result does not hold. This innocent looking result turned out to be fundamental for the progress of number theory. A desire of every number theorist is to find a function $f(n)$ that yields only prime numbers, and the sequence of primes so obtained is infinite. Some known attempts which are only of theoretical importance have been discussed in Sect. 4.8. John Wilson's (1741–1793, England) Theorem states: If n is a prime, then the quantity $((n - 1)! + 1)/n$ is a whole number. Joseph Louis Lagrange (1736–1813, Italy-France) not only completed John Wilson's result: n is prime iff (both necessary and sufficient) n divides $(n - 1)! + 1$, but also proved it; however, his proof uses complicated arguments. In Sect. 4.9, we shall give an elementary proof of the complete result, and because of occurrence of $n!$ in the result we conclude that this result is also only of theoretical interest.

In number theory Christian Goldbach's (1690–1764, Prussia-Russia) Conjecture: Every even $n > 2$ is the sum of two, not necessarily distinct, primes, and is widely known for its simplicity in stating and complexities in proving. In Sect. 4.10, we shall summarize the efforts made in settling Goldbach's conjecture. Primes of the form p and $p + 2$ are called twin primes. For these primes the famous conjecture is: There are infinitely many twin primes. In Sect. 4.11, we shall provide the present status of this conjecture. In Sect. 4.12, we shall consider one of the most important function in number theory, namely, $\pi(x)$, which represents the number of primes less than or equal to a given number x . Karl Friedrich Gauss (1777–1855, Germany) conjectured that $\pi(x)$ is asymptotically equal to the ratio $x/\ln x$. His conjecture now known as the Prime Number Theorem was independently proved by Jacques

Salomon Hadamard (1865–1963, France) and Charles de la Vallée Poussin (1866–1962, Belgium). Since then, several proofs of prime number theorem have been offered, some of these we shall summarize. A pair of integers in which each is the sum of the divisors of the other is called an Amicable Pair, or the Friendly Pair. In Sect. 4.13, we shall discuss Thabit ibn Qurra’s (826–901, Turkey-Iraq) general formula which leads to certain types of amicable pairs, and its generalization due to Euler. Unfortunately, their results require the primality of three numbers in advance. Although more than 1, 227, 319, 870 amicable pairs are known, theoretically it is not known if the number of amicable pairs is finite or infinite.

In Sects. 4.14 and 4.15, we shall respectively discuss Fibonacci (Leonardo of Pisa, around 1170–1250, Italy) and François Édouard Anatole Lucas (1842–1891, France) numbers. For these numbers we shall provide recurrence relations, explicit solutions, identities, and generating functions. We shall notice that Fibonacci numbers occur in nature in many surprising ways. It has been conjectured that there are infinitely many Fibonacci as well as Lucas primes. In Sect. 4.16, we shall provide the construction of Golden Section/Ratio (also known as Divine Proportion) $\varphi = (1 + \sqrt{5})/2$, and show its connection with Fibonacci and Lucas numbers. The number φ is found in nature, art, architecture, poetry, music, and of course mathematics. Psychologists have shown that the golden ratio subconsciously affects many of our choices, such as where to sit as we enter a large auditorium, where to stand on a stage when we address an audience, and so on.

The main aim of Sect. 4.17 is to discuss Gauss Law of Quadratic Reciprocity, which he called the gem of arithmetic, and remained fascinated by it throughout his life. In fact, out of 246 known proofs of this law 8 belongs to Gauss. In Sect. 4.18, we shall prove that there are infinite number of primes of the form $4n - 1$ and $4n + 1$; any number of the form $4n + 3$ cannot be expressed as a sum $a^2 + b^2$ of two perfect squares; and Fermat’s Two Square Theorem: if n is a prime number, then it can be expressed as a unique (except the order) sum of two squares iff either $n = 2$ or $n = 4k + 1$. Fermat’s this result is cited in any discussion of mathematical beauty. In Sect. 4.19, we shall state and partially prove Adrien-Marie Legendre’s (1752–1833, France) Three-Square Theorem: An integer n can be represented as the sum of three squares of integers, i.e., $n = a^2 + b^2 + c^2$ iff n is not of the form $n = 4^h(8k + 7)$ for nonnegative integers h and k . In this result the representation is not necessarily unique. In Sect. 4.20, we shall prove Lagrange’s Four-Square Theorem: Every positive integer can be written as the sum of four integer squares. In this result, the representation is also not necessarily unique.

Keeping in mind that the converse of Fermat’s Little Theorem does not hold, a composite number n is called Carmichael Number (after Robert Daniel Carmichael, 1879–1967, USA) provided n divides $b^n - b$ for all integers b . In Sect. 4.21, we shall provide a characterization of Carmichael numbers. In Sect. 4.22, we shall discuss the importance of the numbers 714 and 715, and the new mathematics that has emerged from these numbers. In Sect. 4.23, we shall discuss Bell Primes, Marie-Sophie Germain (1776–1831, France) Primes, Balanced Primes, Ferdinand Gotthold Max Eisenstein (1823–1852, Germany) Real Primes, Primorial Primes, Fortunate Numbers, Good Primes, Denis Arthur Higgs (1932–2011, England),

and Ramanujan Primes, which are special subsets of prime numbers that have been studied with great interest. In Sect. 4.24, we shall conclude this chapter by answering the necessity to find next larger prime number. It is interesting to note that a few prime numbers were known almost 22,000 years back; Hindus had adequate knowledge of prime, perfect, and amicable numbers, much before the days of Pythagoreans; and Fibonacci numbers were known to Hindus by the name *matrameru* during 500 BC.

An ever fresh result in geometry is Pythagoras (or Pythagorean) Theorem: If a and b are the lengths of the two legs of a right triangle, and c is the length of the hypotenuse, then the sum of the areas of the two squares on the legs equals the area of the square on the hypotenuse, i.e., $a^2 + b^2 = c^2$. This equation has been ranked very high among all mathematical equations, and appreciated throughout the history for its simplicity and variety of applications. In Chap. 5, we shall provide its origin which is at least 5200 years old. For Pythagorean theorem almost 500 different proofs are known; out of these we shall provide five which are elementary and have historical importance. Among these we include a proof owing to President James Abram Garfield (1831–1881). We shall also furnish the converse of Pythagorean theorem. Then we shall detail five important generalizations of Pythagorean theorem which were contributed by Hippocrates of Chios (around 470 BC, Greece), Alexandrian Claudius Ptolemaeus (Ptolemy, around 90–168, Egypt-Greece), Pappus of Alexandria (around 290–350, Egypt, was either Greek or a Hellenized Egyptian), ibn Qurra, and the Law of Cosines which first appeared in Euclid's Book II (Propositions 12 and 13) and explicitly stated by Jemshid al-Kashi (around 1380–1429, Persia). Next, we shall generalize Pythagorean theorem in vector spaces, and show how it encompasses for rectangular solids. We shall also prove three abstract results which are due to Jean Paul de Gua de Malves (1713–1785, France), D.R. Conant (USA) and W.A. Beyer (USA), and Eisso Atzema (USA). Finally, we shall discuss Pythagorean theorem in non-Euclidean geometry. Specifically, we shall present spherical law of cosine which was recorded in the first book on Astronomy *Surya Siddhanta*, hyperbolic law of cosine which was first known to Franz Adolph Taurinus (1794–1874, Germany), Pythagorean theorem in Riemannian geometry which was first given by George Friedrich Bernhard Riemann (1826–1866, Germany) in his doctoral address in 1854, and give reason why Pythagorean theorem fails in Elliptic geometry. We shall conclude this chapter with 11 historical problems and an example that requires Pythagorean theorem.

A set of three positive integers a , b , and c which satisfies Pythagorean relation $a^2 + b^2 = c^2$ is called *Pythagorean triple* and written as an ordered triple (a, b, c) . A triangle whose sides form a Pythagorean triple is called a Pythagorean triangle, which is clearly a right triangle. A Pythagorean triple (a, b, c) is said to be *primitive* if a, b, c have no common divisor other than 1. In Chap. 6, we shall make a systematic investigation of primitive Pythagorean triples. In Sect. 6.2, we shall show that Hindus, Babylonians, Egyptians, and Chinese were having ample knowledge of Pythagorean triples several centuries before Pythagoras. In Sect. 6.3, we shall provide Euclid's proposition which gives the characterization of all primitive Pythagorean triples. This proposition was later proved by several

mathematicians; we shall break the proof in six parts and give complete details. In this section we shall also furnish a table of primitive Pythagorean triples with $c \leq 1000$. In Sect. 6.4, for the primitive Pythagorean triples we shall provide 36 elementary results which can be considered as the modern beginning of the number theory. For example, we shall show that in a primitive Pythagorean triple (a, b, c) either a or b is divisible by 3, either a or b is divisible by 4, and either a, b , or c is divisible by 5, and hence the product ab is divisible by 12, and the product abc is divisible by 60. As an another example, we shall show that perimeter of a primitive Pythagorean triangle and its area are the same only for the Pythagorean triple $(5, 12, 13)$.

In Sect. 6.5, we shall provide triples ensuring the construction of right-angled triangles whose sides are rational numbers. For this, we shall assume that a rational side or rational hypotenuse is given in advance. A Heronian triangle (a, b, c) has integer sides whose area is also an integer. Clearly, every Pythagorean triple is a Heronian triple, and hence there are infinitely many primitive Heronian triples; however, the converse is not true. In Sect. 6.6, for a given Heronian triangle we shall provide Brahmagupta's proportional condition which the triple (a, b, c) must satisfy, and for a given triple (a, b, c) sufficient conditions so that it is a Heronian triangle. A congruent number is a positive integer that is equal to the area of a rational right triangle. In Sect. 6.7, we shall list first ten congruent numbers and provide the simplest rational right triangle for the congruent number 157. So far, to decide if a given positive integer is congruent remains an open number-theoretic problem.

Fermat's claim of 1637 that the equation $a^n + b^n = c^n$ has no positive integer solutions for a, b , and c if $n > 2$ is known as Fermat's Last Theorem. In Sect. 6.8, we shall record the continuous struggle of several outstanding mathematicians for 350 years to prove this result, until Andrew John Wiles (born 1953, England) resolved it in 1994. For this, he employed known theories from many branches of mathematics; his original 200-page-long proof (it would be 1000 pages if all details are provided) was published in 1995 after condensing it to 129 pages. Apparently only very few people understand Andrew Wiles's proof, and the world is waiting for a simpler proof. A tuple of four integers a, b, c and d such that $a^2 + b^2 + c^2 = d^2$ is called Pythagorean quadruple, and (a, b, c, d) is called primitive if the greatest common divisor of its numbers is 1. In Sect. 6.9, we shall provide a few characterizations for the construction of Pythagorean quadruple. In Sect. 6.10, we shall report several identities which not only generalize Pythagorean quadruple but also parameterizes the sum of three cubes into a cube, i.e., of the form $x^3 + y^3 + z^3 = c^3$. In an effort to generalize Fermat's Last Theorem, in 1769, Euler conjectured that $x_1^k + x_2^k + \cdots + x_n^k = c^k$ implies $n \geq k$. From Sect. 6.10 it follows that Euler's conjecture holds for $k = 3$. In Sect. 6.11, we shall provide counterexamples to show that his conjecture is not true for $k = 4$ and $k = 5$. For $k \geq 6$ the validity of the conjecture is unknown. We shall also provide several examples for $4 \leq k \leq 8$ which support Euler's conjecture. Finally, in Sect. 6.12, we shall discuss Eugène Charles Catalan (1814–1894, Belgium-France) and Subbayya Sivasankaranarayana Pillai (1901–1950, India) conjectures. Catalan

conjecture confirms that the only solution in natural numbers of the equation $x^a - y^b = 1$ for $a, b > 1$, $x, y > 0$ is $x = 3, a = 2, y = 2, b = 3$. Pillai's conjecture (which is a generalization of Catalan's conjecture) says for fixed positive integers A, B, C the equation $Ax^n - By^m = C$ has only finitely many solutions (x, y, m, n) with $(m, n) \neq (2, 2)$. So far for the Pillai's conjecture the number of solutions has been calculated only for some particular cases.

Figurative numbers are numbers that can be represented in a geometric pattern, usually by dots/pebbles arranged in various regular and discrete patterns. It has been accepted that Pythagoreans were the first to study triangular and square figurative numbers. Nicomachus of Gerasa (around 60–120, Syria-Greece) in his book *Introduction to Arithmetic* (see [394]) of around 100 AD collected earlier works of Pythagoreans on natural numbers, and presented cubic figurative numbers (solid numbers). Since then, the study of figurative numbers continues to be a source of interest and motivation to both amateur and professional mathematicians. In Chap. 7, we shall study 34 different types of figurative numbers, starting with triangular numbers. For each type of figurative number, we shall provide: recurrence relation (which leads to an infinite sequence), the general term, various equalities, numerous properties, explicit relation with other numbers, necessary condition for a given number to be a figurative number, generating function, sum of first n and inverse of all terms of the sequence, and some possible applications. We shall also provide sums of first n positive integers with positive integer exponents, and some bounds when the exponents are positive fractions. Fermat in 1638 claimed that every positive integer is expressible as at most k , k -gonal numbers (Fermat's Polygonal Number Theorem). His theorem was fully resolved in 1813 by Augustin-Louis Cauchy (1789–1857, France). A difficult triangular case (every positive integer is the sum of three or fewer triangular numbers) was disposed of by Gauss in 1796. In the literature, Gauss result is known as EGPHKA theorem, and he wrote it as EGPHKA! num = $\Delta + \Delta + \Delta$.

One of the greatest discoveries in the whole of mathematics is the invention of irrational numbers, and then their understanding. In Chap. 8, we shall demonstrate that Vedic Ascetics more than 5000 years back were unsuccessful in finding exact values of the numbers $\sqrt{2}$ and π . The ancient records (supported by great philosophers, mathematicians, and historians) stipulate that Vedic Ascetics were also definite that these numbers are incommensurable/irrational. We shall exhibit that the claim of the historians of mathematics that Pythagoras proved the irrationality of $\sqrt{2}$ is only conjectural. In fact, the first geometric proof of the irrationality of $\sqrt{2}$ appeared only in Meno (Socratic dialogue by Plato) almost two hundred years after Pythagoras. Since then several different proofs of the irrationality of $\sqrt{2}$ and in general for $\sqrt{N}$ for any natural number N which is not a perfect square have been given. We have provided some of these important proofs. The next major understanding of irrational numbers came from the scholars of the Islamic Middle East toward the end of the first millennium CE. They started treating irrational numbers as algebraic objects, and most importantly provided a geometric interpretation of rational numbers on a horizontal straight line. Since then research continues for the known as well as unknown/expected irrational numbers, their subset of

transcendental numbers, and their computation to trillions of decimal places, we have detailed some of these advancements. We have also discussed Dedekind-Cantor axiom of the nineteenth century which provides geometric interpretation of all real numbers, and thus completes the Islamic work. Particularly, for the number π we have arranged individual's contributions chronologically to show that each continent of the world has contributed in this fascinating field of mathematics. We have also provided very simple proofs of the irrationality of e and e^2 , and transcendence of e and π .

We conclude this book with the note that mathematically interesting sequences of numbers are those that continue without end. If the primes were finite, they would be of considerably less interest; and if it is established ultimately that the perfect numbers are finite, their interest will become merely historical. Odd and even numbers, the primes and composite numbers, the squares, the cubes, the curious pentagonal numbers, algebraic numbers, irrational numbers, transcendental numbers, all are infinite. These infinite sequences of numbers among the infinite sequences of the natural numbers first suggested the revolutionary idea which is cornerstone of the modern theory of the infinite.

We hope in future readers of this book will justify (at least remember) the statement of Archimedes of Syracuse (287–212 BC, Greece) “the man who first states a *theorem* (poses a problem) deserves as much credit as the man who first proves it.” The present mathematical knowledge has only reached its present high level through the labors of numerous centuries for which one cannot underestimate the influence of every culture, personality, philosophy, region, religion, society, and social status. Of course, the focus of mathematical scholarship has shifted from place to place throughout history.

The main purpose of this book is to create interest among students and teachers at all levels, and hopefully its content should be accessible even to non-mathematicians. In the book we have combined history, philosophy, religion, mathematics, and elementary computation. Only at few places we have used sophisticated mathematical terms, which readers can easily skip without any lack of consistency. We have completely avoided tedious proofs, but illustrated the importance of the results with simple examples. To make this collection stimulating, we have included amusing anecdotes, puzzles, and historical problems. Our book requires a certain degree of intellectual maturity and a willingness to do some thinking on one's own.

A book of this nature cannot be written without deriving many valuable ideas from several sources. We express our indebtedness to all authors, too numerous to acknowledge individually, from whose specialized knowledge we have been benefitted. We have also immensely benefitted from several websites, especially en.wikipedia.org and www-history.mcs.st-andrews.ac.uk. Our sincere thanks to Number Theorists Heng Huat Chan (born 1967, Singapore), Carl Bernard Pomerance (born 1944, USA), and Stephen George Simpson (born 1946, USA) for clarifying doubts during the process of writing this book over the period of more than three years.

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